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PATH INTEGRAL APPROACH TO SUPERINTEGRABLE POTENTIALS ON THE TWO-DIMENSIONAL HYPERBOLOID

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This paper is the third in a series, in which we try to generalize the notion of superintegrable potentials, as known from the flat space, to the case of spaces of constant negative curvature. Path integral approach to superintegrable potentials on the two-dimensional hyperboloid is presented. We find five potentials of the sought type, which possess three functionally independent integrals of motion (observables), and in each case we present the appropriate path integral formulation. We list in the soluble cases the path integral solutions explicitly in terms of the propagators, the Green functions, and the spectral expansions in the wave functions. The coordinate systems on the two-dimensional hyperboloid are discussed in detail. The Stäckel matrix, the Schrödinger operator, the general form of the potential, which must be separable, and relevant observables are constructed for each coordinate system. A special care is taken of the proper generalization of the harmonic oscillator on the hyperboloid, i.e., the Higgs-oscillator, and the Kepler-Coulomb problem. The three remaining potentials are analogues of the Holt potential, the centrifugal potential, and the last one is the potential which is linear in the flat space limit.

Настоящая работа является третьей из серии работ, в которой обобщено понятие суперинтегрируемых потенциалов, известных для плоского пространства на случай пространств постоянной отрицательной кривизны. Сформулирован метод континуального интегрирования для суперинтегрируемых потенциалов на двумерном гиперболоиде. Найдены пять потенциалов искомого типа, которые содержат три функционально независимых интеграла движения (наблюдаемых), и в каждом случае выписаны соответствующие интегралы по траекториям. Описаны все случаи, где с помощью метода континуального интегрирования возможно решение в явном виде на языке пропагаторов, функций Грина и спектральных разложений по волновым функциям. Обсуждаются все возможные ортогональные системы координат на двумерном гиперболоиде. Для каждой из систем координат построены оператор Шредингера и матрица Штекеля, приведены соответствующие интегралы движения. Особое внимание уделено обобщению гармонического осциллятора, или осциллятора Хиггса, и задачи Кеплера — Кулона. Оставшиеся три потенциала являются аналогами потенциала Холта и центробежного, а последняя модель соответствует в пределе плоского пространства линейному потенциалу.

1. INTRODUCTION

In this paper we continue our study of potential problems in quantum mechanics in spaces of constant curvature which are separable in more than one coordinate system. For this kind of potential systems the notion *super-integrable* has been introduced by Evans [6] and Wojciechowski [66], as well as *Smorodinsky-Winternitz potentials*, because the first systematic investigation of such systems was undertaken by Smorodinsky, Winternitz and co-workers in Refs.[10,47,65]. In $\mathbb{R}^2$ there are four potentials of this type [10] which all have three constants (integrals) of motion (including energy), i.e., there are two more operators commuting with the Hamiltonian and with each other. In $\mathbb{R}^3$ there are five maximally superintegrable potentials with five integrals of motion [6,21] and nine minimally superintegrable potentials with four integrals of motion [6,21,23]. On the two-dimensional sphere we have found two superintegrable potentials; and on the three-dimensional sphere, three maximally and four minimally superintegrable potentials [22,23]. Generally, in D dimensions maximally superintegrable potentials have $2D-1$ integrals of motion, respectively observables; and minimally superintegrable potentials, $2D-2$ integrals of motion (this means that the notion minimally superintegrable and integrable cannot be distinguished in two dimensions).

Let us briefly discuss the physical significance of the consideration of separation of variables in more than one coordinate system. The free motion in some homogeneous space is, of course, the most symmetric one, and the search for the number of coordinate systems which allow the separation of the Hamiltonian is equivalent to the investigation of how many inequivalent sets of observables can be found, and there are D integrals of motion. The incorporation of potentials usually removes at least some of the symmetry properties of the space. Well-known examples are spherical systems, and they are most conveniently studied in spherical coordinates. For instance, the isotropic harmonic oscillator in three dimensions is separable in eight coordinate systems, namely in cartesian, spherical, circular polar, circular elliptic, conical, oblate spheroidal, prolate spheroidal, and ellipsoidal coordinates. The Coulomb potential is separable in four coordinate systems, namely in conical, spherical, parabolic, and prolate spheroidal II coordinates (for a comprehensive review with the focus on path integration, e.g., [21]).

The separation of a quantum mechanical problem in more than one coordinate system has the consequence that there are additional integrals of motion and that the discrete spectrum, if it exists, is degenerate. The Noether theorem connects the particular symmetries of the Lagrangian, i.e., the invariances with respect to the dynamical symmetries, with conservation laws in classical mechanics and with observables in quantum mechanics, respectively. In the case of the isotropic harmonic oscillator one has in addition to the conservation of energy and the conservation of the angular momentum, the conservation of the quadrupole momentum; in the case of the Coulomb problem one has in addition to the conservation of energy and the angular momentum, the conservation of the Pauli-Runge-Lenz vector. In total, these conserved quantities add up to five integrals of motion in classical mechanics, respectively observables in quantum mechanics. It is even possible to introduce extra terms in the pure oscillator and Coulomb-, respectively Kepler-problem, in such a way that one still has all these integrals of motion, however, somewhat modified [6].

In our paper [22] we extended the notion of «superintegrability» to spaces of constant positive curvature. One knows that the corresponding *Higgs-oscillators* (as discussed by, e.g., Granovsky et al. [11], Higgs [30], Ikeda and Katayama [32], Katayama [41], Leemon [45], Nishino [53], and [58]) and the *Kepler-Coulomb problem* (c.f., Granovsky et al. [12], Hietarinta [29], Ikeda and Katayama [32], Infeld [33], Infeld and Schild [34], Katayama [41], Kurochkin and Otchik [44], Nishino [53], Otchik and Red'kov [55], Schroedinger [59], Stevenson [62], and Vinitsky et al. [63]) in spaces of non-vanishing constant curvature do have additional constants of motion: the analogues of the flat space. For the Higgs-oscillator it is the Demkov-tensor [3,9,53], and for the Kepler problem it is the analogue of the Pauli-Runge-Lenz vector in a scape of constant curvature, c.f. [30,44,53]. It is also found that the Higgs oscillator and the Kepler-Coulomb problem are the only central systems [32] in spaces of constant curvature. However, additional non-central superintegrable potentials might exist.

In our investigation the path integral turns out to be a very convenient tool to formulate and solve the superintegrable potentials on the hyperboloid, and it provides the natural way in which the analytic structure of the solutions is manifested. Separation of variables in each problem can be done in a straightforward and easy way. There are already some studies of the oscillator problem and the Coulomb problem in spaces of constant curvature. The oscillator problem is not very difficult to solve, including the case where additional radial dependences are taken into account, which is basically path integral problems which are related to the Pöschl-Teller and modified Pöschl —

Teller path integral. The Coulomb problem is somewhat more involved, and the pure case has been discussed by means of path integrals in spherical coordinates by Barut et al. [1] and [16]. In the present investigation these earlier results will be used in the calculations, and no detailed derivations will be given in these cases. The path integral calculation of the Coulomb problem on the hyperboloid in elliptic-parabolic coordinates is completely new, and it turns out that some results of the calculation for the free motion can be used in its solution [20].

However, all former studies have taken into account only central systems and their solutions in spherical variables, which is obvious. Neither a systematic search for alternative descriptions in other coordinate systems has been done, nor a search for further separable potentials. In particular, the Holt potential with a linear term is important, because it allows the incorporation of electric fields. The case of magnetic fields on the two-dimensional hyperboloid has been considered by means of path integrals in [15], and it has been found that in spherical, horicyclic and equidistant coordinates a separation of variables is possible, i.e., in coordinate systems which have one ignorable coordinate [40], i.e., they are non-parametric, and the corresponding solutions are circular, respectively plane waves in this (ignorable) coordinate. Depending on the strength of the magnetic field a finite number of bound states can exist. Such investigations play an important role in the theory of tensor-weighted Laplacians, automorphic forms, determinants of Laplacians and zeta-function regularization, and quantum field theory on (super-) Riemann surfaces, e.g., [20] and references therein.

The contents of this paper are as follows. In the next section we give a short summary of the path integral technique we are using, including for completeness in order to make the paper self-contained the path integral solutions of the Pöschl-Teller and modified Pöschl-Teller potential. In the third section we give an introduction to the formulation and construction of coordinate systems on the two-dimensional hyperboloid. This includes an enumeration of all the coordinate systems according to [20,37,38,54], which separate the Schrödinger equation, respectively the path integral. Furthermore, we list for all coordinate systems the corresponding observable, the Stäckel-matrix, the Hamiltonian, and the general form a potential must have to be separable in the coordinate system, together with its observable.

In Section IV we present the path integral formulations of the superintegrable potentials on the two-dimensional hyperboloid. The two most important are the Higgs-oscillator and the Coulomb problem. We find three more potentials with the required properties. One of them, the potential V_3 is an analogue of the Holt potential [31], the fourth is a centrifugal potential which does not have an analogue on the sphere or in flat space, and the fifth model potential which is linear in the flat space limit. These systems have not been considered in the literature before.

In the fifth Section we summarize and discuss our results. Here we also make some remarks about the problem of ambiguities of the generalization of flat space potentials to spaces of constant curvature. We also present Table 3 to illustrate the correspondence of superintegrable potentials in two dimensions.

2. ELEMENTARY PATH INTEGRAL TECHNIQUES

2.1. Defining the Path Integral

For the construction of the path integral in a curved space we proceed in the canonical way according to Feynman and Hibbs [7], Refs.[20,25], Schulman [60], and references therein. In the following **x** denote D-dimensional cartesian coordinates; **q**, some D-dimensional; s, coordinates on a sphere; $u=(u_0, u_1, u_2)$, coordinates on the two-dimensional hyperboloid, and x, y, z, etc., are one-dimensional coordinates. We start by considering the classical Lagrangian corresponding to the line element $ds^2 = g_{ab}dq^a dq^b$ of the classical motion in some Riemannian space

$$\mathcal{L}_{\mathrm{Cl}}(\mathbf{q}, \dot{\mathbf{q}} = \frac{M}{2}\left(\frac{ds}{dt}\right)^2 - V(\mathbf{q}) = \frac{M}{2} g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}). \tag{2.1}$$

The quantum Hamiltonian is *constructed* by means of

$$H = -\frac{\hbar^2}{2M}\Delta_{LB} + V(\mathbf{q}) = -\frac{\hbar^2}{2M}\frac{1}{\sqrt{q}}\frac{\partial}{\partial q^a} g^{ab}\sqrt{g}\frac{\partial}{\partial q^b} + V(\mathbf{q}) \tag{2.2}$$

as a *definition* of the quantum theory on a curved space. Here are $g=\det(g_{ab})$, $(g^{ab})=(g_{ab})^{-1}$, and $\Delta_{LB}=g^{-1/2}\partial_a g^{ab} g^{1/2}\partial_b$ is the Laplace-Beltrami operator. The scalar product for wave-functions on the manifold reads $(f, g)=\int d\mathbf{q}\sqrt{g}f^*(\mathbf{q})g(\mathbf{q})$, and the momentum operators which are hermitian with respect to this scalar product are given by $p_a = -i\hbar(\partial_{q^a}+\Gamma_a/2)$ with $\Gamma_a=\partial\ln\sqrt{g}/\partial q^a$. In terms of these momentum operators we can rewrite H by using an ordering prescription called product-ordering, where we assume $g_{ab}=h_{ac}h_{cb}$; other lattice formulations like the important midpoint prescription (MP) which corresponds to the Weyl ordering in the Hamiltonian, we do not discuss. Then we obtain for the Hamiltonian (2.2)

$$H = -\frac{\hbar^2}{2M}\Delta_{LB} + V(\mathbf{q}) = \frac{1}{2M} h^{ac} p_a p_b h^{cb} + V(\mathbf{q}) + \Delta V(\mathbf{q}), \tag{2.3}$$

and for the path integral we have

$$K(\mathbf{q}'', \mathbf{q}'; T) = \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}\mathbf{q}(t)\sqrt{g(\mathbf{q})} \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{M}{2} h_{ac}(\mathbf{q}h_{cb}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V(\mathbf{q})\right]dt\right\} \equiv$$

$$\equiv \lim_{N\to\infty}\left(\frac{M}{2\pi i\varepsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int d\mathbf{q}_k\sqrt{g(\mathbf{q}_k)} \times$$

$$\times \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{M}{2\varepsilon} h_{bc}(\mathbf{q}_j)\, h_{ac}(\mathbf{q}_{j-1})\,\Delta q_j^a\,\Delta q_j^b - \varepsilon V(\mathbf{q}_j) - \varepsilon\Delta V(\mathbf{q}_j)\right]\right\}. \tag{2.4}$$

ΔV denotes the well-defined quantum potential

$$\Delta V(\mathbf{q}) = \frac{\hbar^2}{8M}\left[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_b)_{,b} + g^{ab}{}_{,ab}\right] +$$

$$+ \frac{\hbar^2}{8M}\left(2h^{ac}h^{bc}{}_{,ab} - h^{ac}{}_{,a}h^{bc}{}_{,b} - h^{ac}{}_{,b}h^{bc}{}_{,a}\right). \tag{2.5}$$

Here we have used the abbreviations $\varepsilon = (t'' - t')/N \equiv T/N$, $\Delta\mathbf{q}_j = \mathbf{q}_j - \mathbf{q}_{j-1}$, $\bar{q}_j = \frac{1}{2}(\mathbf{q}_j + \mathbf{q}_{j-1})$ for $\mathbf{q}_j = \mathbf{q}(t' + j\varepsilon)$ $(t_j = t' + \varepsilon j,\ \ j = 0,\ldots, N)$ and we interpret the limit $N \to \infty$ as equivalent to $\varepsilon \to 0$, T fixed. The lattice representation can be achieved by exploiting the composition law of the time-evolution operator $U = \exp(-iHT/\hbar)$. Then the discretized path integral emerges in a natural way, and the classical Lagrangian is modified into an effective Lagrangian via $\mathcal{L}_{\text{eff}} = \mathcal{L}_{Cl} - \Delta V$. Note that the factorization of the metric according to $g_{ab} = h_{ac}h_{cb}$ characterizes the h_{ac} as Lamé coefficients [52].

Concerning the space-time transformation technique we do not repeat the relevant formulae once more again, and would like to refer to the literature instead, c.f. [5,25,28,42], and references therein.

2.2. The Pöschl-Teller Potential

As we shall see, we encounter particularly in the case of the Higgs oscillator, the Pöschl-Teller and the modified Pöschl-Teller potentials in our path integral problems. The path integral solution of the Pöschl—Teller potential reads as follows (Böhm and Junker [2], Duru [4], Fischer et al. [8], Inomata et al. [35], Kleinert and Mustapic [43], and [20,27,28], $0 < x < \pi/2$)

$$\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{x}^2 - \frac{\hbar^2}{2M}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2 x} + \frac{\beta^2-\frac{1}{4}}{\cos^2 x}\right)\right]dt\right\} =$$

$$= \sum_{n\in\mathbb{N}_0} e^{-iE_nT/\hbar}\, \phi_n^{(\alpha,\beta)}(x')\, \phi_n^{(\alpha,\beta)}(x''), \tag{2.6}$$

$$= \int_{\mathbb{R}} \frac{dE}{2\pi i}\, e^{-iET/\hbar}\, G_{PT}^{(\alpha,\beta)}(x'', x'; E). \tag{2.7}$$

The bound state wave-functions and the energy spectrum are given by

$$\phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2n+1)\frac{n!\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}\right]^{1/2} \times$$

$$\times (\sin x)^{\alpha+1/2} (\cos x)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos 2x), \tag{2.8}$$

$$E_n = \frac{\hbar^2}{2M}(2n+\alpha+\beta+1)^2. \tag{2.9}$$

The $P_n^{(\alpha,\beta)}$ are Jacobi polynomials [13, p.1035], and the wave-functions $\phi^{(\alpha,\beta)}(x)$ are normalized to unity according to $\int_0^{\pi/2} |\phi_n^{(\alpha,\beta)}(x)|^2\, dx = 1$. The Green function $G_{PT}^{(\alpha,\beta)}(E)$ has the form

$$G_{PT}^{(\alpha,\beta)}(x'', x'; E) = \frac{M}{2\hbar^2}\sqrt{\sin x' \sin x''}\,\frac{\Gamma(m_1 - L_E)\,\Gamma(L_E + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\,\Gamma(m_1 - m_2 + 1)} \times$$

$$\times\left(\frac{1-\cos 2x'}{2}\,\frac{1-\cos 2x''}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\cos 2x'}{2}\,\frac{1+\cos 2x''}{2}\right)^{(m_1+m_2)/2}\times$$

$$\times\,{}_2F_1\left(-L_E+m_1,\,L_E+m_1+1;\,m_1-m_2+1;\,\frac{1-\cos 2x_<}{2}\right)\times$$

$$\times\,{}_2F_1\left(-L_E+m_1,\,L_E+m_1+1;\,m_1+m_2+1;\,\frac{1-\cos 2x_>}{2}\right),\qquad(2.10)$$

where $m_{1,2}=\frac{1}{2}(\beta\pm\alpha)$, $L_E=\frac{1}{2}(\sqrt{2ME}/\hbar-1)$; ${}_2F_1(a,b;c;z)$ is the hypergeometric function [13, p.1039], and $x_>$, $x_<$ denotes the larger, respectively smaller of x', x''.

2.3. The Modified Poschl-Teller Potential

The case of the modified Pöschl-Teller potential is given by [2,8,20,27,28,35,43]

$$\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{M}{2}\dot r^2-\frac{\hbar^2}{2M}\left(\frac{\kappa^2-\frac14}{\sinh^2 r}-\frac{\lambda^2-\frac14}{\cosh^2 r}\right)\right]dt\right\}=$$

$$=\sum_{n=0}^{N_{\max}}\mathrm{e}^{-iE_nT/\hbar}\,\psi_n^{(\kappa,\lambda)*}(r')\,\psi_n^{(\kappa,\lambda)}(r'')+$$

$$+\int_0^\infty dp\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\,\psi_p^{(\kappa,\lambda)*}(r')\,\psi_p^{(\kappa,\lambda)}(r''),\qquad(2.11)$$

$$=\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\,G_{mPT}^{(\kappa,\lambda)}(r'',r',E).\qquad(2.12)$$

The bound states are given by

$$\psi_n^{(\kappa,\lambda)}(r)=N_m^{(\kappa,\lambda)}(\sinh r)^{\kappa+1/2}(\cosh r)^{n-\lambda+1/2}\times$$

$$\times\,{}_2F_1(-n,\lambda-n;1+\kappa;\tanh^2 r),$$

$$N_n^{(\kappa,\lambda)}=\frac{1}{\Gamma(1+\kappa)}\left[\frac{2(\lambda-\kappa-2n-1)\,\Gamma(n+1+\kappa)\,\Gamma(\lambda-n)}{\Gamma(\lambda-\kappa-n)n!}\right]^{1/2},\qquad(2.13)$$

$$E_n=-\frac{\hbar^2}{2M}(2n+\kappa-\lambda+1)^2.\qquad(2.14)$$

Here denote $n = 0, 1, \ldots, N_{\max} = \left[\frac{1}{2}(\lambda - \kappa - 1) \right] \geq 0$, and only a finite number of bound states can exist depending on the strength of the attractive potential through and the repulsive centrifugal term as well. Here $[x]$ denotes the integer part of the real number x. The continuous states are

$$\psi_p^{(\kappa, \lambda)}(r) = N_p^{(\kappa, \lambda)} (\cosh r)^{ip} (\tanh r)^{\kappa + 1/2} \times$$

$$\times {}_2F_1\left(\frac{\lambda + \kappa + 1 - ip}{2}, \frac{\kappa - \lambda + 1 - ip}{2}; 1 + \kappa; \tanh^2 r \right),$$

$$N_p^{(\kappa, \lambda)} = \frac{1}{\Gamma(1 + \kappa)} \frac{\sqrt{p \sinh \pi p}}{2\pi^2} \Gamma\left(\frac{\lambda + \kappa + 1 - ip}{2} \right) \Gamma\left(\frac{\kappa - \lambda + 1 - ip}{2} \right), \tag{2.15}$$

and $E_p = \hbar^2 p^2 / 2M$. The Green function $G_{mPT}^{(\kappa, \lambda)}(E)$ has the form

$$G_{mPT}^{(\kappa, \lambda)}(r'', r'; E) = \frac{M}{2\hbar^2} \frac{\Gamma(m_1 - L_\lambda)\,\Gamma(L_\lambda + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\,\Gamma(m_1 - m_2 + 1)} \times$$

$$\times (\cosh r' \cosh r'')^{-(m_1 - m_2)} (\tanh r' \tanh r'')^{m_1 + m_2 + 1/2} \times$$

$$\times {}_2F_1\left(-L_\lambda + m_1, L_\lambda + m_1 + 1; m_1 - m_2 + 1; \frac{1}{\cosh^2 r_<} \right) \times$$

$$\times {}_2F_1(-L_\lambda + m_1, L_\lambda + m_1 + 1; m_1 + m_2 + 1; \tanh^2 r_>), \tag{2.16}$$

where we have set $m_{1,2} = \frac{1}{2}(\kappa \pm \sqrt{-2ME}/\hbar)$, $L_\lambda = \frac{1}{2}(\lambda - 1)$. We make extensively use of the solutions of the Pöschl-Teller and the modified Pöschl-Teller potentials, respectively.

3. SEPARATION OF VARIABLES AND COORDINATE SYSTEMS ON THE HYPERBOLOID

In this section we discuss separation of variables in the Schrödinger equation, respectively in the path integral, and list the corresponding orthogonal coordinate systems on the two-dimensional hyperboloid $\Lambda^{(2)}$.

3.1. Separation of Variables in the Schrödinger Equation and in the Path Integral

Let us consider the time-independent Schrödinger equation in a Riemannian space

$$H\Psi \equiv \left(-\frac{\hbar^2}{2M}\Delta_{LB} + V \right)\Psi = E\Psi, \tag{3.1}$$

where Δ_{LB} is the Laplace-Beltrami operator as defined in the previous section, assuming that the line-element for an orthogonal coordinate system $\rho = (\rho_1, \dots, \rho_D)$ can be written according to

$$ds^2 = \sum_{i=1}^{D} h_i^2 (d\rho_i)^2, \tag{3.2}$$

and Δ_{LB} can be cast into the form

$$\Delta_{LB} = \sum_{i=1}^{D} \frac{1}{\prod_{j=1}^{D} h_j(\rho)} \frac{\partial}{\partial \rho_i} \left(\frac{\prod_{k=1}^{D} h_k(\rho)}{h_i^2(\rho)} \frac{\partial}{\partial \rho_i} \right). \tag{3.3}$$

As was shown by Moon and Spencer [51] the necessary and sufficient condition for simple separability of the Helmholtz equation, in a D-dimensional Reimannian space with an orthogonal coordinate system ρ, is the factorization of the Lame coefficients h_i according to

$$\frac{\prod_{j=1}^{D} h_j(\rho)}{h_i^2(\rho)} = M_{i1} \prod_{j=1}^{D} f_j(\rho_j) \tag{3.4}$$

such that

$$\left. \begin{aligned} & M_{i1}(\rho_1, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_D) = \frac{\partial S}{\partial \Phi_{i1}} = \frac{S(\rho)}{h_i^2(\rho)}, \\ & \frac{h^{1/2}}{S(\rho)} = \prod_{i=1}^{D} f_i(\rho_i), \qquad h = \prod_{i=1}^{D} h_i(\rho), \end{aligned} \right\} \tag{3.5}$$

where S is the Stäckel determinant [52,57]

$$S(\rho) = \begin{vmatrix} \Phi_{11}(\rho_1) & \Phi_{12}(\rho_1) & \cdots & \Phi_{1D}(\rho_1) \\ \Phi_{21}(\rho_2) & \Phi_{22}(\rho_2) & \cdots & \Phi_{2D}(\rho_2) \\ \vdots & \vdots & & \vdots \\ \Phi_{D1}(\rho_D) & \Phi_{D2}(\rho_D) & \cdots & \Phi_{DD}(\rho_D) \end{vmatrix}, \tag{3.6}$$

and M_{i1} is called the cofactor of Φ_{i1}.

For the separation of the Schrödinger equation a potential V must have the following form

$$V = \sum_i^D \frac{v_i(\rho_i)}{h_i^2}, \tag{3.7}$$

and the separated equations are ($\Psi = \psi_1\psi_2 \ldots \psi_D$)

$$\frac{1}{f_i}\frac{d}{d\rho_i}\left(f_i \frac{d\psi_i}{d\rho_i}\right) + \left(\sum_k \Phi_{ik}\alpha_k - v_i\right)\psi_i = 0. \tag{3.8}$$

Here $\alpha_1 = 2ME/\hbar^2$ and $\alpha_2, \alpha_3, \ldots, \alpha_D$ are the separation constants. By using these equations one can construct the full set of commuting operators for each coordinate system. In [61] the following was proven: If the Schrödinger equation (3.1) admits simple separation of variables in the coordinate system $(\rho_1, \ldots, \rho_D)$, then there exists $D - 1$ linearly independent second degree operators I_k, $k = 2, 3, \ldots, D - 1$ commuting with the Hamiltonian H and with each other, and they have the form

$$I_k = -\sum_{i=1}^D (\Phi^{-1})_{ik}\left[\frac{1}{f_i}\frac{d}{d\rho_i}\left(f_i \frac{d\psi_i}{d\rho_i}\right) + v_i\right]. \tag{3.9}$$

The separation constants $\alpha_2, \alpha_3 \ldots \alpha_D$ are the eigenvalue of these operators, i.e.,

$$I_k \Psi = \alpha_k \Psi. \tag{3.10}$$

Superintegrable systems have the property that they admit not only separation of variables in one coordinate system, but in at least two ones. This has the consequence that the system has additional integrals of motion, and that the discrete spectrum has accidental degeneracies.

The theory of separation of variables allows the formulation of the corresponding separation formula for the path integral. Introducing the (new) momentum operators $P_i = \frac{\hbar}{i}\left(\partial_{\rho_i} + \frac{1}{2}\Gamma_i\right)$, $\Gamma_i = f_i'/f_i$, we then can rewrite the Legendre transformed Hamiltonian as follows [24]

$$H - E = -\frac{\hbar^2}{2M}\Delta_{LB} - E = -\frac{\hbar^2}{2M}\sum_{i=1}^{D}\frac{1}{\prod_{j=1}^{D} h_j}\frac{\partial}{\partial\rho_i}\left(\frac{\prod_{k=1}^{D} h_k}{h_i^2}\frac{\partial}{\partial\rho_i}\right) - E =$$

$$= -\frac{\hbar^2}{2M}\frac{1}{S}\sum_{i=1}^{D}\left[\frac{1}{f_i}\frac{\partial}{\partial\rho_i}\left(f_i\frac{\partial}{\partial\rho_i}\right)\right] - E = -\frac{\hbar^2}{2M}\frac{1}{S}\sum_{i=1}^{D} M_{1i}\left(\frac{\partial^2}{\partial\rho_i^2} + \Gamma_i\frac{\partial}{\partial\rho_i}\right) - E =$$

$$= \frac{1}{S}\sum_{i=1}^{D} M_{1i}\left[\frac{1}{2m}P_i^2 - E\hbar_i^2 + \frac{h^2}{8M}(\Gamma_i^2 + 2\Gamma_i')\right] =$$

$$= \frac{1}{S}\sum_{i=1}^{D} M_{1i}\left[\frac{1}{2m}P_i^2 - \frac{\hbar^2}{2M}\sum_{j=1}^{D}\alpha_j\,\Phi_{ij}(\rho_i) + \frac{\hbar^2}{8M}(\Gamma_i^2 + 2\Gamma_i')\right]. \quad (3.11)$$

We then obtain according to the general theory by means of a space-time transformation the following identity in the path integral ($g = \prod h_i^2$)

$$\int\limits_{\rho(t')=\rho'}^{\rho(t'')=\rho''} \mathcal{D}\rho(t)\sqrt{g}\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}(\mathbf{h}\cdot\dot{\boldsymbol\rho})^2 - \Delta V_{PF}(\rho)\right]dt\right\} =$$

$$= \int\limits_{\rho(t')=\rho'}^{\rho(t'')=\rho''} \mathcal{D}\rho(t)\prod_{i=1}^{D}\sqrt{\frac{S}{M_{1i}}}\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}S\frac{\dot\rho_i^2}{M_{1i}} - \Delta V_i(\rho)\right]dt\right\} =$$

$$= (S'S'')^{\frac{1}{2}(1-D/2)}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}e^{-iET/\hbar}\int_0^\infty ds''\prod_{i=1}^{D}(M_{1i}'M_{1i}'')^{1/4}\int\limits_{\rho_i(0)=\rho_i'}^{\rho_i(s'')=\rho_i''}\mathcal{D}\rho_i(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{M}{2}\dot\rho_i^2 + \frac{\hbar^2}{2M}\sum_{j=1}^{D}\alpha_j\Phi_{ij}(\rho_i) - \frac{\hbar^2}{8M}(\Gamma_i^2 + 2\Gamma_i')\right]ds\right\}. \quad (3.12)$$

Therefore we achieved complete separation of variables in the ρ-path integral.

