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PROPAGATION OF SIGNALS IN SPACES  
WITH AFFINE CONNECTIONS AND METRICS  
AS MODELS OF SPACE-TIME

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INTRODUCTION	1185
NULL (ISOTROPIC) VECTOR FIELDS. DEFINITION AND PROPERTIES	1187
DISTANCE AND VELOCITY IN A $(\bar{L}_n, g)$ SPACE	1195
KINEMATIC EFFECTS RELATED TO THE RELATIVE VELOCITY AND TO THE RELATIVE ACCELERATION	1203
ABERRATION	1212
DOPPLER EFFECT	1213
HUBBLE EFFECT	1218
COMMENTS	1234
CONCLUSIONS	1236
APPENDIX 1	1236
APPENDIX 2	1239
APPENDIX 3	1242
REFERENCES	1257

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## PROPAGATION OF SIGNALS IN SPACES WITH AFFINE CONNECTIONS AND METRICS AS MODELS OF SPACE-TIME

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The propagation of signals in space-time is considered on the basis of the notion of null (isotropic) vector field in spaces with affine connections and metrics  $(\overline{L}_n, g)$  spaces as models of space or space-time. The Doppler effect is generalized for this type of spaces. The notions of the aberration, standard (longitudinal) Doppler effect, and transversal Doppler effect are introduced. On their grounds, the Hubble effect appears as the Doppler effect with explicit forms of the centrifugal (centripetal) and Coriolis velocities and accelerations in spaces with affine connections and metrics. The Doppler effect, Hubble effect and aberration could be used in mechanics of continuous media and in other classical field theories in the same way as a standard Doppler effect is used in classical and relativistic mechanics.

Рассматривается распространение сигналов в пространстве-времени на основе понятия нулевого (изотропного) векторного поля в пространствах с аффинными связностями и метриками  $(\overline{L}_n, g)$ -пространства, используемых в качестве моделей пространства или пространства-времени. Обобщается эффект Доплера для этих типов пространств. Вводятся понятия аберрации, обычного (продольного) и поперечного эффектов Доплера. На их основе эффект Хаббла возникает как эффект Доплера с явными выражениями для центробежных (центростремительных) и кориолисовых скоростей и ускорений в пространствах с аффинными связностями и метриками. Эффект Доплера, эффект Хаббла и аберрация могут быть использованы в механике сплошных сред и других классических полевых теориях таким же образом, как обычный эффект Доплера используется в классической и релятивистской механиках.

### INTRODUCTION

1. Modern problems of relativistic astrophysics as well as of relativistic physics (dark matter, dark energy, evolution of the Universe, measurement of velocities of moving objects, etc.) are related to the propagation of signals in space or in space-time. The basis of experimental data received as a result of observations of the Doppler effect or of the Hubble effect as well as the difficulties in finding out the correct definitions of some basic notions

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such as radial (centripetal, centrifugal) velocity [1] give rise to theoretical considerations about the theoretical status of effects related to detecting signals from emitters moving relatively to observers carrying detectors in their laboratories.

2. In classical physics, the Doppler effect is considered only as a longitudinal effect in a continuous media or in vacuum. It is assumed that the reason for this effect is the relative velocity between emitter and observer. In acoustics, the signals are considered as propagating in a continuous media.

3. In relativistic physics, the Doppler effect is considered in special relativity as a (standard) longitudinal and transversal effect caused by the motion of electromagnetic signals in vacuum with respect to an observer (detector). In general relativity, the Doppler effect is related to the propagation of light in astrophysical systems and to the existence of the red shift relation due to the Hubble effect and Hubble law. The question arises whether the theoretical basis for the use of the Doppler and Hubble effects as tools for check-up of theoretical models in astrophysics and relativistic physics in sophisticated models of space-time such as spaces with affine connections and metrics  $(\overline{L}_n, g)$  spaces) is sufficiently worked out. It has been recently shown that every classical (nonquantized) field theory could be considered as a theory of continuous media [2–5]. On this basis, the propagation of signals in different models of space or of space-time is worth being investigated. From this point of view, questions arise as how the Doppler effect is related to the Hubble effect from the point of view of the kinematic characteristics of a continuous media and, especially, is there a relation between the Doppler and Hubble effects and the relative accelerations between emitters and detectors. In a previous paper [6] the Doppler and Hubble effects are considered on the basis of dimension preconditions with relations to the relative velocity between an emitter and an observer. In this paper we will use the properties of a covariant exponential operator for finding out the change of a null (isotropic) vector field along the world line of an observer (detector). It is assumed that the null vector field is related to the propagation of a signal when at a given time moment in the frame of the observer the emitter and the observer are at rest. After that moment the observer could detect the relative motion of the emitter and observe the frequency shifts of its signals.

4. The notion of null (isotropic) vector field is related to the light propagation described in relativistic electrodynamics on the basis of special and general relativity theories [7–9]. On the other side, the notion of null (isotropic) vector field could be considered in spaces with definite or indefinite metric as a geometric object (contravariant vector field) with specific properties making it useful in the description of the propagation of signals in space or in space-time as well as in geometrical optics based on different mathematical models. Usually, it is assumed that a signal is propagating with limited velocity through a continuous media or in vacuum. The velocity of propagation of signals could be a constant

quantity or a nonconstant quantity depending on the properties of the space or the space-time, where the signals are transmitted and propagated.

5. In the present paper the notion of contravariant null (isotropic) vector field is introduced and considered in spaces with affine connections and metrics  $((\bar{L}_n, g)$  spaces). In Sec.1 the properties of null vector fields are considered on the basis of  $(n - 1) + 1$  representation of non-null (nonisotropic) vector fields orthogonal to each other. In Sec.2 the notions of distance and space velocity are discussed and their relations to null vector fields are investigated. In Sec.3 the kinematic effects (aberration, longitudinal and transversal Doppler effects, and Hubble effect) related to the kinematic characteristics of the relative velocity and the relative acceleration as well as their connections with null vector fields are considered, and the kinematic effects related to the relative velocity and the relative acceleration are recalled. In Sec.4 the aberration of signals is considered as corollary of the change of a null vector field along the world line of an observer. In Sec.5 different types of the Doppler effect are introduced and investigated. In Sec.6 the Hubble effect as a Doppler effect with explicitly given forms of the relative velocities and the relative accelerations is considered. It is shown that the Hubble effect appears as a corollary of the standard (longitudinal) and transversal Doppler effects. On the other side, the Hubble effect is closely related to centrifugal (centripetal) and Coriolis velocities and accelerations. The results discussed in the paper could be important from the point of view of the possible applications of kinematic characteristics in continuous media mechanics as well as in classical (nonquantum) field theories in spaces with affine connections and metrics. Section 7 comprises concluding remarks.

The main results in the paper are given in detail (even in full detail) for those readers who are not familiar with the considered problems. Almost all definitions and abbreviations are explained. The abbreviations and definitions not determined in the paper are identical to those used in [4,5]. The reader is kindly asked to refer to them for more details and explanations of the statements and results only cited in this paper.

## 1. NULL (ISOTROPIC) VECTOR FIELDS. DEFINITION AND PROPERTIES

**1.1. Definition of a Null (Isotropic) Vector Field.** Let us now consider a space with affine connections and metrics  $((\bar{L}_n, g)$  space) [10,11] as a model of a space or of a space-time. In this space the length  $l_v$  of a contravariant vector field  $v \in T(M)$  is defined by the use of the covariant metric tensor field (covariant metric)  $g \in \otimes_{2s}(M)$  as

$$g(v, v) = \pm l_v^2, \quad l_v^2 \geq 0. \quad (1)$$

*Remark.* The sign before  $l_v^2$  depends on the signature  $\text{Sgn } g$  of the covariant metric  $g$ .  $M$  is differentiable manifold,  $\dim M = n$ . A  $(\bar{L}_n, g)$  space is a differentiable manifold  $M$  provided with contravariant and covariant affine connections (whose components differ not only by sign) and metrics.  $T(M) = \cup_{x \in M} T_x(M)$ .  $T_x(M)$  is the tangent space at a point  $x \in M$ .  $\otimes_{2s}(M)$  is the space of covariant symmetric tensors  $g$  of second rank with  $\det g \neq 0$  over  $M$ .

The contravariant vector fields can be divided into two classes with respect to their lengths:

- null or isotropic vector fields with length  $l_v = 0$ ,
- non-null or nonisotropic vector fields with length  $l_v \neq 0$ .

In the case of a positive definite covariant metric  $g$  ( $\text{Sgn } g = \pm n$ ,  $\dim M = n$ ) the null (isotropic) vector field is identically equal to zero, i.e., if  $l_v = 0$ , then  $v = v^i e_i \equiv \mathbf{0} \in T(M)$ ,  $v^i \equiv 0$ .

In the case of an indefinite covariant metric  $g$  ( $\text{Sgn } g < n$  or  $\text{Sgn } g > -n$ ,  $\dim M = n$ ) the null (isotropic) vector field with equal to zero length  $l_v = 0$  can have different from zero components in an arbitrary given basis, i.e., it is not identically equal to zero at the points, where it has been defined, i.e., if  $l_v = 0$ , then  $v \neq \mathbf{0} \in T(M)$ ,  $v = v^i \cdot e_i \in T(M)$  and  $v^i \neq 0$ . In a  $(\bar{L}_n, g)$  space the components  $g_{ij}$  of a covariant metric tensor  $g$  could be written in a local co-ordinate system at a given point of the space as  $g_{ij} = \underbrace{(-1, -1, -1, \dots, +1, +1, +1, \dots)}_{\substack{k \text{ times} \\ l \text{ times}}}$  with  $k + l = n$ .

The signature  $\text{Sgn } g$  is defined as

$$\text{Sgn } g = -k + l = 2l - n = n - 2k, \quad n, k, l \in \mathbf{N}, \quad (2)$$

where  $k = n - l$ ,  $l = n - k$ .

In the relativistic physics for  $\dim M = 4$ , the numbers  $l$  and  $k$  are chosen as  $l = 1$ ,  $k = 3$  or  $l = 3$ ,  $k = 1$  so that  $\text{Sgn } g = -2 \sim (-1, -1, -1, +1)$  or  $\text{Sgn } g = +2 \sim (+1, +1, +1, -1)$ . In general, a  $(\bar{L}_n, g)$  space could be considered as a model of space-time with  $\text{Sgn } g < 0$  and ( $k > l$ ,  $l = 1$ ) or with  $\text{Sgn } g > 0$  and ( $l > k$ ,  $k = 1$ ).

The non-null (nonisotropic) contravariant vector fields are divided into two classes.

1. For  $\text{Sgn } g < 0$ 
  - (a)  $g(v, v) = +l_v^2 > 0$  := time-like vector field  $v \in T(M)$ ,
  - (b)  $g(v, v) = -l_v^2 < 0$  := space-like vector field  $v \in T(M)$ .
2. For  $\text{Sgn } g > 0$ 
  - (a)  $g(v, v) = -l_v^2 < 0$  := time-like vector field  $v \in T(M)$ ,
  - (b)  $g(v, v) = +l_v^2 > 0$  := space-like vector field  $v \in T(M)$ .

Therefore, if we do not fix a priori the signature of the space-time models, we can distinguish a *time-like vector field*  $u$  with

$$\begin{aligned} g(u, u) &= +l_u^2 \quad \text{for } \text{Sgn } g < 0, \\ &= -l_u^2 \quad \text{for } \text{Sgn } g > 0, \end{aligned}$$

or  $g(u, u) = \pm l_u^2$ , and a *space-like vector field*  $\xi_\perp$  with

$$\begin{aligned} g(\xi_\perp, \xi_\perp) &= -l_{\xi_\perp}^2 \quad \text{for } \text{Sgn } g < 0, \\ &= +l_{\xi_\perp}^2 \quad \text{for } \text{Sgn } g > 0, \end{aligned}$$

or  $g(\xi_\perp, \xi_\perp) = \mp l_{\xi_\perp}^2$ . This means that in symbols  $\pm l_\diamond^2$  or  $\mp l_\diamond^2$  ( $\diamond \in T(M)$ ) the sign above is related to  $\text{Sgn } g < 0$  and the sign below is related to  $\text{Sgn } g > 0$ .

*Remark.* Since  $l_\diamond = \pm \sqrt{l_\diamond^2}$ , the signs in this case will be denoted as not related to the signature of the metric  $g$ . Usually, it is assumed that  $l_\diamond = +\sqrt{l_\diamond^2} \geq 0$ .

A non-null (nonisotropic) contravariant vector field  $v$  could be represented by its length  $l_v$  and its corresponding unit vector  $n_v = v/l_v$  as  $v = \pm l_v n_v$  in contrast to a null (isotropic) vector field  $\tilde{k}$  with  $l_{\tilde{k}} = 0$  (the signs here are not related to the signature of the metric  $g$ )

$$v = \pm l_v n_v, \quad g(v, v) = l_v^2 g(n_v, n_v) = \pm l_v^2, \quad g(n_v, n_v) = \pm 1,$$

for time-like unit vector field  $n_v$ , or

$$v = \mp l_v n_v, \quad g(v, v) = l_v^2 g(n_v, n_v) = \mp l_v^2, \quad g(n_v, n_v) = \mp 1,$$

for space-like unit vector field  $n_v$ .