3.2. Coordinate Systems on $\Lambda^{(2)}$

In this subsection we consider the coordinate systems of the two-dimensional hyperboloid defined by

$$u_0^2 - u_1^2 - u_2^2 = u_0^2 - \mathbf{u}^2 = R^2, \qquad u_0 > 0 \tag{3.13}$$

which separate the Schrödinger equation, respectively the path integral on $\Lambda^{(2)}$. The notion $u_0 > 0$ means that we consider only one sheet of the double-sheeted hyperboloid $u_0^2 - \mathbf{u}^2 = R^2$. The enumeration includes the definition of the coordinates, the characteristic operator I, i.e., the operator which commutes with the Hamiltonian, the Stäckel-matrix S, the momentum operators p_i, the Schrödinger operator (Hamiltonian) H, and the general form of the potential which separates in the corresponding coordinates, together with its observable $I^{(V)}$. In the notation of the coordinate systems we follow [38,54] and [64]. The Hamiltonian on $\Lambda^{(2)}$ can be written as

$$H = H_0 + V(u), \qquad H_0 = -\frac{\hbar^2}{2M}\Delta_{LB} = \frac{1}{2MR^2}(K_1^2 + K_2^2 - L_3^2), \tag{3.14}$$

where $K_{1,2}$ are (hyperbolic) angular-momentum operators defined by

$$K_1 = \frac{\hbar}{\mathrm{i}}\left(u_0\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial u_0}\right), \qquad K_2 = \frac{\hbar}{\mathrm{i}}\left(u_0\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_0}\right), \tag{3.15}$$

and L_3 is the angular momentum operator

$$L_3 = \frac{\hbar}{\mathrm{i}}\left(u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1}\right). \tag{3.16}$$

K_1, K_2 are the generators of the Lorentz transformations, and L_3 is the generator of (spatial) rotations in three-dimensional Minkowskian space. They satisfy the commutation relations

$$[K_1, K_2] = \mathrm{i}\hbar L_3, \qquad [K_2, L_3] = -\mathrm{i}\hbar K_1, \qquad [L_3, K_1] = -\mathrm{i}\hbar K_2. \tag{3.17}$$

The Schrödinger equation for the eigenvalue problem for the free motion on the two-dimensional hyperboloid has the form [38]

$$H_0\Psi(u) = E\Psi(u) = \frac{\hbar^2}{2MR^2}\left(p^2 + \frac{1}{4}\right)\Psi(u), \qquad p > 0. \tag{3.18}$$

The spectrum is purely continuous with largest lower bound $E_0 = \hbar^2/8MR^2$ [20].

For the classification of the coordinate system on the two-dimensional hyperboloid we need the Hamiltonian H and another second-order differential operator I which commutes with H. In the following we call the operator I corresponding to this quantum number *(characteristic) observable*, respectively the *characteristic operator.*

In the sequel we only consider *orthogonal* coordinate systems on the two-dimensional hyperboloid. $u \in \Lambda^{(2)}$ is expressed as $u = u(\rho)$, where $\rho = (\rho_1, \rho_2)$ are two-dimensional coordinates on $\Lambda^{(2)}$. For the metric tensor then follows

$$g_{ab} = G_{ik} \frac{\partial u_i}{\partial \rho_a} \frac{\partial u_k}{\partial \rho_b}, \tag{3.19}$$

where G_{ik} is the metric tensor of the ambient space, which is in the present case $G_{ik} = \mathrm{diag}(1, -1, -1, -1)$, and in order that the line element $ds^2 = = \Sigma_{ab}\varepsilon_{ab}\, g_{ab} dq^a dq^b$ is positive definite an appropriate $\varepsilon = \pm 1$ must be taken into account. Actually $\varepsilon_{ab} = \varepsilon_{aa} = -1, \forall_{a,b}$ because the metric tensor is always diagonal. In the following we state for convenience only the explicit form of ds^2.

The nine possible coordinate systems on $\Lambda^{(2)}$ now are the following:
1. The first coordinate system is the *(pseudo-) spherical system*:

$$u_0 = R\cosh\tau, \quad u_1 = R\sinh\tau\cos\varphi, \quad u_2 = R\sinh\tau\sin\varphi \tag{3.20}$$

($\tau > 0$, $\varphi \in [0, 2\pi)$). The characteristic operator is

$$I_S = L_3^2, \tag{3.21}$$

which means that in the flat space limit we obtain the polar system in $\mathbb{R}^2$. The Stäckel-determinant is given by

$$S = \begin{vmatrix} R^2 & -\dfrac{1}{\sinh^2\tau} \\ 0 & 1 \end{vmatrix} = R^2, \tag{3.22}$$

and $f_1 = \sinh\tau, f_2 = 1$. For the line element we have $ds^2 = R^2 (d\tau^2 + \sinh^2\tau d\varphi^2)$, and therefore the momentum operators are given by

$$p_\tau = \frac{\hbar}{i}\left(\frac{\partial}{\partial\tau} + \frac{1}{2}\coth\tau\right), \quad p_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}. \tag{3.23}$$

The Hamiltonian reads

$$H_0 = -\frac{\hbar^2}{2MR^2}\left(\frac{\partial^2}{\partial\tau^2} + \coth\tau\,\frac{\partial}{\partial\tau} + \frac{1}{\sinh^2\tau}\frac{\partial^2}{\partial\varphi^2}\right) =$$

$$= \frac{1}{2MR^2}\left(p_\tau^2 + \frac{1}{\sinh^2\tau}p_\varphi^2\right) + \frac{\hbar^2}{8MR^2}\left(1 - \frac{1}{\sinh^2\tau}\right). \qquad (3.24)$$

A potential separable in pseudospherical coordinates must have the form

$$V(\tau, \varphi) = V_1(\tau) + \frac{V_2(\varphi)}{\sinh^2\tau}, \qquad (3.25)$$

and the corresponding constant of motion, respectively observable, is

$$I_S^{(V)} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial\varphi^2} + V_2(\varphi) = \frac{1}{2M}L_3^2 + V_2(\varphi). \qquad (3.26)$$

Note that the corresponding observable on the two-dimensional sphere $S^{(2)}$ has exactly the same form. In the following the prefix «pseudo» is omitted.
2. The second system is the *equidistant* system. It has the form

$$u_0 = R\cosh\tau_1\cosh\tau_2, \quad u_1 = R\cosh\tau_1\sinh\tau_2, \quad u_2 = R\sinh\tau_1 \qquad (3.27)$$

$(\tau_1, \tau_2 \in \mathbb{R})$. The operator corresponding to this system is

$$I_{EQ} = K_2^2 \qquad (3.28)$$

which characterizes this system as «cartesian»-like, i.e., in the flat space limit we obtain cartesian coordinates, and the K_i operators, $i = 1, 2$, yield the usual $p_i = -i\hbar\partial_{x_i}$ momentum operators. The Stäckel determinant is

$$S = \begin{vmatrix} R^2 & -\dfrac{1}{\cosh^2\tau_1} \\ 0 & 1 \end{vmatrix} = R^2, \qquad (3.29)$$

and $f_1 = \cosh\tau, f_2 = 1$. The line element is given by $ds^2 = R^2(d\tau_1^2 + \cosh^2\tau_1 d^2\tau_2^2)$, and the momentum operators have the form

$$p_{\tau_1} = \frac{\hbar}{i}\left(\frac{\partial}{\partial\tau_1} + \frac{1}{2}\tanh\tau_1\right), \quad p_{\tau_2} = \frac{\hbar}{i}\frac{\partial}{\partial\tau_2}. \qquad (3.30)$$

For the Hamiltonian we obtain

$$H_0 = -\frac{\hbar^2}{2MR^2}\left(\frac{\partial^2}{\partial\tau_1^2} + \tanh\tau_1\frac{\partial}{\partial\tau_1} + \frac{1}{\cosh^2\tau_1}\frac{\partial^2}{\partial\tau_2^2}\right) =$$

$$= \frac{1}{2MR^2}\left(p_{\tau_1}^2 + \frac{1}{\cosh^2\tau_1}p_{\tau_2}^2\right) + \frac{\hbar^2}{8MR^2}\left(1 + \frac{1}{\cosh^2\tau_1}\right). \tag{3.31}$$

A potential on $\Lambda^{(2)}$ separable in equidistant coordinates must have the form

$$V(\tau_1, \tau_2) = V_1(\tau_1) + \frac{V_2(\tau_2)}{\cosh^2\tau_1}, \tag{3.32}$$

and the corresponding observable is given by

$$I_{EQ}^{(V)} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial\tau_2^2} + V_2(\tau_2) = \frac{1}{2M}K_2^2 + V_2(\tau_2). \tag{3.33}$$

3. The third coordinate system is called *horicyclic* system:

$$u_0 = R\,\frac{x^2 + y^2 + 1}{2y}, \qquad u_1 = R\,\frac{x^2 + y^2 - 1}{2y}, \qquad u_2 = R\,\frac{x}{y} \tag{3.34}$$

($y > 0$, $x \in \mathbb{R}$). The characteristic operator is given by

$$I_{HO} = (K_1 - L_3)^2 = K_1^2 + L_3^2 - \{K_1, L_3\}, \tag{3.35}$$

where $\{X, Y\} = XY + YX$ is the anticommutator of two operators X and Y. In the flat space limit this system gives cartesian coordinates. For the Stäckel determinant we get

$$S = \begin{vmatrix} 0 & -1 \\ \frac{R^2}{y^2} & 1 \end{vmatrix} = \frac{R^2}{y^2}, \tag{3.36}$$

and $f_1 = f_2 = 1$. The line element is $ds^2 = R^2(dx^2 + dy^2)/y^2$, and the momentum operators have the form

$$p_x = \frac{\hbar}{i}\frac{\partial}{\partial x}, \qquad p_y = \frac{\hbar}{i}\left(\frac{\partial}{\partial y} - \frac{1}{y}\right). \tag{3.37}$$

Therefore we obtain for the Hamiltonian

$$H_0 = -\frac{\hbar^2}{2MR^2}y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{2MR^2}y(p_x^2 + p_y^2)\,y. \tag{3.38}$$

Note that we have in this case no quantum potential ΔV which is due to the fact that the metric is proportional to $\mathbf{I}_2$. A potential separable in horicyclic coordinates must have the form

$$V(x, y) = V_1(y) + y^2 V_2(x) = V_1(y) + R^2 \frac{V_2(x)}{(u_0 - u_1)^2}, \tag{3.39}$$

and the corresponding observable is given by

$$I_{HO}^{(V)} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + V_2(x) = \frac{1}{2M}(K_1 - L_3)^2 + V_2(x). \tag{3.40}$$

4. The fourth coordinate system is the *elliptic* coordinate system. In algebraic form it is defined as

$$\begin{aligned} u_0^2 &= R^2 \frac{(\rho_1 - a_3)(\rho_2 - a_3)}{(a_1 - a_3)(a_2 - a_3)}, \\ u_1^2 &= R^2 \frac{(\rho_1 - a_2)(\rho_2 - a_2)}{(a_1 - a_2)(a_2 - a_3)}, \\ u_2^2 &= R^2 \frac{(\rho_1 - a_1)(a_1 - \rho_2)}{(a_1 - a_2)(a_1 - a_3)} \end{aligned} \tag{3.41}$$

$(a_3 < a_2 < \rho_2 < a_1 < \rho_1)$. The Stäckel determinant has the form

$$S = \begin{vmatrix} \frac{R^2}{4}\frac{\rho_1}{P(\rho_1)} & -\frac{1}{P(\rho_1)} \\ \frac{R^2}{4}\frac{\rho_2}{P(\rho_2)} & -\frac{1}{P(\rho_2)} \end{vmatrix} = -\frac{R^2}{4}\frac{\rho_1 - \rho_2}{P(\rho_1)\,P(\rho_2)}, \tag{3.42}$$

$f_1 = \sqrt{P(\rho_1)}$, $f_2 = \sqrt{-P(\rho_2)}$, and $P(\rho) = (\rho - a_1)(\rho - a_2)(\rho - a_3)$. After putting

$$\rho_1 = a_1 - (a_1 - a_3)\,\mathrm{dn}^2(\alpha, k), \quad \rho_2 = a_1 - (a_1 - a_2)\,\mathrm{sn}^2(\beta, k'), \tag{3.43}$$

and

$$k^2 = \frac{a_2 - a_3}{a_1 - a_3}, \quad k'^2 = \frac{a_1 - a_2}{a_1 - a_3}, \tag{3.44}$$

with the property $k^2 + k'^2 = 1$, we get

$$u_0 = R\,\mathrm{sn}\,(\alpha, k)\,\mathrm{dn}\,(\beta, k'),$$

$$u_1 = \mathrm{i}R\,\mathrm{cn}\,(\alpha, k)\,\mathrm{cn}\,(\beta, k'), \tag{3.45}$$

$$u_2 = \mathrm{i}R\,\mathrm{dn}\,(\alpha, k)\,\mathrm{sn}\,(\beta, k').$$

Here $\alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K)$, $\beta \in [0, 4K')$, and $\mathrm{sn}\,(\mu, k)$, $\mathrm{cn}\,(\mu, k)$, $\mathrm{dn}\,(\mu, k)$ are the Jacobi elliptic functions [13, p.910] with modulus k, and $K = K(k)$, $K' = K(k')$ are the complete elliptic integrals with k and k' the elliptic moduli. In the elliptic system the characteristic operator has the form

$$I_E = L_3^2 + \sinh^2 f K_2^2, \tag{3.46}$$

with $\sinh^2 f$ as in (3.47), and $2f$ is the distance between the foci. Analogously as for the elliptic system on the two-dimensional sphere we can introduce a *rotated elliptic* (also called *elliptic* II) system [22]. Instead of a trigonometric rotation as for the case on the sphere we must consider in the present case a hyperbolic rotation. We define

$$\sinh^2 f = \frac{a_1 - a_2}{a_2 - a_3} = \frac{k'^2}{k^2}, \quad \cosh^2 f = \frac{a_1 - a_3}{a_2 - a_3} = \frac{1}{k^2}, \tag{3.47}$$

and the rotated elliptic system is then obtained by

$$\begin{pmatrix} u_0' \\ u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \cosh f & \sinh f & 0 \\ \sinh f & \cosh f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_0 \cosh f + u_1 \sinh f \\ u_0 \sinh f + u_1 \cosh f \\ u_2 \end{pmatrix}. \tag{3.48}$$

Explicitly this yield

$$u_0' = \frac{R}{a_2 - a_3}\left(\sqrt{(\rho_1 - a_3)(\rho_2 - a_3)} + \sqrt{(\rho_1 - a_2)(\rho_2 - a_2)}\right)$$

$$= R\left[\frac{1}{k}\,\mathrm{sn}\,(\alpha, k)\,\mathrm{dn}\,(\beta, k') + \mathrm{i}\frac{k'}{k}\,\mathrm{cn}\,(\alpha, k)\,\mathrm{cn}\,(\beta, k')\right],$$

$$u_1' = \frac{R}{a_2 - a_3}\left(\sqrt{\frac{a_1 - a_2}{a_1 - a_3}(\rho_1 - a_3)(\rho_2 - a_3)} + \sqrt{\frac{a_1 - a_3}{a_1 - a_2}(\rho_1 - a_2)(\rho_2 - a_2)}\right)$$

$$= R\left[\frac{k'}{k}\,\mathrm{sn}\,(\alpha, k)\,\mathrm{dn}\,(\beta, k') + \frac{\mathrm{i}}{k}\,\mathrm{cn}\,(\alpha, k)\,\mathrm{cn}\,(\beta, k')\right],$$

$$u_2 = R\sqrt{\frac{(\rho_1 - a_1)(a_1 - \rho_2)}{(a_1 - a_2)(a_1 - a_3)}} = \mathrm{i}R\,\mathrm{dn}\,(\alpha, k)\,\mathrm{sn}\,(\beta, k'). \tag{3.49}$$

In the rotated elliptic system we get

$$I_{E'} = \cosh 2f\, L_3^2 - \frac{1}{2} \sinh 2f\, \{K_1, L_3\}. \tag{3.50}$$

In the flat space limit the elliptic system gives elliptic coordinates in $\mathbb{R}^2$; and the rotated elliptic system, elliptic II coordinates in $\mathbb{R}^2$. If no confusion can arise we do not distinguish in the following the rotated elliptic system by printing the coordinates. For short-hand notation we also omit the moduli. The line element in each case is given by $ds^2 = = R^2(k^2 \operatorname{cn}^2 \alpha + k'^2 \operatorname{cn}^2 \beta)(d\alpha^2 + d\beta^2)$. For the momentum operators we obtain

$$\begin{aligned} p_\alpha &= \frac{\hbar}{i}\left(\frac{\partial}{\partial\alpha} - \frac{k^2 \operatorname{sn}\alpha \operatorname{cn}\alpha \operatorname{dn}\alpha}{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta}\right), \\ p_\beta &= \frac{\hbar}{i}\left(\frac{\partial}{\partial\beta} - \frac{k^2 \operatorname{sn}\beta \operatorname{cn}\beta \operatorname{dn}\beta}{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta}\right), \end{aligned} \tag{3.51}$$

and for the Hamiltonian we have

$$H_0 = -\frac{\hbar^2}{2MR^2}\frac{1}{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right) =$$

$$= \frac{1}{2MR^2}\frac{1}{\sqrt{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta}}(p_\alpha^2 + p_\beta^2)\frac{1}{\sqrt{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta}}. \tag{3.52}$$

A potential separable in elliptic coordinates must have the form

$$V(\alpha, \beta) = \frac{\tilde{V}_1(\alpha) + \tilde{V}_2(\beta)}{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta} = \frac{V_1(\rho_1) + V_2(\rho_2)}{\rho_1 - \rho_2}. \tag{3.53}$$

The observable then is given by

$$\begin{aligned} I_E^{(V)} = &-\frac{\hbar^2}{2M}\frac{1}{\rho_1 - \rho_2}\left(\rho_2 \sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1}\sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1} + \right. \\ &\left. + \rho_1\sqrt{-P(\rho_2)}\frac{\partial}{\partial\rho_2}\sqrt{-P(\rho_2)}\frac{\partial}{\partial\rho_2}\right) + \\ &+ \frac{\rho_2 V_1(\rho_1) + \rho_1 V_2(\rho_2)}{\rho_1 - \rho_2} = \\ = &\frac{1}{2M}(L_3^2 + \sinh^2 f\, K_2^2) + \frac{\rho_2 V_1(\rho_1) + \rho_1 V_2(\rho_2)}{\rho_1 - \rho_2}. \end{aligned} \tag{3.54}$$

Note that the corresponding observable on the two-dimensional sphere has the form

$$I^{(V)}_{E,S^{(2)}} = \frac{1}{2M}(L_1^2 + k'^2 L_2^2) + \frac{\rho_2 V_1(\rho_1) + \rho_1 V_2(\rho_2)}{\rho_1 - \rho_2}, \tag{3.55}$$

with ρ_1, ρ_2 elliptic coordinates on $S^{(2)}$ [22].

5. The fifth coordinate system is the hyperbolic system:

$$\begin{aligned} u_0^2 &= R^2 \frac{(\rho_1 - a_2)(a_2 - \rho_2)}{(a_1 - a_2)(a_2 - a_3)}, \\ u_1^2 &= R^2 \frac{(\rho_1 - a_3)(a_3 - \rho_2)}{(a_1 - a_3)(a_2 - a_3)}, \\ u_2^2 &= R^2 \frac{(\rho_1 - a_1)(a_1 - \rho_2)}{(a_1 - a_2)(a_1 - a_3)} \end{aligned} \tag{3.56}$$

$(\rho_2 < a_3 < a_2 < a_1 < \rho_1)$. The Stäckel determinant is given by

$$S = \begin{vmatrix} \frac{R^2}{4}\frac{\rho_1}{P(\rho_1)} & -\frac{1}{P(\rho_1)} \\ \frac{R^2}{4}\frac{\rho_2}{P(\rho_2)} & -\frac{1}{P(\rho_2)} \end{vmatrix} = -\frac{R^2}{4}\frac{\rho_1 - \rho_2}{P(\rho_1)\,P(\rho_2)}, \tag{3.57}$$

and $f_1 = \sqrt{P(\rho_1)}$, $f_2 = \sqrt{-P(\rho_2)}$. After putting [64]

$$\rho_1 = a_2 - (a_2 - a_3)\,\mathrm{cn}^2(\mu, k), \qquad \rho_2 = a_2 + (a_1 - a_2)\,\mathrm{cn}^2(\eta, k'), \tag{3.58}$$

and

$$k^2 = \frac{a_2 - a_3}{a_1 - a_3}, \qquad k'^2 = \frac{a_1 - a_2}{a_1 - a_3}, \tag{3.59}$$

where $\mu \in (iK', iK' + 2K)$, $\eta \in [0, 4K')$ we get

$$\begin{aligned} u_0 &= -R\,\mathrm{cn}(\mu, k)\,\mathrm{cn}(\eta, k'), \\ u_1 &= iR\,\mathrm{sn}(\mu, k)\,\mathrm{dn}(\eta, k'), \\ u_2 &= iR\,\mathrm{dn}(\mu, k)\,\mathrm{sn}(\eta, k'). \end{aligned} \tag{3.60}$$

The characteristic operator is given by

$$I_H = K_2^2 - \sin^2\alpha L_3^2, \tag{3.61}$$

where $\sin^2\alpha=(a_2-a_3)/(a_1-a_3)$ and 2α is the angle between the two focal lines. In the flat space limit the hyperbolic system gives cartesian coordinates. The line element has the form $ds^2=R^2(k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2\eta)(d\mu^2+d\nu^2)$, and the momentum operators are

$$p_\mu=\frac{\hbar}{i}\left(\frac{\partial}{\partial\mu}-\frac{k^2\operatorname{sn}\mu\operatorname{cn}\mu\operatorname{dn}\mu}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2\eta}\right),$$

$$p_\eta=\frac{\hbar}{i}\left(\frac{\partial}{\partial\eta}-\frac{k^2\operatorname{sn}\eta\operatorname{cn}\eta\operatorname{dn}\eta}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2\eta}\right),\tag{3.62}$$

and for the Hamiltonian we obtain

$$H_0=-\frac{\hbar^2}{2MR^2}\frac{1}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2\eta}\left(\frac{\partial^2}{\partial\mu^2}+\frac{\partial^2}{\partial\eta^2}\right)=$$

$$=\frac{1}{2MR^2}\frac{1}{\sqrt{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2\eta}}(p_\mu^2+p_\eta^2)\frac{1}{\sqrt{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2\eta}}.\tag{3.63}$$

A potential separable in hyperbolic coordinates must have the form

$$V(\mu,\eta)=\frac{\tilde V_1(\mu)+\tilde V_2(\eta)}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2\eta}=\frac{V_1(\rho_1)+V_2(\rho_2)}{\rho_1-\rho_2},\tag{3.64}$$

and the corresponding observable is

$$I_H^{(V)}=-\frac{\hbar^2}{2M}\frac{1}{\rho_1-\rho_2}\left(\rho_2\sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1}\sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1}+\right.$$

$$\left.+\rho_1\sqrt{-P(\rho_2)}\frac{\partial}{\partial\rho_2}\sqrt{-P(\rho_2)}\frac{\partial}{\partial\rho_2}\right)$$

$$+\frac{\rho_2V_1(\rho_1)+\rho_1V_2(\rho_2)}{\rho_1-\rho_2}=$$

$$=\frac{1}{2M}(K_2^2-\sin^2\alpha L_3^2)+\frac{\rho_2V_1(\rho_1)+\rho_1V_2(\rho_2)}{\rho_1-\rho_2}.\tag{3.65}$$

6. The sixth coordinate system is the *semi-hyperbolic* system:

$$u_0^2=\frac{R^2}{2}\left(\frac{1}{\delta}\sqrt{\frac{[(\rho_1-\gamma)^2+\delta^2][(\rho_2-\gamma)^2+\delta^2]}{(a-\gamma)^2+\delta^2}}+\frac{(\rho_1-a)(a-\rho_2)}{[(a-\gamma)^2+\delta^2]}+1\right),$$

$$u_1^2 = \frac{R^2}{2}\left(\frac{1}{\delta}\sqrt{\frac{[(\rho_1-\gamma)^2+\delta^2][(\rho_2-\gamma)^2+\delta^2]}{(a-\gamma)^2+\delta^2}} + \frac{(\rho_1-a)(a-\rho_2)}{[(a-\gamma)^2+\delta^2]} - 1\right),$$

$$u_2^2 = R^2\frac{(\rho_1-a)(a-\rho_2)}{(a-\gamma)^2+\delta^2} \tag{3.66}$$

($\rho_2 < a < \rho_1$, γ, $\delta \in \mathbb{R}$). The characteristic operator has the form

$$I_{SH} = \{K_1, L_3\} - \sinh 2f K_2^2, \tag{3.67}$$

where $\sinh 2f = (a-\gamma)/\delta$ and $2f$ is the distance between the focus of the semihyperbolas and the basis of the equidistants. In the flat space limit the case of $\sinh 2f \to 0$ gives parabolic coordinates; and the case $\sinh 2f \to \infty$, cartesian coordinates. For the Stäckel determinant we obtain

$$S = \begin{vmatrix} \dfrac{R^2}{4}\dfrac{1}{1+\mu_2^2} & -\dfrac{1}{P(\mu_1)} \\ \dfrac{R^2}{4}\dfrac{1}{1+\mu_2^2} & \dfrac{1}{P(\mu_2)} \end{vmatrix} = \frac{R^2}{4}\frac{\mu_1+\mu_2}{P(\mu_1)\,P(\mu_2)}, \tag{3.68}$$

and $f_1 = \sqrt{P(\mu_1)}$, $f_2 = \sqrt{P(\mu_2)}$. The special choice of the parameters $a = \gamma = 0$, $\delta = 1$ together with $\rho_1 = \mu_1 > 0$, $-\rho_2 = \mu_2 > 0$ yields

$$\left.\begin{aligned} u_0^2 &= \frac{R^2}{2}\left(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} + \mu_1\mu_2 + 1\right) \\ &= \frac{R^2}{4}\left[\sqrt{(1-i\mu_1)(1-i\mu_2)} - \sqrt{(1+i\mu_1)(1+i\mu_2)}\right]^2, \\ u_1^2 &= \frac{R^2}{2}\left(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} - \mu_1\mu_2 - 1\right) = \\ &= -\frac{R^2}{4}\left[\sqrt{(1-i\mu_1)(1+i\mu_2)} - \sqrt{(1+i\mu_1)(1-i\mu_2)}\right]^2, \\ u_2^2 &= R^2\mu_1\mu_2. \end{aligned}\right\} \tag{3.69}$$