*Remark.* In the experimental physics, the measurements are related to the lengths and to the directions of a non-null (nonisotropic) vector field with respect to a frame of reference. Since different types of co-ordinates could be used in a frame of reference, the components of a vector field related to these co-ordinates cannot be considered as invariant characteristics of the vector field and on this grounds the components cannot be important characteristics for the vector fields.

After these preliminary remarks, we can introduce the notion of a null (isotropic) vector field

*Definition 1.* A contravariant vector field  $\tilde{k}$  with length zero is called null (isotropic) vector field, i.e.,  $\tilde{k} := \text{null (isotropic) vector field}$  if

$$g(\tilde{k}, \tilde{k}) = \pm l_{\tilde{k}}^2 = 0, \quad l_{\tilde{k}} = |g(\tilde{k}, \tilde{k})|^{1/2} = 0. \tag{3}$$

**1.2. Properties of a Null (Isotropic) Vector Field.** The properties of a null (isotropic) contravariant vector field could be considered in a  $(n - 1) + 1$  invariant decomposition of a space-time by the use of two nonisotropic contravariant

vector fields  $u$  and  $\xi_\perp$ , orthogonal to each other [10], i.e.,  $g(u, \xi_\perp) = 0$ . The contravariant vectors  $u$  and  $\xi_\perp$  are essential elements of the structure of a frame of reference [12] in a space-time.

*1.2.1. Invariant Representation of a Null Vector Field by the Use of a Non-Null Contravariant Vector Field.*

(a) *Invariant projections of a null vector field along and orthogonal to a non-null (nonisotropic) contravariant vector field  $u$ .*

Every contravariant vector field  $\tilde{k} \in T(M)$  could be represented in the form

$$\tilde{k} = \frac{1}{e}g(u, \tilde{k})u + \bar{g}[h_u(\tilde{k})] = k_\parallel + k_\perp, \quad (4)$$

where

$$\begin{aligned} e &= g(u, u) = \pm l_u^2, \\ \bar{g} &= g^{ij}\partial_i\partial_j, & \partial_i\partial_j &= \frac{1}{2}(\partial_i \otimes \partial_j + \partial_j \otimes \partial_i), \\ g &= g_{ij}dx^i dx^j, & dx^i dx^j &= \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i), \\ h_u &= g - \frac{1}{e}g(u) \otimes g(u), & h^u &= \bar{g} - \frac{1}{e}u \otimes u, \\ g(u) &= g_{i\bar{j}}u^{\bar{j}}dx^i, \\ \bar{g}[h_u(\tilde{k})] &= g^{ij}h_{\bar{j}i}\tilde{k}^{\bar{l}}\partial_i := k_\perp, & k_\parallel &:= \frac{1}{e}g(u, \tilde{k})u. \end{aligned} \quad (5)$$

$$g(k_\parallel, k_\perp) = 0, \quad g(u, k_\perp) = 0. \quad (6)$$

Let us now take a closer look at the first term  $k_\parallel$  of the representation of  $\tilde{k}$ .

$$k_\parallel = \frac{1}{e}g(u, \tilde{k})u = \pm \frac{1}{l_u^2}g(u, \tilde{k})u = \pm \frac{1}{l_u}g(u, k_\parallel)\frac{u}{l_u}. \quad (7)$$

If we introduce the abbreviations

$$n_\parallel = \frac{u}{l_u}, \quad \omega = g(u, \tilde{k}) = g(u, k_\parallel), \quad (8)$$

where

$$g(n_\parallel, n_\parallel) = \frac{1}{l_u^2}g(u, u) = \frac{1}{l_u^2}(\pm l_u^2) = \pm 1, \quad (9)$$

$$\omega = g(u, \tilde{k}) = g(u, k_\parallel + k_\perp) = g(u, k_\parallel) = l_u g(n_\parallel, k_\parallel) = l_u g(k_\parallel, n_\parallel), \quad (10)$$

$$k_\parallel := \pm l_{k_\parallel} n_\parallel, \quad (11)$$

$$g(k_\parallel, k_\parallel) = l_{k_\parallel}^2 g(n_\parallel, n_\parallel) = l_{k_\parallel}^2 (\pm 1) = \pm l_{k_\parallel}^2, \quad (12)$$

$$g(k_{\parallel}, n_{\parallel}) = \pm l_{k_{\parallel}} g(n_{\parallel}, n_{\parallel}) = \pm l_{k_{\parallel}} (\pm 1) = l_{k_{\parallel}} = \frac{\omega}{l_u}, \quad (13)$$

then  $k_{\parallel}$  could be expressed as (the signs are not related to the signature of the metric  $g$ )

$$k_{\parallel} = \pm \frac{\omega}{l_u} n_{\parallel} = \pm l_{k_{\parallel}} n_{\parallel}. \quad (14)$$

The vector  $n_{\parallel}$  is a unit vector [ $g(n_{\parallel}, n_{\parallel}) = \pm 1$ ] collinear to  $u$  and, therefore, tangential to a curve with parameter  $\tau$  if  $u = d/(d\tau)$ .

The scalar invariant  $\omega = g(u, \tilde{k})$  is usually interpreted as the frequency of the radiation related to the null vector field  $\tilde{k}$  and propagating with velocity  $u$  with absolute value  $l_u$  with respect to the trajectory  $x^i(\tau)$ . In general relativity  $l_u := c$ , and it is assumed that the radiation is of electromagnetic nature propagating with the velocity of light  $c$  in vacuum. We will come back to this interpretation in the next considerations.