The characteristic operator then has the form

$$I_{SH} = \{K_1, L_3\}, \tag{3.70}$$

which shows that the coordinate system (3.69) yields in the flat space limit *parabolic* coordinates. Note also the relation $u_0u_1 = R^2(\mu_1-\mu_2)/2$. In the following we only consider this special choice of parameters. The line element reads as ($P(\mu)=\mu(1+\mu^2)$)

$$ds^2 = R^2 \frac{\mu_1+\mu_2}{4}\left(\frac{d\mu_1^2}{P(\mu_1)} - \frac{d\mu_2^2}{P(\mu_2)}\right), \tag{3.71}$$

the momentum operators are

$$p_{\mu_i} = \frac{\hbar}{i}\left(\frac{\partial}{\partial\mu_i} + \frac{1}{2(\mu_1+\mu_2)} - \frac{1}{4}\frac{P'(\mu_i)}{P(\mu_i)}\right), \tag{3.72}$$

and for the Hamiltonian we obtain

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2MR^2}\frac{4}{\mu_1+\mu_2}\left(P(\mu_1)\left(\frac{\partial^2}{\partial\mu_1^2} + \frac{P'(\mu_1)}{2P(\mu_1)}\right) - P(\mu_2)\left(\frac{\partial^2}{\partial\mu_2^2} + \frac{P'(\mu_2)}{2P(\mu_2)}\right)\right) = \\ &= \frac{1}{2MR^2}\left(\sqrt{\frac{4P(\mu_1)}{\mu_1+\mu_2}}\,p_{\mu_1}^2\sqrt{\frac{4P(\mu_1)}{\mu_1+\mu_2}} + \sqrt{\frac{-4P(\mu_2)}{\mu_1+\mu_2}}\,p_{\mu_2}^2\sqrt{\frac{-4P(\mu_2)}{\mu_1+\mu_2}}\right) + \\ &\quad + \frac{\hbar^2}{2MR^2}\frac{1}{\mu_1+\mu_2}\left(P''(\mu_1) - P''(\mu_2) - \frac{3P'^2(\mu_1)}{4P(\mu_1)} + \frac{3P'^2(\mu_2)}{4P(\mu_2)}\right). \end{aligned} \tag{3.73}$$

A potential separable in semihyperbolic coordinates must have the form

$$V(\mu_1,\mu_2) = \frac{V_1(\mu_1)+V_2(\mu_2)}{\mu_1+\mu_2}, \tag{3.74}$$

and the corresponding observable is given by

$$\begin{aligned} I_{SH}^{(V)} &= -\frac{\hbar^2}{2M}\frac{1}{\mu_1+\mu_2}\left(-\mu_2\sqrt{P(\mu_1)}\frac{\partial}{\partial\mu_1}\sqrt{P(\mu_1)}\frac{\partial}{\partial\mu_1} + \right. \\ &\quad \left. + \mu_1\sqrt{P(\mu_2)}\frac{\partial}{\partial\mu_2}\sqrt{P(\mu_2)}\frac{\partial}{\partial\mu_2}\right) + \\ &\quad + \frac{\mu_2V_1(\mu_1) - \mu_1V_2(\mu_2)}{\mu_1+\mu_2} = \\ &= \frac{1}{2M}\{K_1, L_3\} + \frac{\mu_2V_1(\mu_1) - \mu_1V_2(\mu_2)}{\mu_1+\mu_2}. \end{aligned} \tag{3.75}$$

7. The seventh coordinate system is called the *elliptic-parabolic* system. It has the form

$$\left.\begin{aligned} u_0 &= \frac{R}{2}\left(\frac{(\rho_1-a_1)(a_1-\rho_2)}{(a_1-a_2)^{3/2}\sqrt{(\rho_1-a_2)(\rho_2-a_2)}}\right. \\ &\left.+\sqrt{\frac{a_1-a_2}{(\rho_1-a_2)(\rho_2-a_2)}}+\sqrt{\frac{(\rho_1-a_2)(\rho_2-a_2)}{a_1-a_2}}\right), \\ u_1 &= \frac{R}{2}\left(\frac{(\rho_1-a_1)(a_1-\rho_2)}{(a_1-a_2)^{3/2}\sqrt{(\rho_1-a_2)(\rho_2-a_2)}}\right. \\ &\left.+\sqrt{\frac{a_1-a_2}{(\rho_1-a_2)(\rho_2-a_2)}}-\sqrt{\frac{(\rho_1-a_2)(\rho_2-a_2)}{a_1-a_2}}\right), \\ u_2 &= R\frac{\sqrt{(\rho_1-a_1)(a_1-\rho_2)}}{a_1-a_2} \end{aligned}\right\} \tag{3.76}$$

$(a_2<\rho_2<a_1<\rho_1)$. The characteristic operator is given by

$$I_{EP}=K_1^2+(a_1-a_2)\,K_2^2+L_3^2-\{K_1,L_3\}. \tag{3.77}$$

Making the special choice $a_1=0$, $a_2=-1$ together with $\rho_1=\tan^2\vartheta$, $\rho_2=-\tanh^2 a$ $(\vartheta\in(-\pi/2,\pi/2), a\in\mathbb{R})$ we obtain

$$\left.\begin{aligned} u_0 &= R\frac{\cosh^2 a+\cos^2\vartheta}{2\cosh a\cos\vartheta} \\ u_1 &= R\frac{\sinh^2 a-\sin^2\vartheta}{2\cosh a\cos\vartheta}, \\ u_2 &= R\tan\vartheta\tanh a. \end{aligned}\right\} \tag{3.78}$$

In this case the characteristic operator has the form

$$I_{EP}=K_1^2+K_2^2+L_3^2-\{K_1,L_3\}=-\hbar^2R^2\Delta_{LB}+2L_3^2-\{K_1,L_3\}, \tag{3.79}$$

which shows that for this choice of the parameters the coordinate system may be characterized as a polar-parabolic system. The Stäckel determinant then has the form

$$S = \begin{vmatrix} -\dfrac{R^2}{\cosh^2 a} & -1 \\ \dfrac{R^2}{\cos^2 \vartheta} & 1 \end{vmatrix} = R^2 \frac{\cosh^2 a - \cos^2 \vartheta}{\cosh^2 a \cos^2 \vartheta}, \tag{3.80}$$

and $f_1 = f_2 = 1$. In the flat space limit we obtain parabolic coordinates. The line element is given by

$$ds^2 = R^2 \frac{\cosh^2 a - \cos^2 \vartheta}{\cosh^2 a \cos^2 \vartheta} (da^2 + d\vartheta^2). \tag{3.81}$$

For the momentum operators we have

$$p_a = \frac{\hbar}{i}\left(\frac{\partial}{\partial a} + \frac{\sinh a \cosh a}{\cosh^2 a - \cos^2 \vartheta} - \tanh a \right),$$

$$p_\vartheta = \frac{\hbar}{i}\left(\frac{\partial}{\partial \vartheta} + \frac{\sin \vartheta \cos \vartheta}{\cosh^2 a - \cos^2 \vartheta} + \tan \vartheta \right), \tag{3.82}$$

and the Hamiltonian reads

$$H_0 = -\frac{\hbar^2}{2MR^2} \frac{\cosh^2 a \cos^2 \vartheta}{\cosh^2 a - \cos^2 \vartheta} \left(\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial \vartheta^2} \right) =$$

$$= \frac{1}{2MR^2} \frac{\cosh a \cos \vartheta}{\sqrt{\cosh^2 a - \cos^2 \vartheta}} (p_a^2 + p_\vartheta^2) \frac{\cosh a \cos \vartheta}{\sqrt{\cosh^2 a - \cos^2 \vartheta}}. \tag{3.83}$$

A potential separable in elliptic-parabolic coordinates must have the form

$$V(a, \vartheta) = \frac{\cosh^2 a \cos^2 \vartheta}{\cosh^2 a - \cos^2 \vartheta} [V_1(a) + V_2(\vartheta)], \tag{3.84}$$

and the observable then is

$$I_{EP}^{(V)} = -\frac{\hbar^2}{2M} \frac{1}{\cosh^2 a - \cos^2 \vartheta} \left(\cosh^2 a \frac{\partial^2}{\partial a^2} + \cos^2 \vartheta \frac{\partial^2}{\partial \vartheta^2} \right) +$$

$$+ \frac{\cosh^2 a V_1(a) + \cos^2 \vartheta V_2(\vartheta)}{\cosh^2 a - \cos^2 \vartheta} =$$

$$= \frac{1}{2M} (K_1^2 + K_2^2 + L_3^2 - \{K_1, L_3\}) + \frac{\cosh^2 a V_1(a) + \cos^2 \vartheta V_2(\vartheta)}{\cosh^2 a - \cos^2 \vartheta}. \tag{3.85}$$

8. The eighth coordinate system is called the *hyperbolic-parabolic* system. It has the form

$$\left.\begin{aligned}
u_0 &= \frac{R}{2}\left(\frac{(\rho_1 - a_1)(a_1 - \rho_2)}{(a_1 - a_2)^{3/2}\sqrt{(\rho_1 - a_2)(\rho_2 - a_2)}}\right.\\
&\quad \left. + \sqrt{\frac{a_1 - a_2}{(\rho_1 - a_2)(a_2 - \rho_2)}} + \sqrt{\frac{(\rho_1 - a_2)(a_2 - \rho_2)}{a_1 - a_2}}\right),\\
u_1 &= \frac{R}{2}\left(\frac{(\rho_1 - a_1)(a_1 - \rho_2)}{(a_1 - a_2)^{3/2}\sqrt{(\rho_1 - a_2)(a_2 - \rho_2)}}\right.\\
&\quad \left. + \sqrt{\frac{a_1 - a_2}{(\rho_1 - a_2)(a_2 - \rho_2)}} - \sqrt{\frac{(\rho_1 - a_2)(a_2 - \rho_2)}{a_1 - a_2}}\right),\\
u_2 &= R\,\frac{\sqrt{(\rho_1 - a_1)(a_1 - \rho_2)}}{a_1 - a_2}
\end{aligned}\right\} \tag{3.86}$$

$(\rho_2 < a_2 < a_1 < \rho_1)$. The characteristic operator is given by

$$I_{HP} = K_1^2 - (a_1 - a_2)\,K_2^2 + L_3^2 - \{K_1, L_3\}. \tag{3.87}$$

Making the special choice $a_1 = 0$, $a_2 = -1$ together with $\rho_1 = \cot^2\vartheta$, $\rho_2 = -\coth^2 b$ ($\vartheta \in (0,\pi)$, $b > 0$), we obtain

$$\left.\begin{aligned}
u_0 &= R\,\frac{\cosh^2 b + \cos^2\vartheta}{2\sinh b\sin\vartheta},\\
u_1 &= R\,\frac{\sinh^2 b - \sin^2\vartheta}{2\sinh b\sin\vartheta},\\
u_2 &= R\cot\vartheta\coth b.
\end{aligned}\right\} \tag{3.88}$$

In this case the characteristic operator reads

$$I_{HP} = K_1^2 - K_2^2 + L_3^2 - \{K_1, L_3\}. \tag{3.89}$$

For the Stäckel determinant we have

$$S = \begin{vmatrix} \dfrac{R^2}{\sinh^2 b} & -1 \\ -\dfrac{R^2}{\sin^2\vartheta} & 1 \end{vmatrix} = R^2\,\frac{\sinh^2 b + \sin^2\vartheta}{\sinh^2 b\sin^2\vartheta}, \tag{3.90}$$

and $f_1 = f_2 = 1$. In the flat space limit we obtain cartesian coordinates from this system. The line element is given by

$$ds^2 = R^2 \frac{\sinh^2 b + \sin^2 \vartheta}{\sinh^2 b \sin^2 \vartheta} (db^2 + d\vartheta^2). \tag{3.91}$$

For the momentum operators we have

$$p_b = \frac{\hbar}{\mathrm{i}} \left(\frac{\partial}{\partial b} + \frac{\sinh b \cosh b}{\sinh^2 b + \sin^2 \vartheta} - \coth b \right),$$

$$p_\vartheta = \frac{\hbar}{\mathrm{i}} \left(\frac{\partial}{\partial \vartheta} + \frac{\sin \vartheta \cos \vartheta}{\sinh^2 b + \sin^2 \vartheta} - \cot \vartheta \right), \tag{3.92}$$

and for the Hamiltonian we get

$$H_0 = -\frac{\hbar^2}{2MR^2} \frac{\sinh^2 b \sin^2 \vartheta}{\sinh^2 b + \sin^2 \vartheta} \left(\frac{\partial^2}{\partial b^2} + \frac{\partial^2}{\partial \vartheta^2} \right) =$$

$$= \frac{1}{2MR^2} \frac{\sinh b \sin \vartheta}{\sqrt{\sinh^2 b + \sin^2 \vartheta}} (p_b^2 + p_\vartheta^2) \frac{\sinh b \sin \vartheta}{\sqrt{\sinh^2 b + \sin^2 \vartheta}}. \tag{3.93}$$

A potential separable in elliptic-parabolic coordinates must have the form

$$V(b, \vartheta) = \frac{\sinh^2 b \sin^2 \vartheta}{\sinh^2 b + \sin^2 \vartheta} [V_1(b) + V_2(\vartheta)], \tag{3.94}$$

and the corresponding observable is

$$I_{HP}^{(V)} = -\frac{\hbar^2}{2M} \frac{1}{\sinh^2 b + \sin^2 \vartheta} \left(\sinh^2 b \frac{\partial^2}{\partial b^2} + \sin^2 \vartheta \frac{\partial^2}{\partial \vartheta^2} \right) +$$

$$+ \frac{\sinh^2 b V_1(b) + \sin^2 \vartheta V_2(\vartheta)}{\sinh^2 b + \sin^2 \vartheta} =$$

$$= \frac{1}{2M} (K_1^2 - K_2^2 + L_3^2 - \{K_1, L_3\}) + \frac{\sinh^2 b V_1(b) + \sin^2 \vartheta V_2(\vartheta)}{\sinh^2 b + \sin^2 \vartheta}. \tag{3.95}$$

9. The ninth and the last system is the *semicircular parabolic* coordinate system:

$$\left.\begin{aligned}
u_0 &= R\left[\frac{(\rho_1-\rho_2)^2}{8[(\rho_1-a)(a-\rho_2)]^{3/2}}+\frac{1}{2}\sqrt{(\rho_1-a)(a-\rho_2)}\right] = R\,\frac{(\xi^2+\eta^2)^2+4}{8\xi\eta},\\
u_1 &= R\left[\frac{(\rho_1-\rho_2)^2}{8[(\rho_1-a)(a-\rho_2)]^{3/2}}-\frac{1}{2}\sqrt{(\rho_1-a)(a-\rho_2)}\right] = R\,\frac{(\xi^2+\eta^2)^2-4}{8\xi\eta},\\
u_2 &= \frac{R}{2}\left(\sqrt{\frac{\rho_1-a}{a-\rho_2}}-\sqrt{\frac{a-\rho_2}{\rho_1-a}}\right) = R\,\frac{\eta^2-\xi^2}{2\xi\eta}
\end{aligned}\right\} \tag{3.96}$$

$(\rho_2<a<\rho_1)$, and we have made the choice $a=0$, $\rho_2=-1/\eta^2$, $\rho_1=1/\xi^2$, $\xi,\eta>0$. The characteristic operator has the form

$$I_{SCP}=\{K_1,K_2\}-\{K_2,L_3\}. \tag{3.97}$$

The Stäckel determinant is given by

$$S=\begin{vmatrix}\dfrac{R^2}{\xi^2} & -1\\ \dfrac{R^2}{\eta^2} & 1\end{vmatrix} = R^2\,\frac{\xi^2+\eta^2}{\xi^2\eta^2}, \tag{3.98}$$

and $f_1=f_2=1$. In the flat space limit this coordinate system gives cartesian coordinates. The line element reads

$$ds^2=R^2\,\frac{\xi^2+\eta^2}{\xi^2\eta^2}\,(d\xi^2+d\eta^2), \tag{3.99}$$

the momentum operators are

$$p_\xi=\frac{\hbar}{i}\left(\frac{\partial}{\partial\xi}+\frac{\xi}{\xi^2+\eta^2}-\frac{1}{\xi}\right),\quad p_\eta=\frac{\hbar}{i}\left(\frac{\partial}{\partial\eta}+\frac{\eta}{\xi^2+\eta^2}-\frac{1}{\eta}\right), \tag{3.100}$$

and for the Hamiltonian we have

$$\begin{aligned}
H_0 &= -\frac{\hbar^2}{2MR^2}\,\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)=\\
&= \frac{1}{2MR^2}\,\frac{\xi\eta}{\sqrt{\xi^2+\eta^2}}\,(p_\xi^2+p_\eta^2)\,\frac{\xi\eta}{\sqrt{\xi^2+\eta^2}}.
\end{aligned} \tag{3.101}$$

Table 1. Coordinate systems on the two-dimensional hyperboloid

Coordinate system observable I	Coordinates	Separates potential	Limiting systems
I. Spherical $\tau > 0,\ \varphi \in [0, 2\pi)$ $I = L_3^2$	$u_0 = R\cosh\tau$ $u_1 = R\sinh\tau\cos\varphi$ $u_2 = R\sinh\tau\sin\varphi$	V_1, V_2 $V_4^{(\omega=0)}$	Polar
II. Equidistant $\tau_{1,2} \in \mathbb{R}$ $I = K_2^2$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2$ $u_2 = R\sinh\tau_1$	V_1, V_4, V_5	Cartesian
III. Horicyclic $y > 0,\ x \in \mathbb{R}$ $I = (K_1 - L_3)^2$	$u_0 = \frac{R}{2y}(x^2 + y^2 + 1)$ $u_1 = \frac{R}{2y}(x^2 + y^2 - 1)$ $u_2 = Rx/y$	V_3, V_4	Cartesian
IV. Elliptic $\alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\beta \in [0, 4K')$ $I = L_3^2 + \sinh^2 f K_2^2$ $I' = \cosh 2f L_3^2 -$ $- 1/2 \sinh 2f\{K_1, L_3\}$	$u_0 = R\,\mathrm{sn}\,\alpha\,\mathrm{dn}\,\beta$ $u_1 = \mathrm{i}R\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta$ $u_2 = \mathrm{i}\,R\,\mathrm{dn}\,\alpha\,\mathrm{sn}\,\beta$	V_1, V_2^* $V_4^{(\omega=0)}$	Elliptic
V. Hyperbolic $\mu \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\eta \in [0, 4K')$ $I = K_2^2 - \sin^2\alpha L_3^2$	$u_0 = -R\,\mathrm{cn}\,\mu\,\mathrm{cn}\,\eta$ $u_1 = \mathrm{i}R\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\eta$ $u_2 = \mathrm{i}R\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\eta$	V_1 $V_4^{(\omega=0)}$	Cartesian
VI. Semi-Hyperbolic $\mu_{1,2} > 0$ $I = \{K_1, L_3\}$	$u_0 = \frac{R}{\sqrt{2}}(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} +$ $+ \mu_1\mu_2 + 1)^{1/2}$ $u_1 = \frac{R}{\sqrt{2}}(\sqrt{(1+\mu_1^2)(1+\mu_2^2)} -$ $- \mu_1\mu_2 - 1)^{1/2}$ $u_2 = R\sqrt{\mu_1\mu_2}$	V_2 $V_4^{(\omega=0)}$	Cartesian** Parabolic

Coordinate system observable I	Coordinates	Separates potential	Limiting systems
VII. Elliptic-parabolic $a \in \mathbb{R}, \vartheta \in (-\pi/2, \pi/2)$ $I = (K_1 - L_3)^2 + K_2^2$	$u_0 = R\dfrac{\cosh^2 a + \cos^2 \vartheta}{2\cosh a \cos \vartheta}$ $u_1 = R\dfrac{\sinh^2 a - \sin^2 \vartheta}{2\cosh a \cos \vartheta}$ $u_2 = R \tan \vartheta \tanh a$	V_2, V_4	Parabolic
VIII. Hyperbolic-parabolic $b > 0, \vartheta \in (0, \pi)$ $I = (K_1 - L_3)^2 - K_2^2$	$u_0 = R\dfrac{\cosh^2 b + \cos^2 \vartheta}{2\sinh b \sin \vartheta}$ $u_1 = R\dfrac{\sinh^2 b - \sin^2 \vartheta}{2\cosh b \sin \vartheta}$ $u_2 = R \cot \vartheta \coth b$	V_4	Cartesian
IX. Semi-circular-parabolic $\xi, \eta > 0$ $I = \{K_1, K_2\} - \{K_2, L_3\}$	$u_0 = R\dfrac{(\xi^2+\eta^2)^2+4}{8\xi\eta}$ $u_1 = R\dfrac{(\xi^2+\eta^2)^2-4}{2\xi\eta}$ $u_2 = R\dfrac{\eta^2-\xi^2}{8\xi\eta}$	V_3, V_5 $V_4^{(\lvert k \rvert = 1/2)}$	Cartesian

*after rotation
**system with whole parameterization

A potential separable in semicircular parabolic coordinates must have the form

$$V(\xi, \eta) = \frac{\xi^2\eta^2}{\xi^2+\eta^2}\,[V_1(\xi) + V_2(\eta)], \tag{3.102}$$

and the corresponding observable is given by

$$I_{SCP}^{(V)} = -\frac{h^2}{2M}\frac{1}{\xi^2+\eta^2}\left(\eta^2\frac{\partial^2}{\partial\eta^2} - \xi^2\frac{\partial^2}{\partial\xi^2}\right) - \frac{\xi^2 V_1(\xi) - \eta^2 V_2(\eta)}{\xi^2+\eta^2} =$$

$$= \frac{1}{2M}(\{K_1, K_2\} - \{K_2, L_3\}) - \frac{\xi^2 V_1(\xi) - \eta^2 V_2(\eta)}{\xi^2+\eta^2}. \tag{3.103}$$

This concludes the enumeration of the coordinate systems on the two-dimensional hyperboloid.

In Table 1 we list the coordinate systems on $\Lambda^{(2)}$, which separate the Schrödinger equation, together with potentials which are separated by coordinate systems, and the limiting cases in $\mathbb{R}^2$, as $R \to \infty$.

4. PATH INTEGRAL FORMULATION OF THE SUPERINTEGRABLE POTENTIALS ON $\Lambda^{(2)}$

In Table 2 we list the superintegrable potentials on the two-dimensional hyperboloid together with the separating coordinate systems, and the corresponding observables. The cases where an explicit path integration is possible are underlined.

4.1. The Higgs-Oscillator

We consider the potential ($k_{1,2} > 0$)

$$V_1(u) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2 + u_2^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2 - \frac{1}{4}}{u_1^2} + \frac{k_2^2 - \frac{1}{4}}{u_2^2}\right), \tag{4.1}$$

which in the four separating coordinate systems has the form

Spherical ($\tau > 0$, $\varphi \in (0, \pi/2)$):

$$V_1(u) = \frac{M}{2}\omega^2 R^2 \tanh^2\tau + \frac{\hbar^2}{2MR^2}\frac{1}{\sinh^2\tau}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right) \tag{4.2}$$

Equidistant ($\tau_1, \tau_2 > 0$):

$$\begin{aligned} &= \frac{M}{2}\omega^2 R^2 \left(1 - \frac{1}{\cosh^2\tau_1 \cosh^2\tau_2}\right) + \\ &\quad + \frac{\hbar^2}{2MR^2}\left(\frac{k_1^2 - \frac{1}{4}}{\cosh^2\tau_1 \sinh^2\tau_2} + \frac{k_2^2 - \frac{1}{4}}{\sinh^2\tau_1}\right) \end{aligned} \tag{4.3}$$

Elliptic ($\alpha \in (iK', iK' + K)$, $\beta \in (0, K')$):

$$= \frac{M}{2}\omega^2 R^2 \left(1 - \frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\right) + \frac{\hbar^2}{2MR^2}\left(\frac{k_1^2 - \frac{1}{4}}{\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta} + \frac{k_2^2 - \frac{1}{4}}{\mathrm{dn}^2\alpha\,\mathrm{sn}^2\beta}\right) \tag{4.4}$$

Table 2. Superintergable potentials on the two-dimensional hyperboloid .