The contravariant vector field  $k_{\perp}$

$$k_{\perp} = \bar{g}[h_u(\tilde{k})]$$

is orthogonal to  $u$  (and  $k_{\parallel}$ , respectively) part of  $\tilde{k}$ . Since

$$g(k_{\parallel}, k_{\parallel}) = g\left(\pm \frac{\omega}{l_u} n_{\parallel}, \pm \frac{\omega}{l_u} n_{\parallel}\right) = \frac{\omega^2}{l_u^2} g(n_{\parallel}, n_{\parallel}) = \pm \frac{\omega^2}{l_u^2} = \pm l_{k_{\parallel}}^2, \quad (15)$$

$$l_{k_{\parallel}} = \frac{\omega}{l_u} > 0, \quad l_{k_{\parallel}}^2 = \frac{\omega^2}{l_u^2}, \quad (16)$$

and

$$g(\tilde{k}, \tilde{k}) = 0, \quad g(k_{\parallel}, k_{\perp}) = 0,$$

we have for  $g(k_{\perp}, k_{\perp})$

$$\begin{aligned} g(\tilde{k}, \tilde{k}) = 0 &= g(k_{\parallel} + k_{\perp}, k_{\parallel} + k_{\perp}) = g(k_{\parallel}, k_{\parallel}) + g(k_{\perp}, k_{\perp}) = \\ &= \pm \frac{\omega^2}{l_u^2} + g(k_{\perp}, k_{\perp}) = \pm \frac{\omega^2}{l_u^2} \mp l_{k_{\perp}}^2 = 0, \end{aligned} \quad (17)$$

and, therefore,

$$l_{k_{\perp}}^2 = \frac{\omega^2}{l_u^2}, \quad l_{k_{\perp}} = \frac{\omega}{l_u} = l_{k_{\parallel}}. \quad (18)$$

*Remark.* Since  $\omega \geq 0$  and  $l_u > 0$ , and at the same time  $l_{k_{\perp}} > 0$ , and  $l_{k_{\parallel}} > 0$ , we have

$$l_{k_{\parallel}} = \frac{\omega}{l_u} = l_{k_{\perp}}.$$

If we introduce the unit contravariant vector  $\tilde{n}_\perp$  with  $g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1$ , then the vector  $k_\perp$  could be written as

$$k_\perp := \mp l_{k_\perp} \tilde{n}_\perp, \quad (19)$$

where

$$g(k_\perp, k_\perp) = l_{k_\perp}^2 g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp l_{k_\perp}^2 = \mp \frac{\omega^2}{l_u^2}, \quad l_{k_\perp}^2 = \frac{\omega^2}{l_u^2}, \quad l_{k_\perp} = \frac{\omega}{l_u}. \quad (20)$$

Therefore,

$$k_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp, \quad k_\parallel = \pm \frac{\omega}{l_u} n_\parallel, \quad (21)$$

$$\tilde{k} = k_\parallel + k_\perp = \pm \frac{\omega}{l_u} (n_\parallel - \tilde{n}_\perp), \quad (22)$$

where

$$g(n_\parallel, \tilde{n}_\perp) = 0, \quad g(k_\parallel, \xi_\perp) = 0, \quad g(u, k_\perp) = 0, \quad (23)$$

$$g(n_\parallel, k_\parallel) = \pm l_{k_\parallel} g(n_\parallel, n_\parallel) = l_{k_\parallel} = \frac{\omega}{l_u}, \quad (24)$$

$$g(\tilde{n}_\perp, k_\perp) = \mp l_{k_\perp} g(n_\perp, n_\perp) = \frac{\omega}{l_u} = l_{k_\parallel} = l_{k_\perp}, \quad (25)$$

$$g(n_\parallel, k_\parallel) = g(\tilde{n}_\perp, k_\perp) = \frac{\omega}{l_u} = l_{k_\parallel} = l_{k_\perp}. \quad (26)$$

*Remark.* The signs, not related to the metric  $g$ , are chosen to be the same with the signs related to the metric  $g$ .

We have now the relations

$$\omega = g(u, \tilde{k}) = l_u g(n_\parallel, k_\parallel) = l_u g(\tilde{n}_\perp, k_\perp). \quad (27)$$

If  $\tilde{n}_\perp$  is interpreted as the unit vector in the direction of the propagation of a signal in the subspace orthogonal to the contravariant vector field  $u$ , and  $l_u$  is interpreted as the absolute value of the velocity of the radiated signal, then  $l_u \tilde{n}_\perp$  is the path along  $\tilde{n}_\perp$  propagated by the signal in a unit time interval. Then

$$\omega = g(u_\perp, k_\perp), \quad u_\perp := l_u \tilde{n}_\perp, \quad g(u, u_\perp) = 0. \quad (28)$$

*1.2.2. Explicit Form of the Vector Field  $k_\perp$  and Its Interpretation.* Let us now consider more closely the explicit form of  $k_\perp$

$$k_\perp = \mp l_{k_\perp} \tilde{n}_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp. \quad (29)$$

(a) In 3-dimensional Euclidean space (as the model of space-time of the Newtonian mechanics) the wave vector  $\mathbf{k}$  is defined as

$$\mathbf{k} = \frac{2\pi}{\lambda} \mathbf{n}, \tag{30}$$

where  $\mathbf{n}$  is the unit 3-vector in the direction of propagation of a signal with absolute value of its velocity  $l_u = \lambda\nu$ . If we express  $\lambda$  by  $\lambda = l_u/\nu$  and put the equivalent expression in this for  $\mathbf{k}$ , we obtain the expression

$$\mathbf{k} = \frac{2\pi\nu}{l_u} \mathbf{n} = \frac{\omega}{l_u} \mathbf{n}, \tag{31}$$

which (up to a sign depending on the signature of the metric  $g$ ) is identical with the expression for  $k_\perp$  for  $n = 3$  if  $k_\perp = \mathbf{k}$ ,  $\mathbf{n} = \tilde{n}_\perp$ , and  $\omega = 2\pi\nu$ .

(b) In 4-dimensional (pseudo) Riemannian space (as a model of space-time of the Einstein theory of gravitation)  $l_u$  is interpreted as the absolute value of the velocity of light in vacuum (normalized by some authors to 1), i.e.,  $l_u = c, 1$ . Then

$$k_\perp = \mp \frac{\omega}{c} \tilde{n}_\perp = \mp \frac{2\pi\nu}{\lambda\nu} \tilde{n}_\perp = \mp \frac{2\pi}{\lambda} \tilde{n}_\perp, \tag{32}$$

and we obtain the expression for the wave vector of light propagation in general relativity, where  $\tilde{n}_\perp$  is the unit vector along the propagation of light in the corresponding 3-dimensional subspace of an observer with world line  $x^i(\tau)$  if

$$u = \frac{d}{d\tau} = l_u n_\parallel, \quad n_\parallel = \frac{1}{l_u} \frac{d}{d\tau}, \tag{33}$$

$l_u$  is the velocity of light measured by the observer.