Potential $V(\mathrm{u})$	Coordinate system	Observables
$V_1(\mathrm{u}) = \frac{M}{2}\omega^2 R^2 \frac{u_1^2+u_2^2}{u_0^2} + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{u_1^2}+\frac{k_2^2-\frac{1}{4}}{u_2^2}\right)$	Spherical Equidistant Elliptic Hyperbolic	$I_1 = \frac{1}{2MR^2}(K_1^2+K_2^2-L_3^2)+V_1(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)$ $I_3 = \frac{1}{2M}K_2^2 - \frac{M}{2}\frac{\omega^2 R^4}{\cosh^2\tau_2} + \frac{h^2}{2M}\frac{k_1^2-\frac{1}{4}}{\sinh^2\tau_2}$
$V_2(\mathrm{u}) = -\frac{\alpha}{R}\left(\frac{0}{\sqrt{u_1^2+u_2^2}}-1\right) + \frac{\hbar^2}{4M\sqrt{u_1^2+u_2^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}+u_1}+\frac{k_2^2-\frac{1}{4}}{\sqrt{u_1^2+u_2^2}-u_1}\right)$	Spherical Elliptic-parabolic Elliptic II Semihyperbolic	$I_1 = \frac{1}{2MR^2}(K_1^2+K_2^2-L_3^2)+V_2(\mathrm{u})$ $I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2\frac{\varphi}{2}}+\frac{k_2^2-\frac{1}{4}}{\cos^2\frac{\varphi}{2}}\right)$ $I_3 = \frac{1}{2M}\{K_1, L_3\} - \alpha R \frac{\mu_2\sqrt{1+\mu_1^2}-\mu_1\sqrt{1+\mu_2^2}}{\mu_1+\mu_2}$ $+ \frac{\hbar^2}{4M}\left[\left(k_1^2+k_2^2-\frac{1}{2}\right)\left(\frac{\mu_1/\mu_2+\mu_2/\mu_1}{\mu_1+\mu_2}\right)+\right.$ $\left.+(k_1^2-k_2^2)\frac{\mu_2^2\sqrt{1+\mu_1^2}+\mu_1^2\sqrt{1+\mu_2^2}}{\mu_1\mu_2(\mu_1+\mu_2)}\right]$

Potential V(u)	Coordinate system	Observables
$V_3(\mathrm{u}) = \dfrac{\alpha}{(u_0-u_1)^2} + \dfrac{M}{2}\omega^2 \dfrac{R^2+4u_2^2}{(u_0-u_1)^4} - \lambda \dfrac{u_2}{(u_0-u_1)^3}$	Horicyclic Semicircular-parabolic	$I_1 = \dfrac{1}{2MR^2}(K_1^2+K_2^2-L_3^2)+V_3(\mathrm{u})$ $I_2 = \dfrac{1}{2M}(K_1-L_3)^2+\alpha+2M\,\omega^2x^2-\lambda x$ $I_3 = \dfrac{1}{2M}(\{K_1,K_2\}-\{K_2,L_3\}) + \dfrac{1}{2}\dfrac{\xi^4(2\alpha+\xi^2\lambda+M\omega^2\xi^4)-\eta^4(2\alpha-\eta^2\lambda+M\omega^2\eta^4)}{\xi^2+\eta^2}$
$V_4(\mathrm{u}) = \dfrac{M}{2}\dfrac{\omega^2}{(u_0-u_1)^2}+\dfrac{\hbar^2}{2M}\dfrac{\kappa^2-\frac{1}{4}}{u_2^2}$ $\lvert\kappa\rvert = 1/2$ $\omega = 0$	Equidistant Horicyclic Elliptic-parabolic Hyperbolic-parabolic Semicircular-parabolic all systems except IX	$I_1 = \dfrac{1}{2MR^2}(K_1^2+K_2^2-L_3^2)+V_4(\mathrm{u})$ $I_2 = \dfrac{1}{2M}(K_1-L_3)^2+\dfrac{\hbar^2}{2M}\dfrac{\kappa^2-\frac{1}{4}}{x^2}$ $I_3 = \dfrac{1}{2M}K_2^2+\dfrac{M}{2}\omega^2 e^{2\tau_2}$
$V_5(\mathrm{u}) = \alpha R\dfrac{u_2}{\sqrt{u_0^2-u_1^2}}$	Equidistant Semicircular-parabolic	$I_1 = \dfrac{1}{2MR^2}(K_1^2+K_2^2-L_3^2)+V_5(\mathrm{u})$ $I_2 = \dfrac{1}{2M}(\{K_1,K_2\}-\{K_2,L_3\})+\dfrac{2\alpha R}{\xi^2+\eta^2}$ $I_3 = K_2^2$

Hyperbolic ($\mu \in (iK', iK'+2K), \eta \in (0, K')$):

$$= \frac{M}{2}\omega^2 R^2\left(1 - \frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\right) + \frac{\hbar^2}{2MR^2}\left(\frac{k_1^2 - \frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta} + \frac{k_2^2 - \frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\right). \quad (4.5)$$

The constants of motion for the potential V_1 are the following

$$\left.\begin{aligned} I_1 &= \frac{1}{2MR^2}(K_1^2 + K_2^2 - L_3^2) + V_1(u), \\ I_2 &= \frac{1}{2M}L_3^2 + \frac{\hbar^2}{2M}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right), \\ I_3 &= \frac{1}{2M}K_2^2 - \frac{M}{2}\frac{\omega^2 R^4}{\cosh^2\tau_2} + \frac{\hbar^2}{2M}\frac{k_1^2 - \frac{1}{4}}{\sinh^2\tau_2}. \end{aligned}\right\} \quad (4.6)$$

We have for V_1 the path integral representations (in the elliptic system we explicitly state the separated path integral formulation $\nu^2 = M^2\omega^2 R^4/\hbar^2 + 1/4$):

$$K^{(V_1)}(u'', u'; T)$$

Spherical:

$$= \frac{1}{R^2}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh\tau \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau^2 + \sinh^2\tau\dot\varphi^2 - \omega^2\tanh^2\tau)\right.\right.$$

$$\left.\left. - \frac{\hbar^2}{2MR^2}\left(\frac{1}{\sinh^2\tau}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi} - \frac{1}{4}\right) + \frac{1}{4}\right]dt\right\} \quad (4.7)$$

Equidistant:

$$= \frac{1}{R^2}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t)\cosh\tau_1 \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_2^2-\omega^2\left(1-\frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\right)\right)-\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\cosh^2\tau_1}\left(\frac{k_1^2-\frac{1}{4}}{\sinh^2\tau_2}+\frac{1}{4}\right)+\frac{k_2^2-\frac{1}{4}}{\sinh^2\tau_1}+\frac{1}{4}\right)\right]dt\right\} \tag{4.8}$$

Elliptic:

$$=\frac{1}{R^2}\int_{\alpha(t')=\alpha'}^{\alpha(t'')=\alpha''}\mathcal{D}\alpha(t)\int_{\beta(t')=\beta'}^{\beta(t'')=\beta''}\mathcal{D}\beta(t)\,(k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left((k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta)(\dot\alpha^2+\dot\beta^2)-\omega^2\left(1-\frac{1}{\mathrm{sn}^2\,\alpha\,\mathrm{dn}^2\,\beta}\right)\right)+\right.\right.$$

$$\left.\left.+\frac{\hbar^2}{2MR^2}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{cn}^2\,\alpha\,\mathrm{cn}^2\,\beta}+\frac{k_1^2-\frac{1}{4}}{\mathrm{dn}^2\,\alpha\,\mathrm{sn}^2\,\beta}\right)\right]dt\right\} \tag{4.9}$$

$$=\frac{e^{-iM\omega^2R^2T/2\hbar}}{R^2}\frac{i}{\hbar}\int_0^\infty dT\,e^{iET/\hbar}\int_0^\infty ds''\int_{\alpha(0)=\alpha'}^{\alpha(s'')=\alpha''}\mathcal{D}\alpha(s)\int_{\beta(0)=\beta'}^{\beta(s'')=\beta''}\mathcal{D}\beta(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{M}{2}(\dot\alpha^2+\dot\beta^2)+R^2(k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta)\,E+\right.\right.$$

$$+\frac{\hbar^2}{2M}\left(\left(\nu^2-\frac{1}{4}\right)\left(\frac{1}{\mathrm{sn}^2\,\alpha}-\frac{k^2}{\mathrm{dn}^2\,\beta}\right)+\right.$$

$$\left.\left.\left.+\left(k_1^2-\frac{1}{4}\right)\left(\frac{k'^2}{\mathrm{cn}^2\,\alpha}+\frac{k^2}{\mathrm{cn}^2\,\beta}\right)-\left(k_2^2-\frac{1}{4}\right)\left(\frac{k'^2}{\mathrm{dn}^2\,\alpha}-\frac{1}{\mathrm{sn}^2\,\beta}\right)\right)\right]ds\right\} \tag{4.10}$$

Hyperbolic:

$$=\frac{1}{R^2}\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\,(k^2\,\mathrm{cn}^2\,\mu+k'^2\,\mathrm{sn}^2\,\eta)\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\left((k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\eta)(\dot\mu^2+\dot\eta^2) - \omega^2\left(1-\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\eta}\right)\right)+\right.\right.$$

$$\left.\left.+\frac{\hbar^2}{2MR^2}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\eta}+\frac{k_2^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\eta}\right)\right]dt\right\}. \tag{4.11}$$

We solve the first two-path integrals explicitly. The two remaining ones are too complicated to allow an explicit solution.

The two-path integral formulations of the Higgs oscillator have a simple structure involving Pöschl-Teller (2.6) and modified Pöschl-Teller path integrals (2.12). We start with the *pure oscillator case*, denoted by $K^{(\omega)}(T)$, in order to demonstrate the relevant techniques involved in the solutions.

4.1.1. Pure Oscillator Case. *Spherical Coordinates.* For the oscillator in spherical coordinates the φ-integration is easily separated [20], and we obtain by using the path integral representation of the modified Pöschl-Teller potential (2.12) the following solution ($\nu^2 = M^2\omega^2R^4/\hbar^2 + 1/4$)

$$K^{(\omega)}(u'', u'; T) = \frac{1}{R^2}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh\tau\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau^2+\sinh^2\tau\dot\varphi^2-\omega^2\tanh^2\tau)-\frac{\hbar^2}{8MR^2}\left(1-\frac{1}{\sinh^2\tau}\right)\right]dt\right\}=$$

$$=\frac{\exp\left[-\frac{i}{\hbar}T\left(\frac{\hbar^2}{8MR^2}+\frac{M}{2}R^2\omega^2\right)\right]}{(R^2\sinh\tau'\sinh\tau'')^{1/2}}\sum_{j\in\mathbb{Z}}\frac{e^{ij(\varphi''-\varphi')}}{2\pi}\times$$

$$\times\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\dot\tau^2-\frac{\hbar^2}{2MR^2}\left(\frac{j^2-\frac14}{\sinh^2\tau}-\frac{\nu^2-\frac14}{\cosh^2\tau}\right)\right]dt\right\}=$$

$$=\sum_{j\in\mathbb{Z}}\left[\sum_{N=0}^{N_{\max}}\Psi_{Nj}^{(\omega)}(\tau',\varphi')\,\Psi_{Nj}^{(\omega)}(\tau'',\varphi'')\,e^{-iE_NT/\hbar}+\right.$$

$$\left.+\int_0^\infty dp\,e^{-iE_pT/\hbar}\,\Psi_{pj}^{(\omega)*}(\tau',\varphi')\,\Psi_{pj}^{(\omega)}(\tau'',\varphi'')\right]. \tag{4.12}$$

The wave functions and the energy spectrum of the discrete contributions have the following form (we introduce the principal quantum number $N = 2n + |j| = 0, 1, \ldots$ where appropriate)

$$\Psi^{(\omega)}_{Nj}(\tau, \varphi; R) = (2\pi \sinh \tau)^{-1/2} S^{(\nu)}_{jN}(\tau; R)\, e^{ij\varphi}, \tag{4.13}$$

$$S^{(\nu)}_{jN}(\tau; R) = \frac{1}{|j|!}\left[\frac{2(\nu - N - 1)\left(\frac{1}{2}(N + |j|)\right)!\,\Gamma\left(\nu - \frac{1}{2}(N - |j|)\right)}{R^2\Gamma\left(\nu - \frac{1}{2}(N + |j|)\right)\left(\frac{1}{2}(N - |j|)\right)!}\right]^{1/2} \times$$

$$\times (\sinh \tau)^{|j| + 1/2}(\cosh \tau)^{(N + |j| + 1)/2 - \nu} \times$$

$$\times\, {}_2F_1\left(-\frac{1}{2}(N - |j|), \nu - \frac{1}{2}(N - |j|); 1 + |j|; \tanh^2 \tau\right), \tag{4.14}$$

with the discrete spectrum given by

$$E_N = -\frac{\hbar^2}{2MR^2}\left[(N - \nu + 1)^2 - \frac{1}{4}\right] + \frac{M}{2}\omega^2 R^2. \tag{4.15}$$

Only a finite number exists with $N_{\max} = [\nu - |j| - 1] \geq 0$. In the flat space limit we obtain for the energy spectrum

$$E_N \simeq \hbar\omega(N + 1). \tag{4.16}$$

The continuous wave functions have the form

$$\Psi^{(\omega)}_{pj}(\tau, \varphi; R) = (2\pi \sinh \tau)^{-1/2}\, S^{(\nu)}_{jp}(\tau; R)\, e^{ij\varphi}, \tag{4.17}$$

$$S^{(\nu)}_{jp}(\tau; R) = \frac{1}{|j|!}\sqrt{\frac{p \sinh \pi p}{2\pi^2 R^2}}\,\Gamma\left(\frac{\nu - |j| + 1 - ip}{2}\right)\Gamma\left(\frac{|j| - \nu + 1 - ip}{2}\right) \times$$

$$\times (\tanh \tau)^{|j| + 1/2}(\cosh \tau)^{ip} \times$$

$$\times\, {}_2F_1\left(\frac{\nu + |j| + 1 - ip}{2}, \frac{|j| - \nu + 1 - ip}{2}; 1 + |j|; \tanh^2 \tau\right), \tag{4.18}$$

with the continuous energy spectrum given by

$$E_p = \frac{\hbar^2}{2MR^2}\left(p^2 + \frac{1}{4}\right) + \frac{M}{2}\omega^2 R^2. \tag{4.19}$$

In the limiting case $\omega \to 0$ $(\nu \to 1/2)$ the potential through vanishes (note that in this case $E_N = 0$ exactly), only the continuous spectrum remains, and we obtain the pure continuous spectrum

$$E_p = \frac{\hbar^2}{2MR^2}\left(p^2 + \frac{1}{4}\right), \tag{4.20}$$

which corresponds to the case where just a radial part is present, and has the same feature as the spectrum of the free motion on $\Lambda^{(2)}$.

Let us finally state the corresponding Green function $G^{(V_1)}(E)$ of the potential V_1. It has the form $(m_{1,2} = (|j| \pm \sqrt{-2ME'R^2}/\hbar)$, $L_\nu = \frac{1}{2}(\nu - 1)$ $E' = E - -\hbar^2/8MR^2 - MR^2\omega^2/2)$

$$G^{(V_1)}(\tau'', \tau', \varphi'', \varphi'; E) = \frac{M}{2\hbar^2}\sum_{j\in\mathbb{Z}} \frac{e^{ij(\varphi''-\varphi')}}{2\pi} \frac{\Gamma(m_1 - L_\nu)\,\Gamma(L_\nu + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\,\Gamma(m_1 - m_2 + 1)}$$

$$\times (\cosh\tau' \cosh\tau'')^{-(m_1 - m_2 + 1/2)} (\tanh r' \tanh r'')^{m_1 + m_2}$$

$$\times {}_2F_1\left(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 - m_2 + 1; \frac{1}{\cosh^2_<}\right)$$

$$\times {}_2F_1(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 + m_2 + 1; \tanh^2\tau_<). \tag{4.21}$$

Equidistant Coordinates. In the case of equidistant coordinates we can separate the corresponding path integrations in an analogous way, however, instead of a simple circular wave dependence in the first step leading to a modified Pöschl-Teller problem, we have in this case two symmetric Rosen-Morse path integral problems [20,43]. This yields

$$\left(\lambda = m_2 - \nu + \frac{1}{2}, m_1 = 0, \ldots, N_{\max}^{(1)} = \left[\nu - \frac{1}{2}\right], m_2 = 0, \ldots, N_{\max}^{(2)} = \left[\lambda - \frac{1}{2}\right]\right)$$

$$K^{(\omega)}(u'', u'; T) = \frac{1}{R^2} \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \cosh\tau_1 \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \times$$

$$\times \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{M}{2} R^2(\dot{\tau}_1^2 + \cosh^2 \tau_1 \dot{\tau}_2^2 - \right.\right.$$

$$\left.\left. - \frac{M}{2} R^2 \omega^2 \left(1 - \frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} \right) - \frac{\hbar^2}{8MR^2} \left(1 + \frac{1}{\cosh^2 \tau_1} \right) \right] dt \right\} =$$

$$= \frac{\exp\left[-\frac{i}{\hbar} T \left(\frac{\hbar^2}{8MR^2} + \frac{M}{2} R^2 \omega^2 \right) \right]}{(R^2 \cosh \tau_1' \cosh \tau_1'')^{1/2}} \times$$

$$\times \left\{ \sum_{m_3=0}^{N_{\max}^{(2)}} \left(m_3 - \nu - \frac{1}{2} \right) \frac{\Gamma(2\nu - m_3)}{m_3!} P_{\nu-1/2}^{m_3 - \nu + 1/2}(\tanh \tau_2'')\, P_{\nu-1/2}^{m_3 - \nu + 1/2}(\tanh \tau_2') \times \right.$$

$$\times \int_{\tau_1(t') = \tau_1'}^{\tau_1(t'') = \tau_1''} \mathcal{D}\tau_1(t) \exp\left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{M}{2} R^2 \dot{\tau}_1^2 + \frac{\hbar^2}{2MR^2} \frac{\left(m_3 - \nu + \frac{1}{2} \right)^2 - \frac{1}{4}}{\cosh^2 \tau_1} \right) dt \right] +$$

$$+ \int_{\mathbb{R}} \frac{dk\, k \sinh \pi k}{\cos^2 \pi\nu + \sinh^2 \pi k} P_{\nu-1/2}^{ik}(\tanh \tau_2'')\, P_{\nu-1/2}^{-ik}(\tanh \tau_2') \times$$

$$\left. \times \int_{\tau_1(t') = \tau_1'}^{\tau_1(t'') = \tau_1''} \mathcal{D}\tau_1(t) \exp\left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{M}{2} R^2 \dot{\tau}_1^2 - \frac{\hbar^2}{2MR^2} \frac{k^2 + \frac{1}{4}}{\cosh^2 \tau_1} \right) dt \right] \right\}$$

$$= \sum_{m_3=0}^{N_{\max}^{(2)}} \left\{ \sum_{m_1=0}^{N_{\max}^{(1)}} e^{-iE_N T/\hbar}\, \Psi_{m_1 m_2}^{(\omega)}(\tau_1'', \tau_2''; R)\, \Psi_{m_1 m_2}^{(\omega)}(\tau_1', \tau_2'; R) + \right.$$

$$\left. + \int_0^\infty dp\, e^{-iE_p T/\hbar}\, \Psi_{pm_2}^{(\omega)}(\tau_1'', \tau_2''; R)\, \Psi_{pm_2}^{(\omega)*}(\tau_1', \tau_2'; R) \right\}$$

$$+ \int_0^\infty dk \int_0^\infty dp\, e^{-iE_p T/\hbar}\, \Psi_{pk}^{(\omega)}(\tau_1'', \tau_2''; R)\, \Psi_{pk}^{(\omega)*}(\tau_1', \tau_2'; R). \qquad (4.22)$$

The $P_\nu^\mu(z)$ are Legendre functions [13, p.999]. The discrete wave functions are given by

$$\Psi^{(\omega)}_{m_1 m_2}(\tau_1, \tau_2; R) = (\cosh\tau_1)^{-1/2} S_{\lambda, m_1}(\tau_1; R)\, \psi_{m_2}(\tau_2), \tag{4.23}$$

$$S_{\lambda, m_1}(\tau_1; R) = \sqrt{\left(m_1 - \lambda - \frac{1}{2}\right)\frac{\Gamma(2\lambda - m_1)}{R^2 m_1!}}\, P_{\lambda-1/2}^{m_1-\lambda+1/2}(\tanh\tau_1), \tag{4.24}$$

$$\psi_{m_2}(\tau_2) = \sqrt{\left(m_2 - \nu - \frac{1}{2}\right)\frac{\Gamma(2\nu - m_2)}{m_2!}}\, P_{\nu-1/2}^{m_2-\nu+1/2}(\tanh\tau_2), \tag{4.25}$$

and the discrete spectrum has the form

$$E_N = -\frac{\hbar^2}{2MR^2}\left[(N - \nu + 1)^2 - \frac{1}{4}\right] + \frac{M}{2}\omega^2 R^2, \quad N = m_1 + m_2. \tag{4.26}$$

The bound state energy levels have exactly the same feature as for spherical coordinates, as it must be. Note that the Legendre functions are actually Gegenbauer polynomials. The continuous wave functions consist of two contributions, first where the quantum number corresponding to τ_2 is discrete, second where it is continuous. For the first set we obtain

$$\Psi^{(\omega)}_{p m_2}(\tau_1, \tau_2; R) = (\cosh\tau_1)^{-1/2} S_{\lambda p}(\tau_1; R)\, \psi_{m_2}(\tau_2), \tag{4.27}$$

$$S_{\lambda p}(\tau_1; R) = \frac{1}{R}\sqrt{\frac{p\sinh\pi p}{\cos^2\pi\lambda + \sinh^2\pi p}}\, P_{\lambda-1/2}^{ip}(\tanh\tau_1), \tag{4.28}$$

with the $\psi_{m_2}(\tau_2)$ as in (4.25), and the continuous spectrum is given by

$$E_p = \frac{\hbar^2}{2MR^2}\left(p^2 + \frac{1}{4}\right) + \frac{M}{2}\omega^2 R^2. \tag{4.29}$$

The second set of the continuous wave functions has the form

$$\Psi^{(\omega)}_{kp}(\tau_1, \tau_2; R) = (\cosh\tau_1)^{-1/2} S_{kp}(\tau_1; R)\, \psi_k(\tau_2), \tag{4.30}$$

$$S_{sk}(\tau_1; R) = \frac{1}{R}\sqrt{\frac{p\sinh\pi p}{\cosh^2\pi k + \sinh^2\pi p}}\, P_{ik-1/2}^{ip}(\tanh\tau_1), \tag{4.31}$$

$$\psi_k(\tau_2) = \sqrt{\frac{k\sinh\pi k}{\cos^2\pi\nu + \sinh^2\pi k}}\, P_{\nu-1/2}^{ik}(\tanh\tau_2), \tag{4.32}$$

with the same continuous spectrum as before. The discrete energy-spectrum in the flat scape limit yields again

$$E_N \simeq \hbar\omega(N+1), \tag{4.33}$$

the continuous wave functions vanish, and the discrete wave functions yield Hermite polynomials, i.e., the well-known result of the two-dimensional oscillator.

The corresponding Green function in equidistant coordinates finally has the form

$$(E' = E - \hbar^2/8MR^2 - MR^2\omega^2/2)$$

$$G^{(V_1)}(\tau_1'', \tau_1', \tau_2'', \tau_2'; E) = \frac{M}{\hbar^2}(\cosh\tau_1' \cosh\tau_1'')^{-1/2} \times$$

$$\times \Bigg\{ \sum_{m_2=0}^{N_{\max}^{(2)}} \left(m_2 - \nu - \frac{1}{2}\right) \frac{\Gamma(2\nu - m_2)}{m_2!} P_{\nu-1/2}^{m_2-\nu+1/2}(\tanh\tau_2'')\, P_{\nu-1/2}^{m_2-\nu+1/2}(\tanh\tau_2') \times$$

$$\times \Gamma\left(\frac{1}{\hbar}\sqrt{-2MR^2E'} - \lambda + \frac{1}{2}\right)\Gamma\left(\frac{1}{\hbar}\sqrt{-2MR^2E'} + \lambda + \frac{1}{2}\right) \times$$

$$\times P_{\lambda-1/2}^{-\sqrt{-2MR^2E'}/\hbar}(\tanh\tau_{1,<})\, P_{\lambda-1/2}^{-\sqrt{-2MR^2E'}/\hbar}(-\tanh\tau_{1,>}) +$$

$$+ \int_{\mathbb{R}} \frac{dk\, k \sinh\pi k}{\cos^2\pi\nu + \sinh^2\pi k} P_{\nu-1/2}^{ik}(\tanh\tau_2'')\, P_{\nu-1/2}^{-ik}(\tanh\tau_2') \times$$

$$\times \Gamma\left(\frac{1}{\hbar}\sqrt{-2MR^2E'} - ik + \frac{1}{2}\right)\Gamma\left(\frac{1}{\hbar}\sqrt{-2MR^2E'} + ik + \frac{1}{2}\right) \times$$

$$\times P_{ik-1/2}^{-\sqrt{-2MR^2E'}/h}(\tanh\tau_{1,<})\, P_{ik-1/2}^{-\sqrt{-2MR^2E'}/h}(-\tanh\tau_{1,>}) \Bigg\}. \tag{4.34}$$

4.1.2. General Case. In order to deal with the general case, we do not repeat the whole procedure once more. The separation of variables in each case is performed in exactly the same way, and the evaluations of the path integrals are similar in comparison to the simple oscillator case, the difference being that the entire structure of the (modified) Pöschl-Teller potential must be taken into account. In particular, this has the consequence that we have to consider wave functions with a definite parity.

Spherical Coordinates. First we consider the path integral representation in spherical coordinates and we obtain ($N = m + n \in \mathbb{N}$) is the principal quantum number, we have set $\lambda_1 = 2m \pm k_1 + k_2 + 1$, $\nu^2 = M^2\omega^2 R^4/2\hbar^2 + 1/4$; the range of the coordinates is given by $\tau > 0$, $\varphi \in (0, \pi/2)$

$$K^{(V_1)}(u'', u'; T) = \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{N_{\max}} e^{-iE_N T/\hbar} \Psi_{nm}^{(V_1)}(\tau'', \varphi''; R) \Psi_{nm}^{(V_1)}(\tau', \varphi'; R) + \right.$$

$$\left. + \int_0^{\infty} dp\, e^{-iE_p T/\hbar} \Psi_{pm}^{(V_1)}(\tau'', \varphi''; R) \Psi_{pm}^{(V_1)*}(\tau', \varphi'; R) \right\}, \tag{4.35}$$

and the corresponding discrete wave functions have the form

$$\Psi_{nm}^{(V_1)}(\tau, \varphi; R) = (\sinh\tau)^{-1/2} S_n^{(\lambda_1, \nu)}(\tau; R)\, \phi_m^{(\pm k_2, \pm k_1)}(\varphi), \tag{4.36}$$

$$\phi_m^{(\pm k_2, \pm k_1)}(\varphi) = \left[2(1 + 2m \pm k_1 \pm k_2 + 1) \frac{m!\Gamma(m \pm k_1 \pm k_2 + 1)}{\Gamma(1 + m \pm k_1)\Gamma(1 + m \pm k_2)} \right]^{1/2} \times$$

$$\times (\sin\varphi)^{1/2 \pm k_2} (\cos\varphi)^{1/2 \pm k_1} P_m^{(\pm k_2, \pm k_1)}(\cos 2\varphi) \tag{4.37}$$

$$S_n^{(\lambda_1, \nu)}(\tau; R) = \frac{1}{\Gamma(1 + \lambda_1)} \left[\frac{2(\nu - \lambda_1 - 2n - 1)\Gamma(n + 1 + \lambda_1)\Gamma(\nu - n)}{R^2\Gamma(\nu - \lambda_1 - n)\, n!} \right]^{1/2} \times$$

$$\times (\sinh\tau)^{\lambda_1 + 1/2} (\cosh\tau)^{n + 1/2 - \nu} {}_2F_1(-n, \nu - n; 1 + \lambda_1; \tanh^2\tau). \tag{4.38}$$

The discrete energy spectrum is given by

$$E_N = -\frac{\hbar^2}{2MR^2}\left[(2N \pm k_1 \pm k_2 - \nu + 2)^2 - \frac{1}{4} \right] + \frac{M}{2}\omega^2 R^2,$$

$$N_{\max} = \left[\frac{1}{2}(\nu - \lambda_1 - 1) \right]. \tag{4.39}$$

In the limit $R \to \infty$ $(\nu \to M\omega R^2/\hbar)$ we obtain

$$E_N \simeq \hbar\omega(2N \pm k_1 \pm k_2 + 2), \tag{4.40}$$

which is the correct behaviour for the corresponding two-dimensional potential maximally superintegrable in $\mathbb{R}^2$ [21]. The continuous wave functions and the corresponding energy spectrum are given by (the $\phi_m^{(\pm k_2, \pm k_1)}(\varphi)$ are the same wave functions as in (4.37))

$$\Psi_{pm}^{(V_1)}(\tau, \varphi; R) = (\sinh\tau)^{-1/2} S_p^{(\lambda_1, \nu)}(\tau; R)\, \phi_m^{(\pm k_2, \pm k_1)}(\varphi), \tag{4.41}$$

$$S_p^{(\lambda_1,\nu)}(\tau; R) = \frac{1}{\Gamma(1+\lambda_1)} \sqrt{\frac{p\sinh \pi p}{2\pi^2 R^2}}\, \Gamma\left(\frac{\nu-\lambda_1+1-ip}{2}\right)\Gamma\left(\frac{\lambda_1-\nu+1-ip}{2}\right)$$
$$\times (\tanh\tau)^{\lambda_1+1/2}(\cosh\tau)^{ip}\times$$
$$\times {}_2F_1\left(\frac{\nu+\lambda_1+1-ip}{2}, \frac{\lambda_1-\nu+1-ip}{2}; 1+\lambda_1; \tanh^2\tau\right), \tag{4.42}$$

$$E_p = \frac{\hbar^2}{2MR^2}\left(p^2+\frac{1}{4}\right)+\frac{M}{2}\omega^2R^2. \tag{4.43}$$

The corresponding Green function $G^{(V_1)}(E)$ of the potential V_1 in the general case has the form ($m_{1,2} = (\lambda_1 \pm \sqrt{-2ME'R^2}/\hbar)$, $L_\nu = \frac{1}{2}(\nu-1)$, $E' = E - \hbar^2/8MR^2 - MR^2\omega^2/2$)

$$G^{(V_1)}(\tau'', \tau', \varphi'', \varphi'; E)$$
$$= \frac{M}{2\hbar^2}\sum_{m\in\mathbb{N}_0} \phi_m^{(\pm k_2, \pm k_1)}(\varphi')\,\phi_m^{(\pm k_2, \pm k_1)}(\varphi'')\frac{\Gamma(m_1-L_\nu)\,\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\,\Gamma(m_1-m_2+1)}$$
$$\times(\cosh r'\cosh r'')^{-(m_1-m_2+1/2)}(\tanh r'\tanh r'')^{m_1+m_2}$$
$$\times {}_2F_1\left(-L_\nu+m_1, L_\nu+m_1+1; m_1-m_2+1; \frac{1}{\cosh^2_<}\right)$$
$$\times {}_2F_1(-L_\nu+m_1, L_\nu+m_1+1; m_1+m_2+1; \tanh^2 r_>)\,. \tag{4.44}$$

Equidistant Coordinates. Next we consider the equidistant coordinate system. Similarly as in the pure oscillator case we obtain a discrete spectrum with energy eigenvalues (4.39), and a set of two continuous wave functions each with energy spectrum (4.43), with principal quantum number $N=m+n$, i.e., we have for the propagator ($\lambda_1=2m\pm k_1\pm k_2+1$, $\nu^2=M^2\omega^2R^4/\hbar^2+1/4$, and $\tau_{1,2}>0$)

$$K^{(V_1)}(u'',u';T)=\sum_{m=0}^{N_{\max}^{(m)}}\left\{\sum_{n=0}^{N_{\max}^{(n)}} e^{-iE_NT/\hbar}\,\Psi_{nm}^{(V_1)}(\tau_1'',\tau_2'';R)\,\Psi_{nm}^{(V_1)}(\tau_1',\tau_2';R)\right.$$

$$\left.+\int_0^\infty dp\, e^{-iE_pT/\hbar}\,\Psi_{pm}^{(V_1)}(\tau_1'',\tau_2'';R)\,\Psi_{pm}^{(V_1)*}(\tau_1',\tau_2';R)\right\}$$

$$+\int_0^\infty dp\int_0^\infty dp\, e^{-iE_pT/\hbar}\,\Psi_{pk}^{(V_1)}(\tau_1'',\tau_2'';R)\,\Psi_{kp}^{(V_1)*}(\tau_1',\tau_2';R). \tag{4.45}$$

Here denote $N_{\max}^{(1)}=\left[\frac{1}{2}(\nu\mp k_1-1)\right]$, $N_{\max}^{(2)}=\left[\frac{1}{2}(\lambda_1\mp k_2-1)\right]$ the maximal number of bound states for the wave functions in τ_2 and τ_1, respectively. The discrete wave functions have the form

$$\Psi_{mn}^{(V_1)}(\tau_1,\tau_2;R)=(\cosh\tau_1)^{-1/2}S_n^{(\pm k_2,\lambda_1)}(\tau_1;R)\,\psi_m^{(\pm k_1,\nu)}(\tau_2), \tag{4.46}$$

$$S_n^{(\pm k_2,\lambda_1)}(\tau_1;R)=\frac{1}{\Gamma(1\pm k_2)}\left[\frac{2(\lambda_1\mp k_2-2n-1)\,\Gamma(n+1\pm k_2)\,\Gamma(\lambda_1-n)}{R^2\Gamma(\lambda_1\mp k_2-n)\,n!}\right]^{1/2}+$$

$$\times(\sinh\tau_1)^{1/2\pm k_2}(\cosh\tau_1)^{n+1/2-\lambda_1}\,{}_2F_1(-n,\lambda_1-n;1\pm k_2;\tanh^2\tau_1), \tag{4.47}$$

$$\psi_m^{(\pm k_1,\nu)}(\tau_2)=\frac{1}{\Gamma(1\pm k_1)}\left[\frac{2(\nu\mp k_1-2m-1)\,\Gamma(m+1\pm k_1)\,\Gamma(\nu-m)}{\Gamma(\nu\mp k_1-m)\,m!}\right]^{1/2}$$

$$\times(\sinh\tau_2)^{1/2\pm k_1}(\cosh\tau_2)^{m+1/2-\nu}\,{}_2F_1(-m,\nu-m;1\pm k_1;\tanh^2\tau_2). \tag{4.48}$$

The first set of continuous states is given by

$$\Psi_{pm}^{(V_1)}(\tau_1,\tau_2;R)=(\cosh\tau_1)^{-1/2}S_n^{(\pm k_2,\lambda_1)}(\tau_1;R)\,\psi_m^{(\pm k_1,\nu)}(\tau_2), \tag{4.49}$$

$$S_p^{(\pm k_2, \lambda_1)}(\tau_1; R) = \frac{1}{\Gamma(1 \pm k_2)} \sqrt{\frac{p \sin \pi p}{2\pi^2 R^2}} \times$$

$$\times \Gamma\left(\frac{\lambda \mp k_2 + 1 - ip}{2}\right) \Gamma\left(\frac{\pm k_2 - \lambda_1 + 1 - ip}{2}\right) \times$$

$$\times (\tanh \tau_1)^{1/2 \pm k_2} (\cosh \tau_1)^{ip} \times$$

$$ {}_2F_1\left(\frac{\lambda_1 \pm k_2 + 1 - ip}{2}, \frac{1 \pm k_2 - \lambda_1 - ip}{2}; 1 \pm k_2; \tanh^2 \tau_1\right), \tag{4.50}$$

with the $\psi_m^{(\pm k_1, \nu)}(\tau_2)$ as in (4.48). The second set is given by

$$\Psi_{kp}^{(V_1)}(\tau_1, \tau_2; R) = (\cosh \tau_1)^{-1/2} S_p^{(\pm k_2, ik)}(\tau_1; R)\, \psi_k^{(\pm k_1, \nu)}(\tau_2), \tag{4.51}$$

$$S_p^{(\pm k_2, ik)}(\tau_1; R) = \frac{1}{\Gamma(1 \pm k_2)} \sqrt{\frac{p \sinh \pi p}{2\pi^2 R^2}} \times$$

$$\times \Gamma\left(\frac{ik \mp k_2 + 1 - ip}{2}\right) \Gamma\left(\frac{\pm k_2 - ik + 1 - ip}{2}\right) \times$$

$$\times (\tanh \tau_1)^{1/2 \pm k_2} (\cosh \tau_1)^{ip} \times$$

$$ {}_2F_1\left(\frac{ik \pm k_2 + 1 - ip}{2}; \frac{1 \pm k_2 - ik - ip}{2}; 1 \pm k_2; \tanh^2 \tau_1\right), \tag{4.52}$$

$$\psi_k^{(\pm k_1, \nu)}(\tau_2) = \frac{1}{\Gamma(1 \pm k_1)} \sqrt{\frac{k \sinh \pi k}{2\pi^2}} \times$$

$$\times \Gamma\left(\frac{\nu \mp k_1 + 1 - ik}{2}\right) \Gamma\left(\frac{\pm k_1 - \nu + 1 - ik}{2}\right) \times$$

$$\times (\tanh \tau_2)^{1/2 \pm k_1} (\cosh \tau_2)^{ik} \times$$

$$ {}_2F_1\left(\frac{\nu \pm k_1 + 1 - ik}{2}, \frac{1 \pm k_1 - \nu - ik}{2}; 1 \pm k_1; \tanh^2 \tau_2\right). \tag{4.53}$$

Let us remark that the wave functions have been normalized in the domains $\varphi \in (0, \pi/2)$ and $\tau > 0$ in the spherical and in $\tau_{1,2} > 0$ in the equidistant system. The positive sign for k_i has to be taken whenever $k_i \geq \frac{1}{2}$ $(i = 1,2)$, i.e., the potential term is repulsive at the origin, and the motion takes place only in the denoted domains. If $0 < |k_i| < \frac{1}{2}$, i.e., the potential term is attractive at the origin, both the positive and the negative sign must be taken into account in the solution. This is indicated by the notion $\pm k_i$ in the formulae. It has also the consequence that for each k_i the motion can take place in the entire domains of the variables on $\Lambda^{(2)}$. In the present case this means that we must (e.g., in the equidistant system) distinguish four cases: i) $\tau_1, \tau_2 > 0$, ii) $\tau_1 > 0$, $\tau_2 \in \mathbb{R}$, iii) $\tau_1 \in \mathbb{R}$, $\tau_2 > 0$, and iv) $(\tau_1, \tau_2) \in \mathbb{R}^2$. In polar coordinates the same feature is recovered by the observation that the Pöschl-Teller barriers are absent for $|k_i| < \frac{1}{2}$.

In elliptic coordinates this feature is taken into account in the following way: Due to $\alpha \in (iK', iK' + K)$, we have $\operatorname{sn}(\alpha, k)$, $\operatorname{icn}(\alpha, k) > k'/k$, $\operatorname{idn}(\alpha, k) \geq 0$, and we see that for $\alpha \in (iK', iK' + K)$, $\beta \in (K', 4K')$, and $u_0 \geq 0$, the variables u_1, u_2 change signs in four respective domains, i.e., $\beta \in (0, K')$, $\beta \in (K', 2K')$, $\beta \in (2K', 3K')$, and $\beta \in (3K', 4K')$. We then have for $\alpha \neq 0$

$$\operatorname{sn}(0, k') = \operatorname{sn}(2K', k') = \operatorname{sn}(4K', k') = 0,$$

$$\operatorname{cn}(K', k') = \operatorname{cn}(3K' k') = 0, \tag{4.54}$$

and $\operatorname{dn}(\beta, k') > 0$, $\beta \in [0, 4K')$. For convenience, we have made the choice $\beta \in (0, K')$ in the following. The situation is similar in the hyperbolic system, where we can choose $\mu \in (iK', iK' + K)$, $\eta \in (0, K')$.

This has the following consequences for the degeneracies of the Higgs oscillator on the pseudosphere. If $0 < k_{1,2} \leq \frac{1}{2}$ we have for each $N = n + m$ four possibilities of parities of the levels, i.e. $(\pm, \pm)$; for the cases $0 < k_1 \leq \frac{1}{2}$ and $k_2 > \frac{1}{2}$ or $0 < k_2 \leq \frac{1}{2}$ and $k_1 > \frac{1}{2}$ we have for each N two possibilities $(\pm)$: for $k_{1,2} > \frac{1}{2}$ there is only one possibility: $(+)$. In all cases the degeneracy is

$d = N + 1 = 2j + 1 \left(j = 0, \frac{1}{2}, 1, \ldots \right)$, coinciding with the dimensions of all relevant discrete irreducible representations of the group SU(1,1). In effect, the negative signs lower the potential energies, and the respective spectrum as well. This is exactly the same behaviour as in the two-dimensional singular oscillator in the flat-space case [10,21,22], and we will keep this notion in the sequel for all following superintegrable potentials.

The Green function of the potential V_1 in equidistant coordinates can be constructed by inserting the corresponding one-dimensional Green function in the variable τ_1 into (4.45). We obtain ($E' = E - \hbar^2/8MR^2 - MR^2\omega^2/2$)

$$G^{(V_1)}(\tau_1'', \tau_1', \tau_2'', \tau_2'; E) = (\cosh\tau_1' \cosh\tau_1'')^{-1/2} \times$$

$$\times \left\{ \sum_{m=0}^{N_{\max}^{(m)}} \psi_m^{(\pm k_1, \nu)}(\tau_2'')\, \psi_m^{(\pm k_1, \nu)}(\tau_2')\, G_{mPT}^{(\pm k_2, \lambda)}(\tau_1'', \tau_1'; E') + \right.$$

$$\left. + \int_0^\infty dk\, \psi_k^{(\pm k_1, \nu)}(\tau_2'')\, \psi_k^{(\pm k_1, \nu)*}(\tau_2')\, G_{mPT}^{(\pm k_2, ik)}(\tau_1'', \tau_1'; E') \right\}, \qquad (4.55)$$

in the notation of (2.12, 2.16).

4.2. The Coulomb Potential

We consider the generalized Coulomb potential on the two-dimensional pseudosphere in the four separating coordinate systems

$$V_2(u) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2 + u_2^2}} - 1 \right) +$$

$$+ \frac{\hbar^2}{4M\sqrt{u_1^2 + u_2^2}} \left(\frac{k_1^2 - \frac{1}{4}}{\sqrt{u_1^2 + u_2^2} + u_1} + \frac{k_2^2 - \frac{1}{4}}{\sqrt{u_1^2 + u_2^2} - u_1} \right) \qquad (4.56)$$

Spherical ($\tau > 0$, $\varphi \in (0, \pi)$):

$$V_2(u) = -\frac{\alpha}{R}(\coth\tau - 1) + \frac{\hbar^2}{8MR^2\sinh^2\tau}\left(\frac{k_1^2 - \frac{1}{4}}{\sin^2\frac{\varphi}{2}} + \frac{k_2^2 - \frac{1}{4}}{\cos^2\frac{\varphi}{2}} \right) \qquad (4.57)$$

Elliptic-Parabolic ($a > 0,\ \vartheta \in (0, \pi/2)$):

$$= -\frac{\alpha}{R}\left(\frac{\cosh^2 a + \cos^2\vartheta}{\cosh^2 a - \cos^2\vartheta} - 1\right) + \\ + \frac{\hbar^2}{2MR^2}\frac{\cosh^2 a \cosh^2\vartheta}{\cosh^2 a - \cos^2\vartheta}\left(\frac{k_1^2 - \frac{1}{4}}{\sin^2\vartheta} + \frac{k_2^2 - \frac{1}{4}}{\sinh^2 a}\right) \tag{4.58}$$

Elliptic II (*algebraic form*, $0 < \rho_2 < a_1 < \rho_1$):

$$= -\frac{\alpha}{R}\left(\frac{\sqrt{(\rho_1 - a_2)(\rho_1 - a_3)} - \sqrt{(\rho_2 - a_2)(\rho_2 - a_3)}}{\rho_1 - \rho_2} - 1\right) - \\ - \frac{\hbar^2}{4M}\left[\left(k_1^2 + k_2^2 - \frac{1}{2}\right)\frac{(a_1 - a_2)(a_1 - a_3)}{\rho_1 - \rho_2}\left(\frac{1}{a_1 - \rho_2} + \frac{1}{\rho_1 - a_1}\right) - \right. \\ \left. - (k_1^2 - k_2^2)\frac{\sqrt{(a_1 - a_2)(a_1 - a_3)}}{a_2 - a_3}\cdot\frac{\sqrt{(\rho_2 - a_2)(\rho_2 - a_3)} + \sqrt{(\rho_1 - a_2)(\rho_1 - a_3)}}{\rho_1 - \rho_2}\right] \tag{4.59}$$

Elliptic II (*Jacobi elliptic function form*, $\alpha \in (\mathrm{i}K', \mathrm{i}K' + K)$, $\beta \in (0, K')$):

$$= -\frac{\alpha}{R}\left(\frac{k^2 \operatorname{sn}\alpha \operatorname{cn}\beta - k' \operatorname{cn}\beta \operatorname{dn}\beta}{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta} - 1\right) + \\ + \frac{h^2}{4M}\left[\frac{k_1^2 + k_2^2 - \frac{1}{2}}{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta}\left(\frac{k'^2}{\operatorname{dn}^2\alpha} - \frac{1}{\operatorname{sn}^2\beta}\right) + \right. \tag{4.60}$$

$$\left. + (k_1^2 - k_2^2)\frac{k'}{k}\frac{k^2 \operatorname{sn}\alpha \operatorname{cn}\alpha + k' \operatorname{cn}\beta \operatorname{dn}\beta}{k^2 \operatorname{cn}^2\alpha + k'^2 \operatorname{cn}^2\beta}\right].$$

Semi-Hyperbolic ($\mu_1, \mu_2 > 0$):

$$= -\frac{\alpha}{R}\left(\frac{\sqrt{1 + \mu_1^2} + \sqrt{1 + \mu_2^2}}{\mu_1 + \mu_2} - 1\right) + \frac{\hbar^2}{4MR^2}\frac{1}{\mu_1 + \mu_2}\left[\left(k_1^2 + k_2^2 - \frac{1}{2}\right)\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) + \right. \\ \left. + (k_1^2 - k_2^2)\left(\frac{\sqrt{1 + \mu_1^2}}{\mu_1} - \frac{\sqrt{1 + \mu_2^2}}{\mu_2}\right)\right]. \tag{4.61}$$

For the constants of motion for the potential V_2 we get

$$I_1 = \frac{1}{2MR^2}(K_1^2 + K_2^2 - L_3^2) + V_2(u),$$

$$I_2 = \frac{1}{2M}L_3^2 + \frac{\hbar^2}{8M}\left(\frac{k_1^2 - \frac{1}{4}}{\sin^2\frac{\varphi}{2}} + \frac{k_2^2 - \frac{1}{4}}{\cos^2\frac{\varphi}{2}}\right),$$

$$I_3 = \frac{1}{2M}\{K_1, L_3\} - \alpha R\,\frac{\mu_2\sqrt{1+m_1^2} - \mu_1\sqrt{1+\mu_2^2}}{\mu_1+\mu_2} +$$

$$+\frac{\hbar^2}{4M}\left[\left(k_1^2 + k_2^2 - \frac{1}{2}\right)\left(\frac{\mu_1/\mu_2 + \mu_2/\mu_1}{\mu_1+\mu_2}\right) +\right.$$

$$\left. + (k_1^2 - k_2^2)\,\frac{\mu_2^2\sqrt{1+\mu_1^2} + \mu_1^2\sqrt{1+\mu_2^2}}{\mu_1\mu_2(\mu_1+\mu_2)}\right]. \qquad (4.62)$$

The path integral formulations have the following form

$$K^{(V_2)}(u'', u'; T).$$

Spherical:

$$= \frac{e^{-i\hbar T/8MR^2}}{R^2}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh\tau\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau^2 + \sinh^2\tau\dot\varphi^2) +\right.\right.$$

$$\left.\left. + \frac{\alpha}{R}(\coth\tau - 1) - \frac{\hbar^2}{8MR^2\sinh^2\tau}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\frac{\varphi}{2}} - \frac{1}{4}\right)\right]dt\right\} \qquad (4.63)$$

Elliptic-Parabolic:

$$= \frac{1}{R^2}\int\limits_{a(t')=a'}^{a(t'')=a''}\mathcal{D}a(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\,\frac{\cosh^2 a - \cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}\times$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}(\dot a^2+\dot\vartheta^2)+\frac{\alpha}{R}\left(\frac{\cosh^2 a+\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}-1\right)-\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2MR^2}\frac{\cosh^2 a\cos^2\vartheta}{\cosh^2 a-\cos^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2\vartheta}+\frac{k_2^2-\frac{1}{4}}{\sinh^2 a}\right)\right]dt\right\} \tag{4.64}$$

Elliptic II:

$$=\frac{1}{R^2}\int\limits_{\alpha(t')=\alpha'}^{\alpha(t'')=\alpha''}\mathcal{D}\alpha(t)\int\limits_{\beta(t')=\beta'}^{\beta(t'')=\beta''}\mathcal{D}\beta(t)\,(k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta)\times$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta)(\dot\alpha^2+\dot\beta^2)+\right.\right.$$

$$+\frac{\alpha}{R}\left(\frac{k^2\,\mathrm{sn}\,\alpha\,\mathrm{cn}\,\beta-k'\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta}{k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta}-1\right)-$$

$$-\frac{\hbar^2}{4MR^2}\left(\frac{k_1^2+k_2^2-\frac{1}{2}}{k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\mathrm{cn}^2\,\beta}\left(\frac{k'^2}{\mathrm{dn}^2\,\alpha}-\frac{1}{\mathrm{sn}^2\,\beta}\right)+\right.$$

$$\left.\left.\left.+(k_1^2-k_2^2)\frac{k'}{k}\frac{k^2\,\mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha+k'\,\mathrm{cn}\,\beta\,\mathrm{dn}\,\beta}{k^2\,\mathrm{cn}^2\,\alpha+k'^2\,\beta}\right)\right]dt\right\} \tag{4.65}$$

Semi-Hyperbolic:

$$=\frac{1}{R^2}\int\limits_{\mu_1(t')=\mu_1'}^{\mu_1(t'')=\mu_1''}\mathcal{D}\mu_1(t)\int\limits_{\mu_1(t')=\mu_1'}^{\mu_1(t'')=\mu_1''}\mathcal{D}\mu_1(t)\frac{\mu_1+\mu_2}{4\sqrt{P(\mu_1)\,P(\mu_2)}}\times$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\mu_1+\mu_2}{4}\left(\frac{\dot\mu_1^2}{P(\mu_1)}-\frac{\dot\mu_2^2}{P(\mu_2)}\right)+\frac{\alpha}{R}\left(\frac{\sqrt{1+\mu_1^2}+\sqrt{1+\mu_2^2}}{\mu_1+\mu_2}-1\right)-\right.\right.$$

$$-\frac{\hbar^2}{4MR^2}\frac{1}{\mu_1+\mu_2}\left(\left(k_1^2+k_2^2-\frac{1}{2}\right)\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)+(k_1^2-k_2^2)\left(\frac{\sqrt{1+\mu_1^2}}{\mu_1}-\frac{\sqrt{1+\mu_2^2}}{\mu_2}\right)\right)-$$

$$\left.\left.-\frac{\hbar^2}{2MR^2}\frac{1}{\mu_1+\mu_2}\left(P''(\mu_1)-P''(\mu_2)-\frac{3P'^2(\mu_1)}{4P(\mu_1)}+\frac{3P'^2(\mu_2)}{4P(\mu_2)}\right)\right]dt\right\}. \tag{4.66}$$

4.2.1. Spherical Coordinates. In order to solve the Coulomb problem in spherical coordinates we start by separating off the φ-path integration which yields $\left(\lambda_1 = m + \frac{1}{2}(1 \pm k_1 \pm k_2)\right)$:

$$K^{(V_2)}(\tau'', \tau', \varphi'', \varphi'; T) = \frac{e^{-i\hbar T/8MR^2}}{R^2(\sinh\tau' \sinh\tau'')^{1/2}} \times$$

$$\times \sum_{m=0}^{\infty} \phi_m^{(\pm k_2, \pm k_1)}\left(\frac{\varphi'}{2}\right) \phi_m^{(\pm k_1, \pm k_1)}\left(\frac{\varphi''}{2}\right) \times \int_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\dot{\tau}^2 + \frac{\alpha}{R}(\coth\tau - 1) - \frac{\hbar^2}{2MR^2}\frac{\lambda_1^2 - \frac{1}{4}}{\sinh^2\tau}\right]dt\right\}. \tag{4.67}$$

Here denote the $\phi_m^{(\pm k_2, \pm k_1)}$ the Pöschl-Teller wave functions (2.6)

$$\phi_m^{(\pm k_2, \pm k_1)}\left(\frac{\varphi}{2}\right) = \left[(1 + 2m \pm k_1 \pm k_2 + 1)\frac{m!\Gamma(m \pm k_1 \pm k_2 + 1)}{\Gamma(1 \pm m \pm k_1)\Gamma(1 + m \pm k_2)}\right]^{1/2}$$

$$\times \left(\sin\frac{\varphi}{2}\right)^{1/2 \pm k_2} \left(\cos\frac{\varphi}{2}\right)^{1/2 \pm k_1} P_m^{(\pm k_2, \pm k_1)}(\cos\varphi). \tag{4.68}$$

The remaining τ-path integration, denoted by $K_m^{(V_2)}(T)$ in the following, is of the form of the Manning-Rosen potential, which in turn can be transformed into the path integral problem of the modified Pöschl-Teller problem. This has been done in [1,16], and will not be repeated here. The corresponding non-lilear transformation has the form

$$\frac{1}{2}(1 - \coth\tau) = -\frac{1}{\sinh^2\tau}, \tag{4.69}$$

accompanied by the time-transformation $dt = ds$, with $f(r) = R^2 \tanh^2 r$. In some sense this transformation can be seen as a one-dimensional realization of the Kustaanheimo-Stiefel transformation [5,42] corresponding to a space of constant negative curvature because it maps the path integral (4.67) via a space-time transformation into the path integral of the modified Pöschl-Teller potential which can be transformed by a simple rearrangement into the path integral of the radial Higgs oscillator. The result has the form ($N = 0, 1, 2, \ldots,$ $N_{\max} = \left[\sqrt{R/a} - \lambda_1 - \frac{1}{2}\right]$, $a = \hbar^2/M\alpha$)

$$K_m^{(V_2)}(\tau'', \tau'; T) = \sum_{n=0}^{N_{\max}} e^{-iE_N T/\hbar} S_n^{(V_2)}(\tau') S_n^{(V_2)}(\tau'') +$$

$$+ \int_0^\infty dp\, e^{-i\hbar p^2 T/2M} S_p^{(V_2)*}(\tau') S_p^{(V_2)}(\tau''), \tag{4.70}$$

with the discrete and continuous energy spectrum, respectively, given by $\left(\tilde{N} = N + \lambda_1 + \frac{1}{2}\right)$

$$E_N = \frac{\alpha}{R} - \hbar^2 \frac{\tilde{N}^2 - \frac{1}{4}}{2MR^2} - \frac{M\alpha^2}{2\hbar^2 \tilde{N}^2}, \tag{4.71}$$

$$E_p = \frac{\hbar^2}{2MR^2}\left(p^2 + \frac{1}{4}\right). \tag{4.72}$$

The bound state wave functions have the form $(\sigma_N = a/R\tilde{N})$

$$S_n^{(V_2)}(\tau; R) = \frac{2^{\lambda_1 + 1/2}}{\Gamma(2\lambda_1 + 1)} \left[\frac{\sigma_N^2 - \tilde{N}^2}{R^2 \tilde{N}^2} \cdot \frac{\Gamma\left(\tilde{N} + \lambda_1 + \frac{1}{2}\right)\Gamma\left(\sigma_N + \lambda_1 + \frac{1}{2}\right)}{\Gamma(\tilde{N} - \lambda_1)\,\Gamma(\sigma_N - \lambda_1)} \right]^{1/2}$$

$$\times (\sinh\tau)^{\lambda_1 + 1/2} e^{i\tau(\sigma_N - \tilde{N})} {}_2F_1\left(-n, \lambda_1 + \frac{1}{2} + \sigma_N; 2\lambda_1 + 1; \frac{2}{1 + \coth\tau}\right). \tag{4.73}$$

The continuous wave functions are $(\tilde{p} = \sqrt{2MR^2(E_p - \alpha/R)}/\hbar)$

$$S_p^{(V_2)}(\tau; R) = \frac{2^{(i/2)(p - \tilde{p}) + \lambda_1 + 1/2}}{\pi\Gamma(2\lambda_1 + 1)} \sqrt{\frac{p \sinh \pi p}{2R^2}}$$

$$\times \Gamma\left(\lambda_1 + \frac{1}{2} + \frac{i}{2}(\tilde{p} - p)\right)\Gamma\left(\lambda_1 + \frac{1}{2} - \frac{i}{2}(\tilde{p} + p)\right)$$

$$\times (\sinh\tau)^{\lambda_1 + 1/2} \exp\left[\tau\left(\frac{i}{2}(\tilde{p} + p) - \lambda_1 - \frac{1}{2}\right)\right]$$

$$\times {}_2F_1\left(\lambda_1 + \frac{1}{2} + \frac{i}{2}(\tilde{p} - p), \lambda_1 + \frac{1}{2} - \frac{i}{2}(\tilde{p} + p); 2\lambda_1 + 1; \frac{2}{1 + \coth\tau}\right). \tag{4.74}$$

The complete wave functions of the generalized Coulomb problem on the two-dimensional pseudosphere in spherical coordinates are thus given by

$$\Psi_{nm}^{(V_2)}(\tau,\varphi;R)=(\sinh\tau)^{-1/2}S_n^{(V_2)}(\tau;R)\,\phi_m^{(\pm k_1,\pm k_2)}\left(\frac{\varphi}{2}\right), \qquad (4.75)$$

$$\Psi_{pm}^{(V_2)}(\tau,\varphi;R)=(\sinh\tau)^{-1/2}S_p^{(V_2)}(\tau;R)\,\phi_m^{(\pm k_1,\pm k_2)}\left(\frac{\varphi}{2}\right). \qquad (4.76)$$

The Green function of the Coulomb problem has the form

$$G^{(V_2)}(\tau'',\tau',\varphi'',\varphi';E)=(\sinh\tau'\sinh\tau'')^{-1/2}\times$$

$$\sum_{m=0}^{\infty}\phi_m^{(\pm k_2,\pm k_1)}\left(\frac{\varphi'}{2}\right)\phi_m^{(\pm k_2,\pm k_1)}\left(\frac{\varphi''}{2}\right)\times\frac{M}{\hbar^2}\,\frac{\Gamma(m_1-L_E)\,\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\,\Gamma(m_1-m_2+1)}\times$$

$$\times\left(\frac{2}{\coth\tau'+1}\cdot\frac{2}{\coth\tau''+1}\right)^{(m_1+m_2+1)/2}\left(\frac{\coth\tau'-1}{\coth\tau'+1}\cdot\frac{\coth\tau''-1}{\coth\tau''+1}\right)^{(m_1-m_2)/2}\times$$

$$\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{\cosh\tau_>-1}{\coth\tau_>+1}\right)\times$$

$$\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{2}{\coth\tau_<+1}\right), \qquad (4.77)$$

where $L_E=\frac{1}{2}(\sqrt{-2MR^2E/\hbar^2+1/4}/\hbar-1)$, and $m_{1,2}=\lambda_1\pm$ $\pm\sqrt{-2mR^2(2\alpha/R+E)-1/4}/h$. This representation can be derived by means of the Green function of the modified Pöschl-Teller potential and the Manning-Rosen potential, c.f. [20] for some details and references therein.