(c) In the general case for  $k_\perp$  as

$$k_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp, \tag{34}$$

$\omega$  could also be interpreted as the frequency of a signal propagating with velocity with absolute value  $l_u$  in a frame of reference of an observer with world line  $x^i(\tau)$ . The unit vector  $\tilde{n}_\perp$  is the unit vector in the direction of the propagation of the signal in the subspace orthogonal to the vector  $u$ . The velocity of the observer is usually defined by the use of the parameter  $\tau$  of the world line under the assumption that  $ds = l_u d\tau$ , where  $ds$  is the distance of the propagation of a signal for the proper time interval  $d\tau$  of the observer

$$u = \frac{d}{d\tau} = \frac{d}{(1/l_u)ds} = l_u \frac{d}{ds}. \tag{35}$$

*Remark.* Usually the velocity of a particle (observer) moving in space-time is determined by its velocity vector field  $u = d/(d\tau)$ , where  $\tau$  is the proper time

of the observer. The parameter  $\tau$  is considered as a parameter of the observer's world line  $x^i(\tau)$ . By the use of  $u$  and its corresponding projection metrics  $h_u$  and  $h^u$ , a contravariant (non-null, nonisotropic) vector field  $\xi$  could be represented in two parts: one part is collinear to  $u$  and the other part is orthogonal to  $u$

$$\xi = \frac{1}{e}g(\xi, u)u + \bar{g}[h_u(\xi)] = \xi_{\parallel} + \xi_{\perp}, \quad (36)$$

where

$$\xi_{\parallel} = \frac{1}{e}g(\xi, u)u, \quad \bar{g}[h_u(\xi)], \quad g(\xi_{\parallel}, \xi_{\perp}) = 0. \quad (37)$$

1. If an observer is moving with velocity  $v = d/(d\bar{\tau})$  on his world line  $x^i(\bar{\tau})$ , then his velocity, considered with respect to the observer with velocity  $u$  and world line  $x^i(\tau)$ , will have two parts  $v_{\parallel}$  and  $v_{\perp}$ , collinear and orthogonal to  $u$ , respectively, at the cross point  $\tau = \bar{\tau}$  of both the world lines  $x^i(\tau)$  and  $x^i(\bar{\tau})$

$$v = \frac{1}{e}g(v, u)u + \bar{g}[h_u(v)] = v_{\parallel} + v_{\perp}. \quad (38)$$

The vector  $v_{\parallel}$  describes the motion of the observer with velocity  $v$  along the world line of the first observer with velocity  $u$ . The vector  $v_{\perp}$  describes the motion of the second observer with velocity  $v$  in direction orthogonal to the world line of the first observer. The vector  $v_{\perp}$  is the velocity of the second observer in the space of the first observer in contrast to the vector  $v_{\parallel}$  describing the change of  $v$  in the time of the first observer.

2. If we consider the propagation of a signal, characterized by its null vector field  $\tilde{k}$ , the interpretation of the vector field  $u$ , tangential to the world line of an observer, changes. The vector field  $u = l_u n_{\parallel}$  is interpreted as the velocity vector field of the signal, propagating in the space-time and measured by the observer at its world line  $x^i(\tau)$  with proper time  $\tau$  as a parameter of this world line. The absolute value  $l_u$  of  $u$  is the size of the velocity of the signal measured along the unit vector field  $n_{\parallel}$  collinear to the tangent vector of the world line of the observer.

3. In Einstein's theory of gravitation (ETG) both interpretations of the vector field  $u$  are put together. On the one side, the vector field  $u$  is interpreted as the velocity of an observer on his world line with parameter  $\tau$  interpreted as the proper time of the observer. On the other side, the length  $l_u$  of the vector field  $u$  is normalized either to  $\pm 1$  or to  $\pm c = \text{const}$ . The quantity  $c$  is interpreted as the light velocity in vacuum. The basic reason for this normalization is the possibility for normalization of every non-null (nonisotropic) vector field  $u$  in the form

$$n_u = \frac{u}{l_u} = n_{\parallel}, \quad \text{where } l_u = |g(u, u)|^{1/2} \neq 0, \quad (39)$$

by the use of its different from zero length ( $l_u \neq 0$ ), defined by means of the covariant metric tensor  $g$ .

Both the interpretations of the vector field  $u$  (as a velocity of an observer or as velocity of a signal) should be considered separately from each other for avoiding ambiguities. The identification of the interpretations should mean that we assume the existence of an observer moving in space-time with velocity  $u$  and emitting (or receiving) signals with the same velocity. Such assumption does not exist in the Einstein theory of gravitation. This problem is worth being investigated, and a clear difference between both interpretations should be found. It is related to the notions of distance and of velocity in spaces with affine connections and metrics.

## 2. DISTANCE AND VELOCITY IN A $(\bar{L}_n, g)$ SPACE

### 2.1. Distance in a $(\bar{L}_n, g)$ Space and Its Relations to the Notion of Velocity.

#### 2.1.1. Distance in a $(\bar{L}_n, g)$ Space for World and Space Lines not Depending on Each Other

1. The distance in a  $(\bar{L}_n, g)$  space between a point  $P \in M$  with co-ordinates  $x^i$  and a point  $\bar{P} \in M$  with co-ordinates  $\bar{x}^i = x^i + dx^i$  is determined by the length of the ordinary differential  $d$ , considered as a contravariant vector field  $d = dx^i \partial_i$  [4]. If we denote the distance as  $ds$  between point  $P$  and point  $\bar{P}$ , then the square  $ds^2$  of  $ds$  could be defined as the square of the length of the ordinary differential  $d$

$$ds^2 = g(d, d) = \pm l_d^2 = g_{ij} dx^i dx^j, \quad l_d^2 \geq 0. \tag{40}$$

2. Let us now consider a two-parametric congruence of curves (a set of not intersecting curves) in a  $(\bar{L}_n, g)$  space

$$x^i = x^i(\tau, r(\tau, \lambda)) = x^i(\tau, \lambda), \tag{41}$$

where the function  $r = r(\tau, \lambda) \in C^r(M)$ ,  $r \geq 2$ , depends on the two parameters  $\tau$  and  $\lambda$ ,  $\tau, \lambda \in \mathbf{R}$ . Then

$$dr = \frac{\partial r(\tau, \lambda)}{\partial \tau} d\tau + \frac{\partial r(\tau, \lambda)}{\partial \lambda} d\lambda \tag{42}$$

and

$$\begin{aligned} dx^i &= \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau} d\tau + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \left( \frac{\partial r(\tau, \lambda)}{\partial \tau} d\tau + \frac{\partial r(\tau, \lambda)}{\partial \lambda} d\lambda \right) = \\ &= \left[ \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau} + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \frac{\partial r(\tau, \lambda)}{\partial \tau} \right] d\tau + \\ &\quad + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \frac{\partial r(\tau, \lambda)}{\partial \lambda} d\lambda, \end{aligned} \tag{43}$$

or

$$dx^i = (u^i + \bar{\xi}^i l_v) d\tau + \bar{\xi}^i \frac{\partial r}{\partial \lambda} d\lambda, \quad (44)$$

where

$$u^i = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau}, \quad l_v = \frac{\partial r(\tau, \lambda)}{\partial \tau}, \quad \bar{\xi}^i = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r}, \quad (45)$$

and

$$d = dx^i \partial_i = d\tau(u + l_v \bar{\xi}) + (\partial r / \partial \lambda) d\lambda \bar{\xi}, \quad (46)$$

$$\frac{d\lambda}{d\tau} = 0, \quad \frac{d\tau}{d\lambda} = 0.$$

*Remark.* Here, the parameters  $\tau$  and  $\lambda$  are assumed to be independent of each of other functions.

If the necessary computations are fulfilled (Appendix 1), the expression for  $ds^2$  follows in the form

$$ds^2 = \pm dr^2 l_u^2 \mp d\tau^2 = \pm l_u^2 d\tau^2 \left(1 - \frac{l_v^2}{l_u^2}\right).$$

3. In the nonrelativistic field theories the distance between two points  $P \in M$  and  $\bar{P} \in M$  is defined as

$$ds^2 = \mp dr^2 \quad (47)$$

and  $l_u = 0$ . This means that the distance between two neighboring points  $P$  and  $\bar{P}$  is the *space* distance measured between them in the rest (proper) reference frame of the observer (with absolute value  $l_u$  of his velocity  $u$  equal to zero). The time parameter  $\tau$  is not considered as a co-ordinate in space-time, but as a parameter, independent of the frame of reference of the observer.

4. In the relativistic field theories and especially in the Einstein theory of gravitation  $dr$  is considered as the space distance between two neighboring points  $P$  and  $\bar{P}$  and  $l_u d\tau$  is interpreted as the distance covered by a light signal in a time interval  $d\tau$ , measured by an observer in his proper frame of reference (when the observer in it is at rest). The quantity  $l_u$  is usually interpreted as the absolute value  $c$  of a light signal in vacuum, i.e.,  $l_u = c$ , or  $l_u$  is normalized to 1, i.e.,  $l_u = 1$ , if the proper time interval  $d\tau$  is replaced with the proper distance interval  $ds = cd\tau$ , i.e.,  $\bar{u} = d/(d\tau)$  is replaced with  $\bar{u}' = d/(cd\tau) = d/(ds)$ ,  $ds = cd\tau$ .

Therefore, there is a difference between the interpretation of the absolute value  $l_u$  of the velocity of an observer in classical and relativistic physics.

(a) In classical physics, from the above consideration, it follows that  $l_u = 0$  (the observer is at rest) and  $ds = dr$  is the distance as a space distance. The

quantity  $l_v$  is the absolute value of the velocity between the observer at rest and a point  $\bar{P}$  in his neighborhood.

(b) In relativistic physics  $l_u = c$ , or  $l_u = 1$ , and  $l_u$  is not the absolute value of the velocity of the observer but the velocity of the light propagation which the observer could measure in his proper frame of reference. If we wish to interpret  $l_u$  as the absolute value of the velocity of the observer himself, we should assume that  $l_u \neq c$  or 1 (if the observer is not moving with the speed of light).

- There is the possibility to identify  $l_u$  with  $l_v$  as the absolute value of the velocity of the observer at a point  $P$  at his world line, measured with respect to a neighboring point  $\bar{P}$  with the same proper time as the point  $P$ . Under this assumption, the ordinary differential becomes a null (isotropic) vector field ( $g(d, d) = 0, l_d = 0, l_u = l_v \neq 0$ ) in the proper frame of reference of the observer.

- We could also interpret  $l_u$  as the absolute value of the velocity of the observer with respect to another frame of reference or

- we could consider  $l_u$  as the absolute value of the velocity of a signal coming to the observer with velocity, different from the velocity of light. On the basis of the last assumption we can describe the propagation of signals with propagation velocity different from the velocity of light (for instance, the propagation of sound signals or (may be) gravitational signals).

If  $l_v = 0$ , then  $\bar{u} = u$  and  $u$  could be

- the velocity vector field  $u = d/(d\tau)$  of an observer ( $l_u \neq 0, u = l_u n_{||}$ ) in his proper frame of reference along his world line. Since in his proper frame of reference the observer is at rest,  $u$  could be interpreted as the velocity of a clock measuring the length (proper time) of the world line by the use of the parameter  $\tau$  or

- the velocity of a signal detected or emitted by the observer.

There is another way for considering the ordinary differential as a contravariant vector field with its relations to the notions of velocity and of space velocity.

### 2.1.2. Distance in a $(\bar{L}_n, g)$ Space for World and Space Lines Depending on Each Other

1. Let us consider the ordinary differential  $d = dx^i \partial_i$  as a contravariant vector field over a two parameter congruence  $x^i = x^i(\tau, \lambda)$ . Then

$$dx^i = \frac{\partial x^i(\tau, \lambda)}{\partial \tau} d\tau + \frac{\partial x^i(\tau, \lambda)}{\partial \lambda} d\lambda = u^i d\tau + \xi_{\perp}^i d\lambda, \quad (48)$$

where

$$u^i = \frac{\partial x^i(\tau, \lambda)}{\partial \tau}, \quad \xi_{\perp}^i = \frac{\partial x^i(\tau, \lambda)}{\partial \lambda}. \quad (49)$$

The ordinary differential will have the form

$$d = dx^i \partial_i = d\tau u + d\lambda \xi_{\perp}, \quad u = u^i \partial_i, \quad \xi_{\perp} = \xi_{\perp}^i \partial_i. \quad (50)$$

If we impose the additional condition

$$g(u, \xi_{\perp}) = 0, \quad (51)$$

we will have the relations

$$g(d, u) = d\tau g(u, u) = ed\tau = \pm l_u^2 d\tau, \quad (52)$$

$$g(\xi_{\perp}, d) = d\lambda g(\xi_{\perp}, \xi_{\perp}) = \mp l_{\xi_{\perp}}^2 d\lambda. \quad (53)$$

2. On the other side, if we consider the projections of  $d$  collinear and orthogonal to  $u$ , we obtain the relations

$$d = \frac{1}{e}g(u, d)u + \bar{g}[h_u(d)] = d_{\parallel} + d_{\perp}, \quad (54)$$

where

$$d_{\parallel} = \frac{1}{e}g(u, d)u, \quad d_{\perp} = \bar{g}[h_u(d)]. \quad (55)$$

For the explicit form of  $d$  as  $d = d\tau u + d\lambda \xi_{\perp}$ , we have (under the condition  $g(u, \xi_{\perp}) = 0$ )

$$d_{\parallel} = \frac{1}{e}g(u, d)u = \frac{1}{e}ed\tau u = d\tau u, \quad (56)$$

$$\begin{aligned} d_{\perp} = \bar{g}[h_u(d)] &= \bar{g}[h_u(d\tau u + d\lambda \xi_{\perp})] = \\ &= \bar{g}[h_u(d\lambda \xi_{\perp})] = d\lambda \bar{g}[h_u(\xi_{\perp})] = d\lambda \xi_{\perp}. \end{aligned} \quad (57)$$

Therefore, the representation of  $d$  as  $d = d\tau u + d\lambda \xi_{\perp}$  is the representation in its collinear and orthogonal to  $u$  parts.