Let us make some remarks concerning the pure Coulomb case. The calculation is almost the same with only minor differences: The wave functions $\phi_m^{(\pm k_2,\pm k_1)}\left(\frac{\varphi}{2}\right)$ are replaced by circular waves, i.e., $e^{ij\varphi}/\sqrt{2\pi}$ with $\varphi=[0,2\pi)$. This then has the consequence that the modified angular momentum number has the form $\lambda_1=|j|$. Everything else remains the same.

4.2.2. Elliptic-Parabolic Coordinates. In order to deal with the path integral (4.64) we perform a time substitution $dt=ds(\cosh^2 a-$ $-\cos^2\vartheta)/\cosh^2 a\cos^2\vartheta\, ds$ according to, e.g., [20,27,42] and references therein, such that the new pseudo-time s'' can be introduced via the constraint $\int_0^{s''} ds\,(\cosh^2 a - \cos^2\vartheta)/\cosh^2 a \cos^2\vartheta = T = t'' - t'$. We therefore obtain

$$K^{(V_2)}(a'', a', \vartheta'', \vartheta'; T) =$$

$$= \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} e^{-iET/\hbar} \int_0^\infty ds'' \int_{a(0)=a'}^{a(s'')=a''} \mathcal{D}a(s) \int_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''} \mathcal{D}\vartheta(s) \times$$

$$\times \exp\left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{M}{2}(\dot{a}^2 + \dot{\vartheta}^2) - \right.\right.$$

$$\left.\left. - \frac{h^2}{2M}\left(\frac{k_1^2 - \frac{1}{4}}{\sin^2\vartheta} + \frac{\beta^2 - \frac{1}{4}}{\cos^2\vartheta} + \frac{k_2^2 - \frac{1}{4}}{\sinh^2 a} - \frac{\nu^2 - \frac{1}{4}}{\cosh^2 a} \right) \right] ds \right\}, \tag{4.78}$$

where $\beta^2 = \frac{1}{4} - 2MER^2/\hbar^2$, $\nu^2 = \frac{1}{4} + 2MR^2(2\alpha/R - E)/\hbar^2$. The analysis of this path integral is rather involved and we first consider the pure Coulomb case, denoted by $K^{(\alpha)}(T)$.

Pure Coulomb Case. We observe that in the pure Coulomb case the path integral (4.78) yields a symmetric Pöschl-Teller potential path integral in $\vartheta \in (-\pi/2, \pi/2)$, and a symmetric Rosen-Morse potential path integral in $a \in \mathbb{R}$. The solution consists of two contributions corresponding to the discrete and continuous spectrum, i.e.,

$$K^{(\alpha)}(a'', a', \vartheta'', \vartheta'; T) = K^{(\alpha)}_{\text{disc}}(a'', a', \vartheta'', \vartheta'; T) + K^{(\alpha)}_{\text{cont}}(a'', a', \vartheta'', \vartheta'; T),$$

$$= \sum_{m_1 m_2} e^{-iE_{m_1 m_2} T/\hbar}\, \Psi^{(\alpha)}_{m_1 m_2}(a', \vartheta'; R)\, \Psi^{(\alpha)}_{m_1 m_2}(a', \vartheta''; R) +$$

$$+ \int_0^\infty dk \int_0^\infty dp\, e^{-iE_p T/\hbar}\, \Psi^{(\alpha)*}_{kp}(a', \vartheta'; R)\, \Psi^{(\alpha)}_{kp}(a'', \vartheta''; R). \tag{4.79}$$

In order to obtain the discrete spectrum contribution to (4.78) we insert the spectral expansions of the discrete spectrum of the symmetric Pöschl-Teller and the symmetric Rosen-Morse potential. This yields

$$K^{(\alpha)}_{\text{disc}}(a'', a', \vartheta'', \vartheta'; T) = \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} e^{-iET/\hbar} \sum_{m_1 m_2} \int_0^\infty ds'' \times$$

$$\times \exp\left\{ -\frac{i}{\hbar}\frac{\hbar^2}{2M}\left[\left(m_1 + \beta + \frac{1}{2}\right)^2 - \left(m_2 - \nu + \frac{1}{2}\right)^2 \right] s'' \right\} \times$$

$$\times \sqrt{\cos\vartheta' \cos\vartheta''} \left(m_1 + \beta + \frac{1}{2}\right) \frac{\Gamma(m_1 + 2\beta + 1)}{m_1!} P^{-\beta}_{\beta + m_1}(\sin\vartheta')\, P^{-\beta *}_{\beta + m_1}(\sin\vartheta'') \times$$

$$\times \left(m_2 - \nu - \frac{1}{2}\right) \frac{\Gamma(2\nu - m_2)}{m_2!} P^{m_2 - \nu + 1/2}_{\nu - 1/2}(\tanh a')\, P^{m_2 - \nu + 1/2}_{\nu - 1/2}(\tanh a''). \quad (4.80)$$

Performing the s''-integration gives the quantization condition for the bound states:

$$\left(m_2 - \nu + \frac{1}{2}\right)^2 = \left(m_1 + \beta + \frac{1}{2}\right)^2, \quad (4.81)$$

and therefore the bound state energy levels have the following form ($N = (m_1 + m_2)/2$ is the principal quantum number)

$$E_N = \frac{\alpha}{R} - \hbar^2 \frac{\left(N + \frac{1}{2}\right)^2 - \frac{1}{4}}{2MR^2} - \frac{M\alpha^2}{2\hbar^2\left(N + \frac{1}{2}\right)^2}. \quad (4.82)$$

Considering the residuum in (4.80) we obtain the bound state wave functions

$$\Psi^{(\alpha)}_{m_1 m_2}(a, \vartheta; R) =$$

$$= \left[\frac{1}{2R^2}\left(\frac{M\alpha R}{\hbar^2 N^2} - 1\right)\left(m_1 - \beta - \frac{1}{2}\right) \frac{\Gamma(2\beta - m_1)}{m_1!} \right]^{1/2} P^{m_1 - \beta + 1/2}_{\beta - 1/2}(\tanh a) \times$$

$$\times \left[\left(m_2 + \nu + \frac{1}{2}\right) \frac{\Gamma(2\nu + m_2 + 1)}{m_2!} \right]^{1/2} P^{-\nu}_{m_2 + \nu}(\sin\vartheta). \quad (4.83)$$

The analysis of the continuous spectrum is somewhat more involved. We proceed in a similar way as in [20], where the same calculation was done for the free motion in elliptic-parabolic coordinates on $\Lambda^{(2)}$. We obtain by using the

Green function representations of the symmetric Pöschl-Teller and the symmetric Rosen-Morse potential [43]

$$\int_{a(t')=a'}^{a(t'')=a''} \mathcal{D}a(t) \int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \frac{\cosh^2 a - \cos^2 \vartheta}{\cosh^2 a \cos^2 \vartheta} \times$$

$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[\frac{M}{2} R^2 \frac{\cosh^2 a - \cos^2 \vartheta}{\cosh^2 a \cos^2 \vartheta} (\dot{a}^2 + \dot{\vartheta}^2) + \frac{\alpha}{R} \left(\frac{\cosh^2 a + \cos^2 \vartheta}{\cosh^2 a - \cos^2 \vartheta} - 1 \right) \right] dt \right\} \times$$

$$= \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} \mathrm{e}^{-\mathrm{i}ET/\hbar} \int_0^\infty ds'' \int_{a(0)=a'}^{a(s'')=a''} \mathcal{D}a(s) \int_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''} \mathcal{D}\vartheta(s) \times$$

$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^{s''} \left[\frac{M}{2} (\dot{a}^2 + \dot{\vartheta}^2) - \frac{\hbar^2}{2M} \frac{\beta^2 - \frac{1}{4}}{\cos^2 \vartheta} - \frac{\nu^2 - \frac{1}{4}}{\cosh^2 a} \right] ds \right\} =$$

$$= \frac{1}{2} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} \mathrm{e}^{-\mathrm{i}ET/\hbar} \int_0^\infty ds'' \int_{\mathbb{R}} \frac{dE'}{2\pi \mathrm{i}} \mathrm{e}^{-\mathrm{i}E's''/\hbar} \times$$

$$\times \frac{M}{\hbar^2} \sqrt{\cos \vartheta' \cos \vartheta''}\, \Gamma(\beta - M_{E'}) \Gamma(M_{E'} - \beta + 1) P_{M_{E'}}^{-\beta}(-\sin \vartheta_<) P_{M_{E'}}^{-\beta}(\sin \vartheta_>) \times$$

$$\times \sum_{\varepsilon=\pm 1} \int_0^\infty \frac{dk\, k \sinh \pi k}{\cos^2 \pi\nu + \sinh^2 \pi k} P_{\nu-1/2}^{\mathrm{i}k}(\varepsilon \tanh a'') P_{\nu-1/2}^{-\mathrm{i}k}(\varepsilon \tanh a') \mathrm{e}^{-\mathrm{i}\hbar k^2 s''/2M},$$

$$+ (a \leftrightarrow \vartheta). \tag{4.84}$$

with $M_{E'} = -\frac{1}{2} + \sqrt{2ME'}/\hbar$, and we have written the kernel $K_{\mathrm{cont}}^{(\alpha)}(s'')$ according to

$$K_{\mathrm{cont}}^{(\alpha)}(a'', a', \vartheta'', \vartheta'; s'') = K_a(a'', a'; s'') \cdot K_\vartheta(\vartheta'', \vartheta'; s'')$$

$$= \frac{1}{2} K_a(a'', a'; s'') \cdot \int_{\mathbb{R}} \frac{dE'}{2\pi \mathrm{i}} \mathrm{e}^{-\mathrm{i}E's''/\hbar} G_\vartheta(\vartheta'', \vartheta'; E')$$

$$+ \frac{1}{2} K_\vartheta(\vartheta'', \vartheta'; s'') \cdot \int_{\mathbb{R}} \frac{dE'}{2\pi \mathrm{i}} \mathrm{e}^{-\mathrm{i}E's''/\hbar} G_a(a'', a'; E'), \tag{4.85}$$

and, of course, both contributions must be taken into account which turn out to be equivalent. Note that (4.85) actually corresponds up to the additional

dE-integration to the continuous part of the Green function $G^{(\alpha)}(E)$, whereas (4.80) corresponds to its discrete contribution. The Green function expression (4.85) is evaluated by means of the relation for the Legendre functions [46, p.170]

$$P_\nu^{-\mu}(-y) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[P_\nu^{\mu}(-y)\cos\pi\mu - \frac{2}{\pi} Q_\nu^{\mu}(-y)\sin\pi\mu \right]$$

$$= \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \frac{\sin\pi\mu\, P_\nu^{\mu}(y) + \sin\pi\nu\, P_\nu^{\mu}(-y)}{\sin\pi(\nu+\mu)}. \tag{4.86}$$

Thus we obtain for the ϑ-dependent part along the cut $\beta = -ip$, where $E = \hbar^2(p^2+1/4)/2MR^2$

$$\psi_{kp}(\vartheta'')\,\psi_{kp}^*(\vartheta') \propto \frac{1}{i\pi}\left[\Gamma\left(\frac{1}{2}+ik+ip\right)\Gamma\left(\frac{1}{2}-ik-ip\right)\times \right.$$

$$\times P_{ik-1/2}^{ip}(-\sin\vartheta'')\,P_{ik-1/2}^{ip}(\sin\vartheta') -$$

$$\left. -\Gamma\left(\frac{1}{2}+ik-ip\right)\Gamma\left(\frac{1}{2}-ik+ip\right)P_{ik-1/2}^{-ip}(-\sin\vartheta'')\,P_{ik-1/2}^{-ip}(\sin\vartheta')\right] =$$

$$= \frac{p\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\sum_{\varepsilon=\pm k} P_{ik-1/2}^{ip}(\varepsilon\sin\vartheta'')\,P_{ik-1/2}^{-ip}(\varepsilon\sin\vartheta'). \tag{4.87}$$

We must insert the representation (4.85) into (4.84), and we find that the $ds''dE'$-integration gives $E' = -\hbar^2k^2/2M$. Hence we obtain the following wave functions and energy spectrum of the continuous spectrum ($\tilde{p}^2 = -\nu^2$, $p^2 = -\beta^2\varepsilon$, $\varepsilon' = \pm 1$)

$$\Psi_{k,p}^{(\alpha)}(a,\vartheta;R) = \frac{1}{R}\sqrt{\frac{p\sinh\pi pk\sinh\pi k}{(\cosh^2\pi k+\sinh^2\pi p)(\cosh^2\pi k+\sinh^2\pi\tilde{p})}}\times$$

$$\times\sqrt{\cos\vartheta}\,P_{ik-1/2}^{ip}(\varepsilon\sin\vartheta)\,P_{i\tilde{p}-1/2}^{ik}(\varepsilon'\tanh a), \tag{4.88}$$

$$E_p = \frac{\hbar^2}{2MR^2}\left(p^2+\frac{1}{4}\right). \tag{4.89}$$

Generalized Coulomb Case. To analyze the general case we proceed in an analogous way. For the discrete spectrum we expand the ϑ-path integration into Pöschl-Teller potential wave functions $\Psi_{n_1}^{(\pm k_1,\beta)}(\vartheta)$, and the a-path integration into the bound state contribution of the modified Pöschl-Teller potential wave functions of $\psi_{n_2}^{(\pm k_2,\nu)}(a)$ (2.12). The emerging Green function representation $G_{\rm disc}^{(V_2)}(E)$ of $K_{\rm disc}^{(V_2)}(T)$ has poles which are determined by the equation

$$(2n_1 \pm k_1 + \beta + 1)^2 = (2n_2 \pm k_2 - \nu + 1)^2. \tag{4.90}$$

Solving this equation for $E_{n_1 n_2}$ yields exactly the energy spectrum (4.71), with the principal quantum number $N = n_1 + n_2 + 1 + \frac{1}{2}(\pm k_1, \pm k_2)$. Taking the residuum gives the bound state wave functions.

For the analysis of the continuous spectrum we proceed again in an analogous way as for the pure Coulomb case, the only difference being that we must insert now the entire Green functions of the Pöschl-Teller (2.6) and modified Pöschl-Teller problems (2.12), instead of the corresponding symmetric cases. For this purpose one constructs the Green function $G^{(V_2)}(E)$ in elliptic-parabolic coordinates by considering the ds''-integration following from (4.78) with the solutions of the Pöschl-Teller and modified Pöschl-Teller potential, respectively. It can be cast into the following form (c.f. also [20] for some more details concerning the proper Green function analysis)

$$G^{(V_2)}(a'', a', \vartheta'', \vartheta'; E) = \frac{1}{2}\sum_{n_2} \psi_{n_2}^{(\pm k_2,\nu)}(a'')\, \psi_{n_2}^{(\pm k_2,\nu)}(a') \times$$

$$\times\, G_{PT}^{(\pm k_1,\beta)}(\vartheta'', \vartheta'; E')\Big|_{E' = \hbar^2(2n_1 \pm k_1 + \beta + 1)^2/2MR^2} +$$

$$+ \frac{1}{2}\int_0^\infty dk\, \psi_k^{(\pm k_2,\nu)}(a'')\, \psi_k^{(\pm k_2,\nu)*}(a')\, G_{PT}^{(\pm k_1,\beta)}(\vartheta'', \vartheta'; E')\Big|_{E' = -\hbar^2 k^2/2MR^2}$$

$$+ \text{[appropriate term with } a \text{ and } \vartheta \text{ interchanged]}, \tag{4.91}$$

in the notation of (2.6, 2.10, 2.12) and (2.16). Analyzing the poles and cuts in a similar way as for the pure Coulomb case we therefore obtain with E_N as in (4.71) and E_p as in (4.72)

$$K^{(V_2)}(a'', a', \vartheta'', \vartheta'; T) = \sum_{n_1, n_2} e^{-iE_N T/\hbar}\, \Psi_{n_1 n_2}^{(V_2)}(a'', \vartheta''; R)\, \Psi_{n_1 n_2}^{(V_2)}(a', \vartheta'; R) +$$

$$+ \int_0^\infty dk \int_0^\infty dp\, e^{-iE_p T/\hbar}\, \Psi_{kp}^{(V_2)}(a'', \vartheta''; R)\, \Psi_{kp}^{(V_2)*}(a', \vartheta'; R), \tag{4.92}$$

where the bound state wave functions are given by

$$\Psi^{(V_2)}_{n_1 n_2}(a, \vartheta; R) = \sqrt{\frac{1}{2R^2}\left(\frac{MR\alpha}{\hbar^2 N^2} - 1\right)}\,\psi^{(\pm k_2, \nu)}_{n_2}(a)\,\phi^{(\pm k_1, \beta)}_{n_1}(\vartheta), \tag{4.93}$$

$$\psi^{(\pm k_2, \nu)}_{n_2}(a) = \frac{1}{\Gamma(1 \pm k_2)}\left[\frac{2(\nu \mp k_2 - 2n_2 - 1)\,\Gamma(n_2 + 1 \pm k_2)\,\Gamma(\nu - n_2)}{n_2!\,\Gamma(\nu \mp k_2 - n_2)}\right]^{1/2} \times$$

$$\times (\sinh a)^{1/2 \pm k_2}(\cosh a)^{n_2 + 1/2 - \nu}\; {}_2F_2(-n_2, \nu - n_1; 1 \pm k_2; \tanh^2 a), \tag{4.94}$$

$$\phi^{(\pm k_1, \beta)}_{n_1}(\vartheta) = \left[2(\beta \pm k_1 + 2n_1 + 1)\frac{n_2!\,\Gamma(\beta \pm k_1 + n_1 + 1)}{\Gamma(n_1 \pm k_1 + 1)\,\Gamma(n_1 + \beta + 1)}\right]^{1/2} \times$$

$$\times (\sin\vartheta)^{1/2 \pm k_1}(\cos\vartheta)^{\beta + 1/2} P^{(\pm k_1, \beta)}_{n_1}(\cos 2\vartheta). \tag{4.95}$$

The continuous states have the form

$$\Psi^{(V_2)}_{kp}(a, \vartheta; R) = \frac{1}{R}\,\psi^{(\pm k_2, \tilde{p})}_{k}(a)\,\Phi^{(\pm k_1, p)}_{k}(\vartheta), \tag{4.96}$$

$$\psi^{(\pm k_2, \tilde{p})}_{k}(a) =$$

$$= \frac{\Gamma\left[\frac{1}{2}(1 \pm k_2 + i\tilde{p} + ik)\right]\Gamma\left[\frac{1}{2}(1 \pm k_2 + i\tilde{p} - ik)\right]}{\Gamma(1 \pm k_2)}\sqrt{\frac{k \sinh \pi k}{2\pi^2}}\,(\tanh a)^{\pm k_2 - 1/2} \times$$

$$\times (\cosh a)^{ik}\; {}_2F_1\left(\frac{1 \pm k_2 + i\tilde{p} + ik}{2}, \frac{1 \pm k_2 - i\tilde{p} + ik}{2}, 1 \pm k_2; \tanh^2 a\right), \tag{4.97}$$

$$\Phi^{(\pm k_1, p)}_{k}(\vartheta) = \frac{\Gamma\left[\frac{1}{2}(1 \pm k_1 + ip + ik)\right]\Gamma\left[\frac{1}{2}(1 \pm k_1 + ip - ik)\right]}{\Gamma(1 \pm k_1)} \times$$

$$\times \sqrt{\frac{k \sinh \pi k}{2\pi^2}}\,(\tan\vartheta)^{\pm k_1 - 1/2} \times (\cos\vartheta)^{ip + 1 \pm k_1} \times$$

$$\times\; {}_2F_1\left(\frac{1 \pm k_2 + ip + ik}{2}, \frac{1 \pm k_1 - ip + ik}{2}; 1 \pm k_2; -\sin^2\vartheta\right). \tag{4.98}$$

The special case of the pure Coulomb potential follows from the consideration of the corresponding special cases in (2.6, 2.10, 2.12) and (2.16). This completes the discussion of the Coulomb problem on the two-dimensional hyperboloid in the soluble cases. The cases of elliptic II and semihyperbolic coordinates are not tractable by path integration.

4.3. The Potential V_3

We consider the potential V_3 in its separating coordinate systems:

$$V_3(u) = \frac{\alpha}{(u_0-u_1)^2} + \frac{M}{2}\omega^2 \frac{R^2+4u_2^2}{(u_0-u_1)^4} - \lambda \frac{u_2}{(u_0-u_1)^3}, \tag{4.99}$$

Horiocyclic ($y>0, x\in \mathbb{R}$):

$$= \frac{y^2}{R^2}\left[\alpha + \frac{M}{2}\omega^2(4x^2+y^2) - \lambda x\right], \tag{4.100}$$

Semicircular-Parabolic ($\xi, \eta > 0$):

$$= \frac{1}{R^2}\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left[\alpha(\xi^2+\eta^2) - \frac{1}{2}\lambda(\eta^4-\xi^4) + \frac{M}{2}\omega^2(\xi^6+\eta^6)\right]. \tag{4.101}$$

V_3 corresponds to the Holt potential plus a linear term [21,31], i.e., plus an electric field, in the flat space limit $\mathbb{R}^2$. The constants of motion for potential V_3 have the form

$$\left.\begin{aligned} I_1 &= \frac{1}{2MR^2}(K_1^2+K_2^2-L_3^2) + V_3(u),\\ I_2 &= \frac{1}{2M}(K_1-L_3)^2 + \alpha + 2M\omega^2x^2 - \lambda x,\\ I_3 &= \frac{1}{2M}(\{K_1,K_2\} - \{K_2,L_3\})\\ &+ \frac{1}{2}\frac{\xi^4(2\alpha+\xi^2\lambda+M\omega^2\xi^4) - \eta^4(2\alpha-\eta^2\lambda+M\omega^2\eta^4)}{\xi^3+\eta^2}. \end{aligned}\right\} \tag{4.102}$$

We obtain the following two path integral representations

$$K^{(V_3)}(u'', u'; T),$$

Horicyclic:

$$= \frac{1}{R^2} \int_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^2} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \times$$

$$\times \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{M}{2} R^2 \frac{\dot{x}^2 + \dot{y}^2}{y^2} - \frac{y^2}{R^2} \left(\alpha + \frac{M}{2} \omega^2 (4x^2 + y^2) - \lambda x \right) \right] dx \right\}. \quad (4.103)$$

Semicircular-Parabolic:

$$= \frac{1}{R^2} \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) \frac{\xi^2 + \eta^2}{\xi^2 \eta^2} \times$$

$$\exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{M}{2} R^2 \frac{\xi^2 + \eta^2}{\xi^2 \eta^2} (\dot{\xi}^2 + \dot{\eta}^2) - \right.\right.$$

$$\left.\left. - \frac{\xi^2 \eta^2}{R^2} \left(\alpha - \frac{\lambda}{2} (\eta^2 - \xi^2) + \frac{M}{2} \omega^2 (\xi^4 + \eta^4 - \xi^2 \eta^2) \right) \right] dt \right\}. \quad (4.104)$$

The path integral (4.104) in semicircular parabolic coordinates is not tractable. The path integral (4.103) is solved in the following way: We shift the variable x according to $x \to z = x - \lambda / 4M\omega^2$. The emerging path integral problem is the path integral of an harmonic oscillator yielding the separation

$$K^{(V_3)}(u'', u'; T) = \frac{1}{R} \sum_{m=0}^{\infty} \left(\frac{2M\omega}{\pi\hbar} \right)^{1/2} \frac{1}{2^m m!} \times$$

$$\times H_m\left(\sqrt{\frac{2M\omega}{\hbar}} z' \right) H_m\left(\sqrt{\frac{2M\omega}{\hbar}} z'' \right) \exp\left[-\frac{M\omega}{\hbar} (z'^2 + z''^2) \right]$$

$$\times \int_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y} \exp\left[\frac{iM}{2\hbar} \int_{t'}^{t''} \left(R^2 \frac{\dot{x}^2 + \dot{y}^2}{y^2} - \frac{y^2}{R^2} (E_{\alpha,\omega,\lambda} + \omega^2 y^2) \right) dt \right], \quad (4.105)$$

with the quantity $E_{\alpha,\omega,\lambda}$ given by

$$E_{\alpha,\omega,\lambda} = \alpha + 2\hbar\omega \left(m + \frac{1}{2} \right) - \frac{\lambda^2}{8M\omega^2}. \quad (4.106)$$

The $H_m(x)$ are Hermite polynomials [13, p.1033]. A path integral like this was calculated in [18], and we must distinguish two cases, first $E_{\alpha,\omega,\lambda} > 0$, and second $E_{\alpha,\omega,\lambda} < 0$. In the first case only a continuous spectrum occurs, whereas in the second case bound states can exist with the number of levels given by $n = 0, 1, \ldots, N_{max} = [\,|E_{\alpha,\omega,\lambda}|/2\hbar\omega - 1/2]$. From the explicit form of $E_{\alpha,\omega,\lambda}$ we see that it can be arranged that at least some bound states exist. Therefore we obtain the following path integral solution for V_3 in horicyclic coordinates

$$(\nu = -\mathrm{i}\sqrt{2MR^2E/\hbar^2 - 1/4})$$

$$K^{(V_3)}(u'', u'; T) = \sum_{m=0}^{\infty} \psi_m(x')\,\psi_m(x'') \int_{\mathbb{R}} \frac{dE}{2\pi\mathrm{i}}\, \mathrm{e}^{-\mathrm{i}ET/\hbar} \times$$

$$\times \frac{\Gamma\left[\frac{1}{2}(1+\nu+E_{\alpha,\omega,\lambda}/\hbar\omega)\right]}{\sqrt{y'y''}\,\hbar\omega\Gamma(1+\nu)} W_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\,\nu/2}\left(\frac{M\omega}{h}y_>^2\right)\times$$

$$\times M_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\,\nu/2}\left(\frac{M\omega}{\hbar}y_<^2\right) \tag{4.107}$$

$$= \sum_{m=0}^{\infty} [\,\sum_{n=0}^{N_{max}} \mathrm{e}^{-\mathrm{i}E_pT/\hbar}\, \Psi_{nm}^{(V_3)}(x', y'; R)\,\Psi_{nm}^{(V_3)}(x'', y''; R) +$$

$$+ \int_0^{\infty} dp\, \mathrm{e}^{-\mathrm{i}E_pT/\hbar}\, \Psi_{pm}^{(V_3)*}(x', y'; R)\,\Psi_{pm}^{(V_3)}(x'', y''; R)]. \tag{4.108}$$

$M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ are Whittaker functions [13, p.1059]. The bound state wave functions have the form

$$\Psi_{nm}^{(V_3)}(x, y; R) = \psi_n(y; R)\,\psi_m(x), \tag{4.109}$$

$$\psi_n(y; R) = \sqrt{\frac{2n!(|E_{\alpha,\omega,\lambda}|/\hbar\omega - 2n - 1)y}{R^2\Gamma(|E_{\alpha,\omega,\lambda}|/\hbar\omega - n)}} \left(\frac{M\omega}{\hbar}y^2\right)^{|E_{\alpha,\omega,\lambda}|/2\hbar\omega - n - 1/2} \times$$

$$\times \exp\left(-\frac{M\omega}{2\hbar}y^2\right) M_n^{(|E_{\alpha,\omega,\lambda}|/\hbar\omega - 2n - 1)}\left(\frac{M\omega}{\hbar}y^2\right), \tag{4.110}$$

$$\psi_m(x) = \left(\frac{2M\omega}{\pi\hbar\, 2^{2m}(m!)^2} \right)^{1/4} \times$$

$$\times H_m\left(\sqrt{\frac{2M\omega}{\hbar}} \left(x - \frac{\lambda}{8\omega^2} \right) \right) \exp\left(-\frac{M\omega}{\hbar} \left(x - \frac{\lambda}{8\omega^2} \right)^2 \right) \qquad (4.111)$$

with the discrete energy spectrum given by

$$E_n = \frac{\hbar^2}{8MR^2} - \frac{\hbar^2}{2MR^2} \left(\frac{|E_{\alpha,\omega,\lambda}|}{\hbar\omega} - 2n - 1 \right)^2. \qquad (4.112)$$

The continuous wave functions and the corresponding energy spectrum have the form

$$\Psi_{nm}^{(V_3)}(x, y; R) = \psi_m(x)\, \psi_p(y; R) \qquad (4.113)$$

$$\psi_p(y; R) = \sqrt{\frac{\hbar}{M\omega} \frac{p \sinh \pi p}{2\pi^2 R^2 y}}\, \Gamma\left[\frac{1}{2} \left(1 + ip + \frac{E_{\alpha,\omega,\lambda}}{\hbar\omega} \right) \right] \times$$

$$\times W_{-E_{\alpha,\omega,\lambda}/2\hbar\omega,\, ip/2} \left(\frac{M\omega}{\hbar} y^2 \right), \qquad (4.114)$$

$$E_p = \frac{\hbar^2}{2MR^2} \left(p^2 + \frac{1}{4} \right), \qquad (4.115)$$

with the $\psi_m(x)$ as in (4.111). The Green function $G^{(V_3)}(E)$ of the potential V_3 can be read off from (4.107). This concludes the discussion of V_3.