3. Since

$$\begin{aligned} g(d, d) = ds^2 &= g(d\tau u + d\lambda \xi_{\perp}, d\tau u + d\lambda \xi_{\perp}) = \\ &= d\tau^2 g(u, u) + d\lambda^2 g(\xi_{\perp}, \xi_{\perp}) = \pm l_u^2 d\tau^2 \mp l_{\xi_{\perp}}^2 d\lambda^2, \end{aligned} \quad (58)$$

it follows for the line element  $ds$

$$ds^2 = \pm l_u^2 d\tau^2 \mp l_{\xi_{\perp}}^2 d\lambda^2 = \pm l_u^2 d\tau^2 \left( 1 - \frac{l_{\xi_{\perp}}^2 d\lambda^2}{l_u^2 d\tau^2} \right). \quad (59)$$

If we chose the contravariant nonisotropic (non-null) vector field  $\xi_{\perp}$  as a unit vector field, i.e., if  $l_{\xi_{\perp}} = 1$ ,  $\xi_{\perp} = \tilde{n}_{\perp}$ ,  $g(\tilde{n}_{\perp}, \tilde{n}_{\perp}) = \mp 1 = \mp l_{\xi_{\perp}}^2$ , then

$$ds^2 = \pm l_u^2 d\tau^2 \left( 1 - \frac{1}{l_u^2} \frac{d\lambda^2}{d\tau^2} \right). \quad (60)$$

If we, further, interpret  $d\lambda$  as a distance along a curve with a tangent vector  $\xi_\perp$ , orthogonal to  $u$ , we can define and interpret the expression

$$\frac{d\lambda}{d\tau} = l_v \quad (61)$$

as the 3-dimensional space velocity of a material point along the curve  $x^i(\tau, \lambda)$  with tangential vector  $\xi_\perp = \tilde{n}_\perp$  along the curve  $x^i(\tau_0, \lambda)$ . Then

$$ds^2 = \pm l_u^2 d\tau^2 \left( 1 - \frac{l_v^2}{l_u^2} \right). \quad (62)$$

*Remark.* Here, it is assumed that the parameter  $\lambda$  is depending on the parameter  $\tau$ , i.e.,  $\lambda = \lambda(\tau)$ ,  $\tau = \tau(\lambda)$ , and  $d\lambda/d\tau \neq 0$ . In the opposite case, where  $\lambda$  and  $\tau$  are parameters independent of each other,  $d\lambda/d\tau = 0$ .

The quantity  $l_u$  is interpreted as the absolute value of the velocity of a signal (in the relativity theory it is interpreted as the absolute value of the velocity of light in vacuum). The parameter  $\tau$  is interpreted as the proper time of an observer moving on a trajectory  $x^i(\tau, \lambda_0)$  interpreted as his world line  $x^i(\tau, \lambda_0)$ . The quantity  $l_v$  is interpreted as the absolute value of the velocity of a material point moving along a space distance  $\lambda$  from the trajectory of the observer  $x^i(\tau, \lambda_0)$ .

4. If we now turn back to the general case, when  $l_{\xi_\perp} \neq 1$ ,  $\xi_\perp \neq \tilde{n}_\perp$ , we have the relation

$$ds^2 = \pm l_u^2 d\tau^2 \left[ 1 - \frac{1}{l_u^2} \left( \frac{l_{\xi_\perp}^2 d\lambda^2}{d\tau^2} \right) \right]. \quad (63)$$

Then we can introduce the abbreviation

$$dl^2 := l_{\xi_\perp}^2 d\lambda^2, \quad (64)$$

interpreted as the square of the *space distance* of a point in  $(n-1)$ -dimensional subspace of  $M$  ( $\dim M = n$ ), ( $n = 4$ ) from the trajectory (world line) of the observer with proper time  $\tau$  and tangential vector  $u$ , orthogonal to  $\xi_\perp = l_{\xi_\perp} \tilde{n}_\perp$ . The square  $ds^2$  of the distance  $ds$  in the  $n$ -dimensional manifold  $M$  will have the form

$$ds^2 = \pm l_u^2 d\tau^2 \left[ 1 - \frac{1}{l_u^2} \frac{dl^2}{d\tau^2} \right]. \quad (65)$$

If we again denote

$$\frac{dl^2}{d\tau^2} := \tilde{l}_v^2, \quad (66)$$

we obtain

$$ds^2 = \pm l_u^2 d\tau^2 \left( 1 - \frac{\tilde{l}_v^2}{l_u^2} \right), \quad (67)$$

where  $\bar{l}_v$  could be interpreted again as the absolute value of the space velocity of a point moving at a space distance  $dl$  from the world line  $x^i(\tau, \lambda_0)$  of the observer. In this general case, the parameter  $\lambda$  is not interpreted as a space distance. Instead of  $d\lambda$  the quantity  $dl = l_{\xi_\perp} d\lambda$  has this interpretation.