4.4. The Potential V_4

We consider the potential V_4 in its separating coordinate systems *Equidistant* ($\tau_1 > 0, \tau_2 \in \mathbb{R}$):

$$V_4(u) = \frac{M}{2} \frac{\omega^2}{(u_0 - u_1)^2} + \frac{\hbar^2}{2M} \frac{\kappa^2 - \frac{1}{4}}{u_2^2} =$$

$$= \frac{M}{2R^2} \frac{\omega^2}{\cosh^2 \tau_1} e^{2\tau_2} + \frac{\hbar^2}{2MR^2} \frac{\kappa^2 - \frac{1}{4}}{\sinh^2 \tau_1} \qquad (4.116)$$

Horicyclic ($y > 0, x > 0$):

$$= \frac{M}{2R^2}\omega^2 y^2 + \frac{\hbar^2}{2MR^2} y^2 \frac{\kappa^2 - \frac{1}{4}}{x^2} \tag{4.117}$$

Elliptic-Parabolic ($b > 0, \vartheta \in (0, \pi/2)$):

$$= \frac{M}{2R^2}\omega^2 \cosh^2 a \cos^2 \vartheta + \frac{\hbar^2}{2MR^2}\left(\kappa^2 - \frac{1}{4}\right)\cot^2 \vartheta \coth^2 a \tag{4.118}$$

Elliptic-Hyperbolic ($b > 0, \vartheta \in (0, \pi/2)$):

$$= \frac{M}{2R^2}\omega^2 \sinh^2 b \sin^2 \vartheta + \frac{\hbar^2}{2MR^2}\left(\kappa^2 - \frac{1}{4}\right)\tan^2 \vartheta \tanh^2 b \tag{4.119}$$

Semicircular-Parabolic ($|\kappa| = 1/2, \xi, \eta > 0$):

$$= \frac{M}{2R^2}\omega^2 \xi^2 \eta^2. \tag{4.120}$$

For the constants of motion of the potential V_4 we find

$$\left.\begin{aligned} I_1 &= \frac{1}{2MR^2}(K_1^2 + K_2^2 - L_3^2) + V_4(u), \\ I_2 &= \frac{1}{2M}(K_1 - L_3)^2 + \frac{\hbar^2}{2M}\frac{\kappa^2 - \frac{1}{4}}{x^2}, \\ I_3 &= \frac{1}{2M}K_2^2 + \frac{M}{2}\omega^2 e^{2\tau_2}. \end{aligned}\right\} \tag{4.121}$$

We discuss the corresponding solutions in the five coordinate systems only briefly because this potential seems not to be rather important. Also, the methods how to evaluate such path integrals have been presented already in earlier investigations, c.f. [20,24]. In particular, for the elliptic-, hyperbolic-parabolic, and semicircular parabolic we argue along the lines of Ref.[20], where also more details can be found. The path integral evaluations in equidistant and horicyclic coordinates are easy to do.

4.4.1. Equidistant Coordinates. We start with the path integral representation in equidistant coordinates. We consider

$$K^{(V_4)}(u'', u'; T) = \frac{1}{R^2}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t)\cosh\tau_1 \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t)\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_2^2)-\frac{\omega^2 e^{2\tau_2}}{R^2\cosh^2\tau_1}-\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2MR^2}\frac{\kappa^2-\frac{1}{4}}{\sinh^2\tau_1}-\frac{\hbar^2}{8MR^2}\left(1+\frac{1}{\cosh^2\tau_1}\right)\right]dt\right\} \tag{4.122}$$

$$=\int_0^\infty dk\int_0^\infty dp\,\exp\left[-\frac{i\hbar T}{2MR^2}\left(p^2+\frac{1}{4}\right)\right]\times$$

$$\times\Psi_{pk}^{(V_4)*}(\tau_1',\tau_2';R)\,\Psi_{pk}^{(V_4)}(\tau_1'',\tau_2'';R). \tag{4.123}$$

The path integral in the coordinate τ_2 is a path integral for the Liouville potential [24], and the remaining path integral in τ_1 is again of the form of a modified Pöschl-Teller potential path integral (2.12). Therefore the separation procedure and the path integral evaluations are straightforward. The spectrum is purely continuous and the wave functions are given by ($\tilde k = mR\omega/\hbar$)

$$\Psi_{pk}^{(V_4)}(\tau_1,\tau_2;R)=(\cosh\tau_1)^{-1/2}S_p(\tau_1;R)\,\psi_k(\tau_2), \tag{4.124}$$

$$S_p(\tau_1;R)=\frac{1}{\Gamma(1+\kappa)}\sqrt{\frac{p\sinh\pi p}{2\pi^2R^2}}\,\Gamma\left(\frac{ik-\kappa+1-ip}{2}\right)\Gamma\left(\frac{\kappa-ik+1-ip}{2}\right)\times$$

$$\times(\tanh\tau_1)^{1/2+\kappa}(\cosh\tau_1)^{ip}\times$$

$$\times{}_2F_1\left(\frac{ik+\kappa+1-ip}{2},\frac{1+\kappa-ik-ip}{2};1+\kappa;\tanh^2\tau_1\right), \tag{4.125}$$

$$\psi_k(\tau_2)=\sqrt{\frac{2k\sinh\pi k}{\pi^2}}\,K_{ik}(\tilde k\,e^{\tau_2}). \tag{4.126}$$

$K_\nu(z)$ is a modified Bessel function [13, p.952]. The corresponding Green function in these coordinates is given by (in the notation of (2.16) with $/L_\lambda=\frac{1}{2}(ik-1)$)

$$G^{(V_4)}(u'',u';E)=\frac{2}{\pi^2}\int_0^\infty dk\,\sinh\pi k\,K_{ik}(\tilde k\,e^{\tau_2'})\,K_{ik}(\tilde k\,e^{\tau_2''})\times$$

$$\times \frac{M}{2\hbar^2} \frac{\Gamma(m_1 - L_\lambda)\,\Gamma(L_\lambda + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\,\Gamma(m_1 - m_2 + 1)} \times$$

$$\times (\cosh\tau_1' \cosh\tau_1'')^{-(m_1 - m_2 + 1/2)} (\tanh\tau_1' \tanh\tau_1'')^{m_1 + m_2 - 1/2} \times$$

$$\times {}_2F_1\left(-L_\lambda + m_1, L_\lambda + m_1 + 1; m_1 - m_2 + 1; \frac{1}{\cosh^2\tau_{1,<}}\right) \times$$

$$\times {}_2F_1(-L_\lambda + m_1, L_\lambda + m_1 + 1; m_1 + m_2 + 1; \tanh^2\tau_{1,>}). \qquad (4.127)$$

4.4.2. Horicyclic Coordinates. In horicyclic coordinates we see that in the x-variable we have a radial path integral with a repulsive centrifugal barrier. Therefore we obtain ($\tilde{k}^2 = k^2 + M^2R^2\omega^2/\hbar^2$)

$$K^{(V_4)}(u'', u'; T) = \frac{1}{R^2} \int\limits_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^2} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \times$$

$$\times \exp\left[\frac{i}{\hbar}\int\limits_{t'}^{t''}\left(\frac{M}{2}R^2\frac{\dot{x}^2 + \dot{y}^2}{y^2} - \frac{\omega^2 y^2}{R^2} - \frac{\hbar^2}{2MR^2}y^2\frac{\kappa^2 - \frac{1}{4}}{x^2}\right)dt\right] \qquad (4.128)$$

$$= \sqrt{x'x''y'y''} \int\limits_0^\infty k\,dk\, J_\kappa(kx')\, J_\kappa(kx'') \frac{2M}{\hbar^2} \int\limits_{\mathbb{R}} \frac{dE}{2\pi i} e^{iET/\hbar} \times$$

$$\times I_{-i\sqrt{2MR^2E/\hbar^2 + 1/4}}(\tilde{k}y_<)\, K_{i\sqrt{2MR^2E/\hbar^2 + 1/4}}(\tilde{k}y_>) \qquad (4.129)$$

$$= \frac{\sqrt{x'x''y'y''}}{R^2} \int\limits_0^\infty k\,dk\, J_\kappa(kx')\, J_\kappa(kx'') \times$$

$$\times \frac{2}{\pi^2} \int\limits_0^\infty dp\, p \sinh \pi p \exp\left[-\frac{i\hbar T}{2MR^2}\left(p^2 + \frac{1}{4}\right)\right] K_{ip}(\tilde{k}y')\, K_{ip}(\tilde{k}y''). \qquad (4.130)$$

The $I_\nu(z)$ and $J_\nu(z)$ are (modified) Bessel functions [13, p.951]. In the path integral for the horicyclic system we simply do the x-path integration (a radial path integral [25,56]) and we find that the remaining y-path integral looks exactly as for the free motion with just the separation parameter k shifted by

$M^2R^2\omega^2/\hbar^2$. A path integral like this has been already discussed in, e.g., [20] and references therein, which is not repeated here, and the solutions (4.129, 4.130) for the Green function and the spectral expansion follow immediately.

4.4.3. Elliptic- and Hyperbolic-Parabolic Coordinates. In the following two path integral representations we, first, state the solutions, and second, give a short description how these solutions can be obtained. In elliptic-parabolic coordinates we have an explicit solution only for $|\kappa| = \frac{1}{2}$, and we obtain for that case

$$(k_\rho = MR\omega/\hbar)$$

$$K^{(V_4)}(u'', u'; T) = \frac{1}{R^2} \int\limits_{a(t')=a'}^{a(t'')=a''} \mathcal{D}a(t) \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \frac{\cosh^2 a - \cos^2\vartheta}{\cosh^2 a \cos^2\vartheta} \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\cosh^2 a - \cos^2\vartheta}{\cosh^2 a\cos^2\vartheta}(\dot{a}^2+\dot{\vartheta}^2) - \frac{\omega^2}{R^2}\cosh^2 a\cos^2\vartheta - \right.\right.$$

$$\left.\left. - \frac{\hbar^2}{2MR^2}\left(\kappa^2 - \frac{1}{4}\right)\cot^2\vartheta\coth^2 a\right]dt\right\} \tag{4.131}$$

$$= \frac{1}{R^2}\sqrt{\cos\vartheta'\cos\vartheta''} \sum_{\varepsilon,\varepsilon'=\pm 1} \int_0^\infty dp \sinh\pi p \int_0^\infty \frac{dk\, k \sinh\pi k}{(\cosh^2\pi k + \sinh^2\pi p)^2} \times$$

$$\times e^{-i\hbar T(p^2+1/4)/2MR^2} S^{ik(1)}_{ip-1/2}(\varepsilon\tanh a''; ik_\rho)\, S^{ik(1)*}_{ip-1/2}(\varepsilon\tanh a'; ik_\rho) \times$$

$$\times \mathrm{ps}^{ip}_{ik-1/2}(\varepsilon'\sin\vartheta''; -k_\rho^2)\, \mathrm{ps}^{ip*}_{ik-1/2}(\varepsilon'\sin\vartheta'; -k_\rho^2). \tag{4.132}$$

The $\mathrm{ps}^\mu_\nu(z)$ and $S^{\mu(1)}_\nu(z)$ are spheroidal functions [50, p.236, p.289]. In hyperbolic-parabolic coordinates we have an exact solution only for $|\kappa| = 1/2$ and we obtain in this case $(k_\rho = MR\omega/\hbar)$

$$K^{(V_4)}(u'', u'; T) = \frac{1}{R^2} \int\limits_{b(t')=b'}^{b(t'')=b''} \mathcal{D}b(t) \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \frac{\sinh^2 b + \sin^2\vartheta}{\sinh^2 b \sin^2\vartheta} \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\sinh^2 b+\sin^2\vartheta}{\sinh^2 b\sin^2\vartheta}(\dot b^2+\dot\vartheta^2)-\frac{\omega^2}{R^2}\sinh^2 b\sin^2\vartheta-\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2MR^2}\left(\kappa^2-\frac{1}{4}\right)\tan^2\vartheta\tanh^2 b\right]dt\right\} \qquad (4.133)$$

$$=\frac{1}{R^2}\sqrt{\sinh b'\sinh b''\sin\vartheta'\sin\vartheta''}\times$$

$$\times\sum_{\varepsilon=\pm 1}\int_0^\infty dp\int_0^\infty\frac{dk\,k\sinh\pi k}{\cosh^2\pi k+\sinh^2\pi p\cosh\pi(p-k)}\,e^{-i\hbar T(p^2+1/4)/2MR^2}\times$$

$$\times S^{ip(1)}_{ik-1/2}(\cosh b'';ik_\rho)\,S^{ip(1)*}_{ik-1/2}(\cosh b';ik_\rho)\times$$

$$\times\operatorname{ps}^{ip}_{ik-1/2}(\varepsilon\cos\vartheta'';-k_\rho^2)\,\operatorname{ps}^{ip*}_{ik-1/2}(\varepsilon\cos\vartheta';-k_\rho^2). \qquad (4.134)$$

The path integral representation in these two coordinate systems for V_4 are similar to the path integral representations of the free motion on $\Lambda^{(3)}$ in elliptic-parabolic and hyperbolic-parabolic coordinates, respectively. Let us sketch the solution of the former. Performing a time transformation yields a path integral which looks like the path integral in flat space in the oblate spheroidal coordinate systems, i.e.,

$$K^{(V_4)}(a'',a',\vartheta'',\vartheta';T)=$$

$$=\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}e^{-iET/\hbar}\int_0^\infty ds''\int_{a(0)=a'}^{a(s'')=a''}\mathcal{D}a(s)\int_{\vartheta(0)=\vartheta'}^{\vartheta(s'')=\vartheta''}\mathcal{D}\vartheta(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{M}{2}(\dot a^2+\dot\vartheta^2)-\omega^2(\cosh^2 a-\cos^2\vartheta)\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2M}\left(\frac{\kappa^2-\frac{1}{4}}{\sinh^2 a}-\frac{\lambda^2-\frac{1}{4}}{\cosh^2 a}+\frac{\kappa^2-\frac{1}{4}}{\sin^2\vartheta}+\frac{\lambda^2-\frac{1}{4}}{\cos^2\vartheta}\right)\right]ds\right\}, \qquad (4.135)$$

where $\lambda=\sqrt{\frac{1}{4}-2MR^2E/\hbar^2}$. This path integral could be solved provided we knew the solution of the path integral representation in prolate spheroidal coordinates in $\mathbb{R}^4$. However, this is not the case, and therefore we are restricted to the case $|\kappa| = 1/2$ which is solvable using the result of the free motion on $\Lambda^{(3)}$ in elliptic-parabolic coordinates. Because λ is for $E > \hbar^2/8MR^2$ purely imaginary we cannot apply the oblate spheroidal path integral identity of [20] in a simple way. We must find a proper analytic continuation, and instead we construct this analytic continuation heuristically. Since the (a, ϑ)-path integration in (4.132) corresponds for $\omega = 0$ to the path integral on $\Lambda^{(2)}$ in elliptic-parabolic coordinates we look for those spheroidal wave functions [20,50] which have for the parameter $\omega = 0$ the limit of the wave functions of this system and we find for

$$\mathrm{ps}_\nu^\mu(x;0) = P_\nu^\mu(x), \quad (|x| \le 1), \quad S_\nu^{\mu(1)}(z;0) = \mathcal{P}_\nu^\mu(z), \quad |z| \ge 1. \tag{4.136}$$

Putting everything together yields the conjectural result (4.132). The case of hyperbolic-parabolic system (4.134) is done in an analogous way.

4.4.4. Semicircular-Parabolic Coordinates. In semicircular-parabolic coordinates the potential separates only for $|k| = 1/2$ and we obtain $(q = M\omega/\hbar)$

$$K^{(V_4)}(u'', u'; T) =$$

$$= \frac{1}{R^2} \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) \frac{\xi^2+\eta^2}{\xi^2\eta^2} \times$$

$$\exp\left\{ \frac{iM}{2\hbar} \int_{t'}^{t''} \left[R^2 \frac{\xi^2+\eta^2}{\xi^2\eta^2} (\dot{\xi}^2 + \dot{\eta}^2) \frac{\omega^2}{R^2} \xi^2\eta^2 \right] dt \right\} \tag{4.137}$$

$$= \sum_\pm \frac{1}{\pi^2 q^2 R^2} \int_0^\infty dp\, p(\sinh \pi p)^2 \int_0^\infty dk\, k \left| \Gamma\left[\frac{1}{2}(1 \pm k^2/2q + ip) \right] \right|^4 \times$$

$$e^{-i\hbar T(p^2+1/4)/2MR^2}\, W_{\pm k^2/4q,\, ik/2}(q\xi''^2) \times$$

$$\times W_{\pm k^2/4q,\, ik/2}(q\xi'^2)\, W_{\pm k^2/4q,\, ip/2}(q\eta''^2)\, W_{\pm k^2/4q,\, ip/2}(q\eta'^2). \tag{4.138}$$

This path integral is solved in the following way: After a time transformation we obtain

$$K^{(V_4)}(\xi'', \xi'', \eta'', \eta'; T) =$$

$$= \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} e^{-\mathrm{i}ET/\hbar} \int_0^\infty ds'' \int_{\xi(0)=\xi'}^{\xi(s'')=\xi''} D\xi(s) \int_{\eta(0)=\eta'}^{\eta(s'')=\eta''} \mathcal{D}\eta(s) \times$$

$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^{s''} \left[\frac{M}{2} ((\dot{\xi}^2 + \dot{\eta}^2) - \omega^2(\xi^2 + \eta^2)) - \hbar^2 \frac{\lambda^2 - \frac{1}{4}}{2M} \left(\frac{1}{\xi^2} + \frac{1}{\eta^2} \right) \right] ds \right\} \quad (4.139)$$

$$= \sqrt{\xi'\xi''} \frac{1}{2} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} e^{-\mathrm{i}ET/\hbar} \int_0^\infty ds'' \int_{\mathbb{R}} \frac{dE'}{2\pi\mathrm{i}} e^{-\mathrm{i}E's''/\hbar} \times$$

$$\times \frac{M\omega}{\mathrm{i}\hbar \sin \omega s''} \exp\left[-\frac{M\omega}{2\mathrm{i}\hbar} (\xi'^2 + \xi''^2) \cot \omega s'' \right] I_\lambda \left(\frac{M\omega\xi'\xi''}{\mathrm{i}\hbar \sin \omega s''} \right) \times$$

$$\times \frac{\Gamma\left[\frac{1}{2}(1 + \lambda - E'/\hbar\omega) \right]}{\hbar\omega\Gamma(1 + \lambda)} W_{E'/2\hbar\omega, \lambda/2} \left(\frac{M\omega}{\hbar} \eta_>^2 \right) \times$$

$$\times M_{E'/2\hbar\omega, \lambda/2} \left(\frac{M\omega}{\hbar} \eta_<^2 \right) + (\xi \leftrightarrow \eta), \quad (4.140)$$

where $\lambda^2 = \frac{1}{4} - 2MR^2E/\hbar^2$; we must take into account a term with ξ and η interchanged. One uses the path integral solution of the radial harmonic oscillator [56], where for the ξ-dependent part we expand the propagator by means of

$$I_\lambda(z) = \frac{\hbar^2}{\pi^2 MR^2} \int_0^\infty \frac{dp\, p \sinh \pi p}{\hbar^2 \left(p^2 + \frac{1}{4} \right) / 2MR^2 - E} K_{\mathrm{i}p}(z), \quad (4.141)$$

and the integral representation [13, p.729]

$$W_{\chi, \frac{\mu}{2}}(a) W_{\chi, \frac{\mu}{2}}(b) = \frac{2\sqrt{ab}\, t}{\Gamma\left(\frac{1+\mu}{2} - \chi \right) \Gamma\left(\frac{1-\mu}{2} - \chi \right)} \int_0^\infty e^{-\frac{a+b}{2} \cosh v} \times$$

$$\times K_\mu(\sqrt{ab} \sinh v) \left(\cosh \frac{v}{2} \right)^{2\chi} dv. \quad (4.142)$$

In the η-dependent part one uses the Green function for the radial harmonic oscillator (c.f. [25] for the functional measure formulation)

$$\frac{\mathrm{i}}{h}\int_0^\infty dT\, \mathrm{e}^{\mathrm{i}ET/h} \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\, \mu_\lambda[r^2] \exp\left[\frac{\mathrm{i}M}{2h}\int_{t'}^{t''} (\dot{r}^2-\omega^2 r^2)\, dt\right]=$$

$$=\frac{\Gamma\left[\frac{1}{2}(1+\lambda-E/h\omega)\right]}{h\omega\sqrt{r'r''}\,\Gamma(1+\lambda)}\, W_{E/2h\omega,\,\lambda/2}\left(\frac{M\omega}{h}r_>^2\right) M_{E/2h\omega,\,\lambda/2}\left(\frac{M\omega}{h}r_<^2\right), \quad (4.143)$$

and the relation [13, p.1062]

$$W_{\lambda,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\lambda\right)}M_{\lambda,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\lambda\right)}M_{\lambda,-\mu}(z). \quad (4.144)$$

If we have (4.140) as it stands we obtain the Green function $G^{(V_4)}$ $(\xi'',\xi',\eta'',\eta';E)$, together with the prescription $E'=-h^2k^2/2M$. The final result (4.138) is then obtained by conbining on the one hand, where $E'=-k^2h^2/2M$ and re-inserting

$$\frac{M\omega}{h^2}\int_0^\infty ds''\frac{ds''}{\sin\omega s''}\exp\left[-\mathrm{i}\frac{E's''}{h}-\frac{M\omega}{2\mathrm{i}h}(\xi'^2+\xi''^2)\cot\omega s''\right]I_\lambda\left(\frac{M\omega\xi'\xi''}{\mathrm{i}h\sin\omega s''}\right)=$$

$$=\frac{1}{\pi^2 q}\int_0^\infty\frac{dp\sinh\pi p}{p^2h^2/2M-E}\left|\Gamma\left[\frac{1}{2}\left(1+\mathrm{i}p-\frac{k^2}{2q}\right)\right]\right|^2\times$$

$$\times W_{-k^2/4q,\,\mathrm{i}p/2}(q\xi'^2)\, W_{-k^2/4q,\,\mathrm{i}p/2}(q\xi''^2) \quad (4.145)$$

and on the other $(q=MR\omega/h)$

$$W_{k^2/4q,\,\mathrm{i}p/2}(q\eta'^2)\left[\frac{\Gamma\left[\frac{1}{2}(1+\mathrm{i}p-k^2/2q)\right]}{h\omega\Gamma(1+\mathrm{i}p)}M_{k^2/4q,\,\mathrm{i}p/2}(q\eta''^2)-\right.$$

$$\left.-\frac{\Gamma\left[\frac{1}{2}(1-\mathrm{i}p-k^2/2q)\right]}{h\omega\Gamma(1-\mathrm{i}p)}M_{k^2/4q,\,-\mathrm{i}p/2}(q\eta''^2)\right]=$$

$$= \frac{iM}{\pi h^2 q} \sinh \pi p \left| \Gamma\left[\frac{1}{2}\left(1 + ip - \frac{k^2}{2q} \right)\right]\right|^2 \times$$

$$\times W_{-k^2/4q,\, ip/2}(q\eta'^2)\, W_{-k^2/4q,\, ip/2}(q\eta''^2). \tag{4.146}$$

4.5. The Potential V_5

We consider the potential V_5 in its two separating coordinate systems

Equidistant ($\tau_1, \tau_2 \in \mathbb{R}$):

$$V_5(u) = \alpha R \frac{u_2}{\sqrt{u_0^2 - u_1^2}} = \alpha R \tanh \tau_1 \tag{4.147}$$

Semicircular-Parabolic ($\xi, \eta > 0$):

$$= \alpha R \frac{\xi^2 \eta^2}{\xi^2 + \eta^2}\left(\frac{1}{\xi^2} - \frac{1}{\eta^2} \right). \tag{4.148}$$

The constants of motion for the potential V_5 are the following

$$\left.\begin{aligned} I_1 &= \frac{1}{2MR^2}(K_1^2 + K_2^2 - L_3^2) + V_5(u), \\ I_2 &= \frac{1}{2M}(\{K_1, K_2\} - \{K_2, L_3\}) + \frac{2\alpha R}{\xi^2 + \eta^2}, \\ I_3 &= K_2^2. \end{aligned}\right\} \tag{4.149}$$

We have the following two path integral representations

$$K^{(V_5)}(u'', u'; T)$$

Equidistant:

$$= \frac{1}{R^2} \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \cosh \tau_1 \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \times$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\tau_1^2+\cosh^2\tau_1\dot\tau_1^2)-\alpha R\tanh\tau_1-\right.\right.$$

$$\left.\left.\frac{\hbar^2}{8MR^2}\left(1+\frac{1}{\cosh^2\tau_1}\right)\right]dt\right\} \qquad (4.150)$$

Semicircular-Parabolic:

$$=\frac{1}{R^2}\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\,\frac{\xi^2+\eta^2}{\xi^2\eta^2}\times$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\frac{\xi^2+\eta^2}{\xi^2\eta^2}.(\dot\xi^2+\dot\eta^2)-\alpha R\frac{\eta^2-\xi^2}{\xi^2+\eta^2}\right]dt\right\}. \qquad (4.151)$$

4.5.1. Equidistant Coordinates. After separating off the τ_2-path integration we obtain a pure scattering Rosen-Morse potential, a path integral problem which has been solved in [14,43]. Therefore we get

$$K^{(V_5)}(u'',u';T)=\frac{\mathrm{e}^{-\mathrm{i}\hbar T/8MR^2}}{R}(\cosh\tau_1'\,\cosh\tau_1'')^{-1/2}\times$$

$$\times\int_{\mathbb{R}}\frac{dk}{2\pi}\mathrm{e}^{ik(\tau_2''-\tau_2')}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\times$$

$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2\dot\tau_1^2-\alpha R\tanh\tau_1-\frac{\hbar^2}{2MR^2}\frac{k^2+\frac{1}{4}}{\cosh^2\tau_1}\right)dt\right] \qquad (4.152)$$

$$=(\cosh\tau_1'\,\cosh\tau_1'')^{-1/2}\int_{\mathbb{R}}\frac{dk}{2\pi}\mathrm{e}^{ik(\tau_2''-\tau_2')}\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}\times$$

$$\times\frac{M}{\hbar^2}\frac{\Gamma(m_1-L_k).\Gamma(L_k+m_1+1)}{\Gamma(m_1+m_2+1)\,\Gamma(m_1-m_2+1)}\times$$

$$\times\left(\frac{1-\tanh\tau_1'}{2}\cdot\frac{1-\tanh\tau_1''}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\tanh\tau_1'}{2}\cdot\frac{1+\tanh\tau_1''}{2}\right)^{(m_1+m_2)/2}\times$$

$$\times {}_2F_1\left(-L_k+m_1, L_k+m_1+1; m_1+m_2+1; \frac{1+\tanh\tau_{1,>}}{2}\right)\times$$

$$\times {}_2F_1\left(-L_k+m_1, L_k+m_1+1; m_1-m_2+1; \frac{1-\tanh\tau_{1,<}}{2}\right) \qquad (4.153)$$

$$=\int_{\mathbb{R}} dk \int_0^\infty dp\, e^{-iE_pT/\hbar}\, \Psi_{pk}^{(V_5)}(\tau_1'', \tau_2''; R)\, \Psi_{pk}^{(V_5)*}(\tau_1', \tau_2'; R). \qquad (4.154)$$

Here denote $L_k=-2ik-\frac{1}{2}$, $m_{1,2}=\sqrt{m/2}(\sqrt{-\alpha R-E-E_0}\pm\sqrt{\alpha R-E-E_0})/\hbar$, $E_0=\hbar^2/8MR^2$, and (4.153) is the Green function corresponding to the path integral (4.150). The wave functions and the energy-spectrum of the continuous states are (where $\pm$ distinguishes between incoming and outgoing scattering states, respectively)

$$\Psi_{pk}^{(V_5)}(\tau_1, \tau_2; R)=(2\pi\cosh\tau_1)^{-1/2}\, S_p^{(\pm)}(\tau_1; R)\, e^{ik\tau_2}, \qquad (4.155)$$

$$S_p^{(\pm)}(\tau_1; R)=\frac{1}{R\,\Gamma(1+m_1\pm m_2)}\,\frac{\sqrt{M\sinh(\pi|m_1\pm m_2|)/2}}{\hbar\,|\sin\pi(m_1+L_k)|}\times$$

$$\times\left(\frac{1+\tanh\tau_1}{2}\right)^{(m_1+m_2)/2}\left(\frac{1-\tanh\tau_1}{2}\right)^{(m_1-m_2)/2}\times$$

$$\times {}_2F_1\left(m_1+L_k+1, m_1-L_k; 1+m_1\pm m_2; \frac{1\pm\tanh\tau_1}{2}\right), \qquad (4.156)$$

$$E_p=\frac{\hbar^2}{2M}\left(p^2+\frac{1}{4}\right)-\alpha R. \qquad (4.157)$$

4.5.2. Semicircular-Parabolic Coordinates. In the semicircular-parabolic system we obtain after a time transformation ($\lambda_{1,2}=1/4-2M(ER^2\pm\alpha R)/\hbar^2$)

$$K^{(V_5)}(u'', u'; T)=\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\, e^{-iET/\hbar}\int_0^\infty ds'' \int_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{M}{2}\left((\dot\xi^2+\dot\eta^2)-\frac{\hbar^2}{2MR^2}\left(\frac{\lambda_1^2-\frac{1}{4}}{\xi^2}-\frac{\lambda_2^2-\frac{1}{4}}{\eta^2}\right)\right)\right]ds\right\}$$

$$= \frac{M^2}{\mathrm{i}\hbar^3}\sqrt{\xi'\xi''\eta'\eta''}\int\limits_{\mathbb{R}}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int\limits_0^\infty\frac{ds''}{s''}\int\limits_{\mathbb{R}}\frac{dE'}{2\pi\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}E's''/\hbar}\times$$

$$\times I_\lambda\left(\sqrt{-2ME'}\,\frac{\xi_<}{\hbar}\right)K_{\lambda_1}\left(\sqrt{-2ME'}\,\frac{\xi_>}{\hbar}\right)\times$$

$$\times\exp\left[-\frac{M}{2\mathrm{i}\hbar s''}(\eta'^2+\eta''^2)\right]I_{\lambda_2}\left(\frac{M\eta'\eta''}{\mathrm{i}\hbar s''}\right). \tag{4.158}$$

The corresponding wave functions are obtained in a similar way as in [20] for the free motion on $\Lambda^{(2)}$ in semicircular-parabolic coordinates by analysing the Green function (4.158) on the cut, which finally yields ($\tilde{p}_{1,2} = -\mathrm{i}\sqrt{p^2 \pm 2MR\alpha/\hbar^2}$)

$$K^{(V_5)}(u'',u';T) = \frac{\sqrt{\xi'\xi''\eta'\eta''}}{4\pi^2}\int\limits_0^\infty k\,dk\int\limits_0^\infty dp\,p\sinh^2\pi p\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\times$$

$$\times\,[H^{(1)}_{-\mathrm{i}\tilde{p}_2}(k\eta')\,H^{(1)}_{\mathrm{i}\tilde{p}_2}(k\eta'')\,K_{\mathrm{i}\tilde{p}_1}(k\xi')\,K_{\mathrm{i}\tilde{p}_1}(k\xi'')+$$

$$+\,K_{\mathrm{i}\tilde{p}_2}(k\eta')\,K_{\mathrm{i}\tilde{p}_2}(k\eta'')\,H^{(1)}_{-\mathrm{i}\tilde{p}_1}(k\xi')\,H^{(1)}_{\mathrm{i}\tilde{p}_1}(k\xi'')], \tag{4.159}$$

with E_p as in (4.157), and the even and odd wave functions can be read off from the spectra expansion. The $H^{(1)}_\nu(z)$ are Hankel functions [13, p.952].

5. SUMMARY AND DISCUSSION

In this paper we have performed an investigation about superintegrable potentials on the two-dimensional hyperboloid. We have found that the two most important potentials, the oscillator and the Coulomb potential, admit separation of variables in four coordinate systems. Each problem is exactly solvable in two coordinate systems, the oscillator in spherical and equidistant coordinates, the Coulomb problem in spherical and elliptic parabolic coordinates. We have also stated the corresponding Green functions.

These particular features are not too surprising. In the flat space limit the spherical system yields two-dimensional polar coordinates, and both problems in $\mathbb{R}^2$ are separable in this coordinate system. The equidistant system yields in the flat space limit cartesian coordinates, and the oscillator in $\mathbb{R}^2$ is separable in cartesian coordinates. The elliptic-parabolic system yields parabolic coordinates (as the semihyperbolic system) and the Coulomb problem in $\mathbb{R}^2$ is separable in parabolic coordinates. The elliptic system on $\Lambda^{(2)}$ gives the elliptic system in $\mathbb{R}^2$, the oscillator is separable in this coordinate system, but does not admit an analytic solution in terms of usually known higher transcendental functions. In fact, the solution of the pure harmonic oscillator in $\mathbb{R}^2$ can be given in terms of Ince polynomials [39,48]. The hyperbolic system on $\Lambda^{(2)}$ also yields the cartesian system.

Furthermore, the elliptic II system on $\Lambda^{(2)}$ gives the elliptic II coordinate system in $\mathbb{R}^2$, and the Coulomb problem in $\mathbb{R}^2$ is separable in this coordinate system [49].

We have seen that the situation concerning separation of variables of these two potentials in the found coordinate system is very similar in flat space [6,10,21], on the sphere [22], and on the hyperboloid. The most significant difference being that on the sphere there are less, and on the hyperboloid more possibilities.

We have also stated explicitly the relevant Green functions of the potentials. This includes the simple and general Higgs oscillator, the Coulomb potential, and for V_3, V_4 and V_5 in several coordinate system representations. In particular, from the spectral expansions in horicyclic coordinates, one can show with the integral representations [46, pp.732,819]

$$\mathcal{P}_{\nu-1/2}\left(\frac{a^2+b^2+c^2}{2ab}\right)=\frac{4\sqrt{ab}}{\pi^2}\cos\nu\pi\int_0^\infty dk\, K_\nu(ak)\, K_\nu(bk)\cos ck, \tag{5.1}$$

$$\mathcal{Q}_{\nu-1/2}\left(\frac{a^2+b^2+c^2}{2ab}\right)=\int_0^\infty dp'\,\frac{p'\tanh\pi p'}{\nu^2+p'^2}\,\mathcal{P}_{ip'-1/2}\left(\frac{a^2+b^2+c^2}{2ab}\right), \tag{5.2}$$

that the Green function for the free motion on the two-dimensional hyperboloid has the form [17,24]

$$G(u'',u';E)=\frac{m}{\pi\hbar^2}\,\mathcal{Q}_{-1/2-i\sqrt{2MR^2E/\hbar^2-1/4}}\left(\frac{(x''-x')^2+y'^2+y''^2}{2y'y''}\right). \tag{5.3}$$

The Green function is a function of the invariant distance $d(u'',u')$ on $\Lambda^{(2)}$ only, i.e., $G(u'',u';E)=G(\cosh d('',u');E)$. A similar consideration can be made for the corresponding path integral representations of the free motion on $\Lambda^{(2)}$ in spherical [17,19,26] and semicircular parabolic coordinates [20].

Let us add some remarks concerning potentials which are separable in the semihyperbolic coordinate system. We consider the potential ($\mu_{1,2} > 0$)

$$V_6(u) = \kappa u_0 u_1 + \frac{M}{2}\omega^2\left(4\frac{u_0^2 u_1^2}{R^2} + u_2^2\right) + \frac{\hbar^2}{2M}\frac{k_2^2 - \frac{1}{4}}{u_2^2} \tag{5.4}$$

$$= \frac{R^2}{\mu_1 + \mu_2}\left[\frac{\kappa}{2}(\mu_1^2 - \mu_2^2) + \frac{M}{2}\omega^2(\mu_1^3 + \mu_2^3) + \frac{\hbar^2}{2MR^2}\left(k_1^2 - \frac{1}{4}\right)\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\right]. \tag{5.5}$$

The specific features of the potential characterize it as a Holt potential plus a linear term, i.e., with an electric field [21,31]. From the flat space case [21] we know that a potential like (5.4) is separable in cartesian and parabolic coordinates. On the hyperboloid (5.4) is separable in the semihyperbolic coordinate system (3.69). The semihyperbolic system can have two flat-space limits, the cartesian and the parabolic coordinate system, however, on the hyperboloid they correspond to two realizations of the same system.

The only potential which is separable in the equidistant and semihyperbolic system is

$$V_4^{(\omega=0)}(u) = V_7(u) = \frac{\hbar^2}{2M}\frac{k_2^2 - \frac{1}{4}}{u_2^2}, \tag{5.6}$$

and it turns out to be separable in eight coordinate systems, which is almost trivial. It can be exactly solved in six coordinate systems, but the difference in comparison to the free motion on $\Lambda^{(2)}$ is insignificant, and we omit these solutions.

Another potential which is separable in the semihyperbolic system has the form ($\mu_{1,2} > 0$)

$$V_8(u) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2 + u_2^2}} - 1\right) +$$

$$+ \frac{\beta_1\sqrt{\sqrt{u_0^2u_1^2 + u_2^2R^2} + u_0u_1} + \beta_2\sqrt{\sqrt{u_0^2u_1^2 + u_2^2R^2} - u_0u_1}}{2R\sqrt{u_0^2u_1^2 + u_2^2R^2}} = \tag{5.7}$$

$$= -\frac{\alpha}{R}\left(\frac{\sqrt{1 + m_1^2} + \sqrt{1 + \mu_2^2}}{\mu_1 + \mu_2} - 1\right) + \frac{1}{R}\frac{\beta_1\sqrt{\mu_1} + \beta_2\sqrt{\mu_2}}{\mu_1 + \mu_2}. \tag{5.8}$$

We mention this potential because in the flat space limit it yields

$$V_4(x, y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\beta_1\sqrt{\sqrt{x^2+y^2}+x}+\beta_2\sqrt{\sqrt{x^2+y^2}-x}}{2\sqrt{x^2+y^2}}, \tag{5.9}$$

which is separable in mutually orthogonal parabolic coordinate systems. Such a notion on the hyperboloid does not make sense. Two of such systems can be transformed into each other by a rotation, and hence they are equivalent. In the flat space limit, however, they yield two mutually parabolic systems, as it must be. Therefore our findings of potentials on the two-dimensional hyperboloid which are separable in more than one coordinate system can be summarized as follows:

1. We have found the generalized oscillator and Coulomb systems, each of them is separable in four coordinate systems.

2. We have found a Holt potential version on the hyperboloid, which is separable in horicyclic and semicircular parabolic coordinates. However, both coordinate systems lead in the flat space limit to the cartesian system.

3. The two other superintegrable potentials known from $\mathbb{R}^2$ could be formulated in terms of coordinates on the hyperboloid and are both separable only in the semihyperbolic systems. They yield the proper flat space limit, where the semihyperbolic system gives parabolic coordinates, and the missing separating coordinate systems emerge in this process as well.

4. We have found the simple potential $V_4(u)$ which is separable in four, respectively five (depending on the parameters) coordinate systems. The flat space limit of this potential is trivial, i.e., $V_4 \propto 1/y^2$ $(R \to \infty)$, which is separable in all four coordinate systems in $\mathbb{R}^2$, let alone that the pure $1/u_2^2$-potential only alters the corresponding radial quantum numbers in its eight separating coordinate system in comparison to the free motion.

5. We have found the potential V_5 which is separable in horicyclic and semicircular-parabolic coordinates. Its flat space limit is the linear potential, i.e., $V_5 \to \alpha x$ $(R \to \infty)$, which is separable in cartesian and parabolic coordinates.

6. The potentials (5.4, 5.7) are the proper generalizations of the Holt potential and the modified Coulomb potential (5.9) of $\mathbb{R}^2$, where both potentials are superintegrable, i.e., separable in cartesian and parabolic, respectively mutually parabolic coordinate systems. However, on the hyperboloid $\Lambda^{(2)}$ they are only separable and the corresponding coordinate systems are not distinguishable from each other. They are only distinguishable in the flat space limit $R \to \infty$.

Table 3. Correspondence of superintegrable potentials in two dimensions

$V_{\Lambda^{(2)}}(u)$	#Systems	$V_{\mathbb{R}^2}(\mathbf{x})$	#Systems	$V_{S^{(2)}}(\mathbf{s})$	#Systems
$V_1(u)$	4(3)	$V_1(\mathbf{x})$	3	$V_1(\mathbf{s})$	2(3)
$V_2(u)$	4(3)	$V_3(\mathbf{x})$	3	$V_2(\mathbf{s})$	2(3)
$V_3(u)$	2(1)	$\frac{M}{2}\omega^2(4x^2+y^2)-\lambda x$	2	—	
$V_4^{(\omega=0)}(u)$	8(4)	$\frac{\hbar^2}{2M}\frac{\kappa^2-1/4}{x^2}$	4	$\frac{\hbar^2}{2M}\frac{\kappa^2-1/4}{s_1^2}$	2(4)
$V_5(u)$	2(1)	αx	2	—	
$V_6(u)$	1(2)	$V_2(\mathbf{x})$	2	—	
$V_8(u)$	1(2)	$V_5(\mathbf{x})$	2	—	

7. We cannot say for sure if we really have found all possible superintegrable potentials on the hyperboloid. For a systematic search one must solve differential equations which emerge from the general form of a potential separable in a particular coordinate system, and then change variables. Becuase there are nine coordinate systems on the hyperboloid which separate the Schrödinger equation, there are $8! = 40320$ of such differential equations. This is not tractable, and one has to look for alternative procedures, for instance physical arguments. In this respect, we have found the relevant potentials which matters from a physical point of view, and which are the analogues of the flat space limit $\mathbb{R}^2$. This can be summarized in Table 3, where the enumeration of the potentials in $\mathbb{R}^2$ is according to [21]; and the enumeration of the potentials on $S^{(2)}$, according to [22]. In parenthesis we have indicated the number of limiting coordinate systems for $R \to \infty$, and constants in this limit are not taken into account. We see that the correspondence for the superintegrable systems on the hyperboloid and in flat space is complete, whereas the correspondence with the sphere is not complete. Note that adding to $V_3(u)$ the (constant!) term $\frac{\hbar^2}{2MR^2}(\kappa^2 - 1/4)$ reproduces for $R \to \infty$ the Holt potential $V_2(x)$!

8. Our discussion lacks a proper treatment of the alternative flat space limit, i.e., the limit of the two-dimensional Minkowski space, respectively the two-dimensional pseudo-Euclidean space. We do not know anything about superintegrable systems in this space. The free motion has been discussed in [20,36], and the separation of variables of the Schrödinger equation, respectively the path integral, is possible in ten coordinate systems. It is therefore desirable to construct and study appropriate superintegrable systems, an oscillator and a Coulomb potential in particular, in this space. Studies along these lines will be the subject of a future publication.

In a forthcoming publication we will deal with superintegrable potentials on the three-dimensional hyperboloid. This will include a detailed discussion of the relevant coordinate systems and the constants of motion. Concerning maximally superintegrable potentials like the oscillator and the Coulomb potential the situation is similar as in $\mathbb{R}^3$ and on the sphere, however, there are more coordinate systems which admit separation of variables for these two potentials. This property is due to the fact that on $\Lambda^{(3)}$ there exist 34 coordinate systems which admit separation of variables in the Schrödinger, respectively Helmholtz equation [54].

The situation is surprisingly different for minimally superintegrable potentials due to the subgroup structure of SO(3,1), i.e., we have $SO(3,1) \supset SO(2,1)$, $SO(3,1) \supset E(2)$ and $SO(3,1) \supset SO(3)$. This means that all potentials which are maximally superintegrable in the corresponding subspace are minimally superintegrable on $\Lambda^{(3)}$, and this property increases the number of potentials considerably.

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REFERENCES

1. **Barut A.O., Inomata A., Junker G.** — J.Phys.A: Math.Gen., 1990, vol.23, p.1179.
2. **Böhm M., Junker G.** — J.Math.Phys., 1987, vol.28, p.1978.
3. **Demkov Yu.N.** — Sov.Phys.JETP, 1954, vol.26, p.757.
4. **Duru I.H.** — Phys.Rev.D, 1984, vol.30, p.2121.
5. **Duru I.H., Kleinert H.** — Fortschr.Phys., 1982, vol.30, p.401; Phys.Lett.B, 1979, vol.84, p.185.
6. **Evans N.W.** — Phys.Rev.A, 1990, vol.41, p.5666.
7. **Feynman R.P., Hibbs A.** — Quantum Mechanics and Path Integrals, McGraw Hill, New York, 1965.

8. **Fischer W., Leschke H., Müller P.** — Ann.Phys.(N.Y.), 1993, vol.227, p.206.
9. **Fradkin D.M.** — Prog.Theor.Phys., 1967, vol.37, p.798.
10. **Fris J., Mandrosov V., Smorodinsky Ya.A., Uhlir M., Winternitz P.** — Phys.Lett., 1965, vol.16, p.354.
Fris J., Mandrosov V., Smorodinsky Ya.A., Uhlir M., Winternitz P. — Sov. J. Nucl. Phys., 1967, vol.4, p.444.
11. **Granovsky Ya.A., Zhedanov A.S., Lutzenko I.M.** — Theor.Math.Phys., 1992, vol.91, p.474.
12. **Granovsky Ya.A., Zhedanov A.S., Lutzenko I.M.** — Theor.Math.Phys., 1992, vol.91, p.604.
13. **Gradsteyn I.S., Ryzhik I.M.** — Table of Integrals, Series and Products, Academic Press, New York, 1980.
14. **Grosche C.** — J.Phys.Math.Gen., 1989, vol.22, p.5073.
15. **Grosche C.** — Ann.Phys.(N.Y.), 1990, vol.201, p.258.
16. **Grosche C.** — Ann.Phys.(N.Y.), 1990, vol.204, p.208.
17. **Grosche C.** — Fortschr.Phys., 1990, vol.38, p.531.
18. **Grosche C.** — J.Phys.Math.Gen., 1990, vol.23, p.4885.
19. **Grosche C.** — J.Phys.Math.Gen., 1992, vol.25, p.4211.
20. **Grosche C.** — Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae. DESY Report, DESY 95-021, February 1995 (Habilitationsschrift), pp.228, to be published by World Scientific.
21. **Grosche C., Pogosyan G.S., Sissakian A.N.** — Fortschr.Phys., 1995, vol.43, p.453.
22. **Grosche C., Pogosyan G.S., Sissakian A.N.** — Fortschr.Phys., 1995, vol.43, p.523.
23. **Grosche C., Pogosyan G.S., Sissakian A.N.** — Path Integral Discussion for Smorodinsky-Winternitz Potentials: IV. The Three-Dimensional Hyberboloid. DESY Report, in preparation.
24. **Grosche C., Steiner F.** — Phys.Lett.A, 1087, vol.123, p.319.
25. **Grosche C., Steiner F.** — Zeitschr.Phys., 1987, vol.C36, p.699.
26. **Grosche C., Steiner F.** — Ann.Phys.(N.Y.), 1988, vol.182, p.120.
27. **Grosche C., Steiner F.** — J.Math.Phys., 1995, vol.36, p.2354.
28. **Grosche C., Steiner F.** — Table of Feynman Path Integrals. To appear in: Springer Tracks in Modern Physics (1996).
29. **Hietarinta J.** — Phys.Rep., 1987, vol.147, p.87.
30. **Higgs P.W.** — J.Phys.A: Math.Gen., 1979, vol.12, p.309.
31. **Holt C.R.** — J.Math.Phys., 1982, vol.23, p.1037.
32. **Ikeda M., Katayama N.** — Tensor, 1982, vol.38, p.37.
33. **Infeld L.** — Phys.Rev., 1941, vol.59, p.737.
34. **Infeld L., Schild A.** — Phys.Rev., 1945, vol.67, p.121.
35. **Inomata A., Kuratsuji H., Gerry C.C.** — Path Integrals and Coherent States of SU(2) and SU(1,1), World Scientific, Singapore, 1992.
36. **Kalnins E.G.** — SIAM J.Math.Anal., 1975, vol.6, p.340.
37. **Kalnins E.G.** — Separation of Variables for Riemannian Spaces of Constant Curvature, Longman Scientific & Technical, Essex, 1986.
38. **Kalnins E.G., Miller W., Jr.** — J.Math.Phys., 1974, vol.15, p.1263.
39. **Kalnins E.G., Miller W., Jr.** — J.Math.Phys., 1975, vol.16, p.1512.
40. **Kalnins E.G., Miller W., Jr.** — J.Math.Phys., 1978, vol.19, p.1233.

41. **Katayama N.** — Nuovo Cimento, 1992, vol.B107, p.763.
42. **Kleinert H.** — Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics, World Scientific, Singapore, 1990.
43. **Kleinert H., Mustapic I.** — J.Math.Phys., 1992, vol.33, p.643.
44. **Kurochkin Yu.A., Otchik V.S.** — DAN BSSR, 1979, vol.23, p.987.
45. **Leemon H.I.** — J.Phys.A: Math.Gen., 1979, vol.12, p.489.
46. **Magnus W., Oberhettinger F., Soni R.** — Formulas and Theorems for the Special Functions of Mathematical Physics, Springer, Berlin, 1966.
47. **Makarov A.A., Smorodinsky J.A., Valiev Kh., Winternitz P.** — Nuovo Cimento A, 1967, vol.52, p.1061.
48. **Mardoyan L.G., Pogosyan G.S., Sissakian A.N., Ter-Antonyan V.M.** — Theor.Math.Phys., 185, vol.65, p.1113.
49. **Mardoyan L.G., Pogosyan G.S., Sissakian A.N., Ter-Antonyan V.M.** — Theor.Math.Phys., 1984, vol.61, p.1021.
50. **Meixner J., Schäfke F.W.** — Mathieusche Funktionen und Sphäroidfunktionen, Springer, Berlin, 1954.
51. **Moon F., Spencer D.** — Proc.Amer.Math.Soc., 1952, vol.3, p.635.
52. **Morse P.M., Feshbach H.** — Methods of Theoretical Physics, McGraw-Hill, New York, 1953.
53. **Nishino Y.** — Math.Japonica, 1972, vol.17, p.59.
54. **Olevskii M.N.** — Math.Sb., 1950, vol.27, p.379.
55. **Otchik V.S., Red'kov V.M.** — Quantum Mechanical Kepler Problem in Space with Constant Curvature, Minsk Preprint No.298, 1983, p.47;
Bogush A.A., Otchik V.S., Red'kov V.M. — Vesti Akad.Nauk.BSSR, 1983, vol.3, p.56.
Bogush A.A., Kurochkin Yu.A., Otchik V.S. — DAN BSSR, 1980, vol.24, p.19.
56. **Peak D., Inomata A.** — J.Math.Phys., 1969, vol.10, p.1422.
57. **Perelomov A.M.** — Integrable Systems of Classical Mechanics and Lie Algebras, Birkhäuser, Basel, 1990.
58. **Pogosyan G.S., Sissakian A.N., Vinitsky S.I.** — In: «Frontiers of Fundamental Physics», p.429—436, eds: M.Barone and F.Selleri, Plenum Publishing, New York, 1994.
59. **Schrödinger E.** — Proc.Roy.Irish Soc., 1941, vol.46, p.9; Proc.Roy.Irish Soc., 1941, vol.46, p.183; Proc.Roy.Irish Soc., 1941, vol.47, p.53.
60. **Schulman L.S.** — Techniques and Applications of Path Integration, John Wiley & Sons, New York, 1981.
61. **Smorodinsky Ya.A., Tugov I.I.** — Sov.Phys.JETP, 1966, vol.23, p.434.
62. **Stevenson A.F.** — Phys.Rev., 1941, vol.59, p.842.
63. **Vinitsky S.I., Mardoyan L.G., Pogosyan G.S., Sissakian A.N., Strizh T.A.** — Phys. At. Nucl., 1993, vol.56, p.321.
64. **Winternitz P., Lukac I., Smorodinsky Ya.A.** — Sov. J. Nucl. Phys., 1968, vol.7, p.139.
65. **Winternitz P., Smorodinsky Ya.A., Uhlir M., Fris I.** — Sov. J. Nucl. Phys., 1967, vol.4, p.444.
66. **Wojciechowski S.** — Phys.Lett.A, 1983, vol.95, p.279.