In general, we do not need to search for interpretation of an expression as a space velocity in the above considerations if we consider only the structure of the square  $ds^2$  of the space-time distance  $ds$

$$ds^2 = \pm(l_u^2 d\tau^2 - l_{\xi_\perp}^2 d\lambda^2) = \pm(l_u^2 d\tau^2 - dl^2). \quad (68)$$

If a signal with absolute value  $l_u$  of its velocity is covering a space distance  $dl$  with  $dl^2 = l_{\xi_\perp}^2 d\lambda^2$  in the proper time interval  $d\tau$  of the observer, then  $ds^2 = g(d, d) = 0$  and the ordinary differential becomes a null (isotropic) vector field, where

$$ds^2 = 0, \quad l_u^2 d\tau^2 = l_{\xi_\perp}^2 d\lambda^2 = dl^2, \quad (69)$$

$$dl = \pm l_{\xi_\perp} d\lambda, \quad l_u d\tau = \pm l_{\xi_\perp} d\lambda = dl, \quad (70)$$

$$d\tau = \frac{dl}{l_u} = \pm \frac{l_{\xi_\perp}}{l_u} d\lambda, \quad l_{\xi_\perp} d\lambda = \pm dl, \quad l_u > 0, \quad l_{\xi_\perp} > 0. \quad (71)$$

## 2.2. Measuring a Distance in $(\bar{L}_n, g)$ Spaces

A. If the notion of distance is introduced in a space-time, modeled by the  $(\bar{L}_n, g)$  space, we have to decide *what is the meaning of the vector field  $u$  as a tangent vector to a trajectory interpreted as the world line of an observer*. On the basis of the above considerations, we have four possible interpretations for the meaning of the vector field  $u$  as:

1. Velocity vector field of a propagating signal in space-time identified with the tangent vector field  $u$  at the world line of an observer. The signal is detected or emitted by the observer on his world line and the absolute value  $l_u$  of  $u$  is identified with the absolute value of the velocity of the signal in- or outcoming to the observer.

2. Velocity vector field of an observer moving in space-time. In this case  $l_u \neq 0$  and the space-time should have a definite metric, i.e.,  $\text{Sgn } g = \pm n$ ,  $\dim M = n$  (for instance, motion of an observer in the Euclidean space considered as a model of space-time). The observer, moving in space-time, could consider processes happened in its subspace orthogonal to his velocity. The observer will move in a flow and consider the characteristics of the flow from his own frame of reference.

3. Velocity of a clock moving in space-time and determining the proper time by a periodical process in the frame of reference of an observer. The velocity  $u$  of the periodical process in the clock in space-time is with fixed absolute value  $l_u$ , i.e.,  $l_u = \text{const}$ . The time interval  $d\tau$  measured by the clock corresponds to the length  $ds$  of its world line, i.e.,  $d\tau^2 = \pm \text{const } ds^2$ . Under the assumption

for the constant velocity of the periodical process in the clock, we consider the periodical process as indicator for the time interval  $d\tau$  in the proper frame of reference of the clock and of the observer, respectively.

4. Velocity  $u$  of an  $(n - 1)$ -dimensional subspace moving in time with  $l_u \neq 0$ . If the subspace deforms in some way, the deformations reflect on the kinematic characteristics of the vector field  $u$ , and  $u$  is used as an indicator for the changing properties of the subspace, considered as the space of an observer (laboratory) where a physical system is investigated. This type of interpretation requires not only the existence of the velocity vector field  $u$  with  $l_u \neq 0$  but also the existence of (at least one) orthogonal to  $u$  vector field  $\xi_{\perp}$ ,  $g(u, \xi_{\perp}) = 0$ , lying in the orthogonal to  $u$  subspace  $T^{\perp u}(M)$ .

All indicated interpretations could be used in solving different physical problems related to motions of physical systems in space-time.

B. After introducing the notion of distance, the question arises *how a space distance between two points in a space could be measured*. We could distinguish three types of measurements:

1. Direct measurements by using a measuring device (e.g., a roulette, a linear (running) meter, yard-measure-stick, etc.)

2. Direct measurements by sending signals from a basic point to a fixed point of space and detecting at the basic point the reflected by the fixed point signal.

3. Indirect measurement by receiving signals from a fixed point of space without sending a signal to it.

Let us now consider every type of measurements more closely.

1. *Direct measurements by using a measuring device* (Fig. 1).

The space distance between two points  $A$  and  $B$  in a space could be measured by a second observer moving from point  $A$  (where the first observer is at rest) to point  $B$  in space. At the same time, the second observer moves in time from point  $B$  to point  $B'$ . The space distance measured by the observer with world line  $AA'$  could be denoted as  $\Delta r = AB$  and the time period passed as  $\Delta\tau = AA'$ . This is a direct measurement of the space distance  $AB = \Delta r$  from point  $A$  to point  $B$  in the space during the time  $AA' = \Delta\tau$ . It is *assumed* that point  $A$  and point  $B$  are at rest during the measurement. Instead of measuring the space distance  $AB$ , the observers measure the space distance  $A'B'$  which exists at the time  $\tau + \Delta\tau$  if the measurement has begun at the time  $\tau$  from the point of the first observer with world line  $AA'$ .

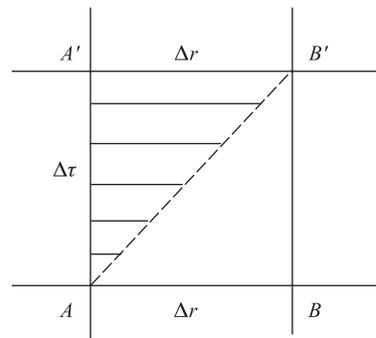


Fig. 1. Direct measurement of space distance by the use of a measuring device

the time  $\tau + \Delta\tau$  if the measurement has begun at the time  $\tau$  from the point of the first observer with world line  $AA'$ .

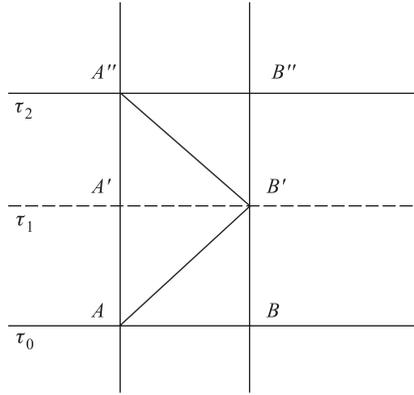


Fig. 2. Direct measurement by sending signals from a basic point to a fixed point in space and detecting the reflected signals

the time  $\tau(A') = \tau_1$ , where  $\tau(B') = \tau_1$ ,  $\tau(B'') = \tau_2$ . At the time ( $\tau_2$ ) the point  $B(\tau_0)$  will be moved in the time to point  $B''(\tau_2)$ . The signal will be propagated

(a) for the time interval  $AA' = \tau_1 - \tau_0$  to the point  $B'$  at the time  $\tau_1$  at the space distance  $\Delta r = l_u(\tau_1 - \tau_0)$ , where  $l_u$  is the velocity of the signal measured by the observer with world line  $AA'$ ;

(b) for the time interval  $A'A''$  from point  $B'$  at the time  $\tau_1$  to the point  $A''$  at the time  $\tau_2$  at a space distance  $l_u(\tau_2 - \tau_1)$ . The whole space distance covered by the signal in the time interval  $AA'A'' = \Delta\tau = \tau_2 - \tau_0$  is  $l_u(\tau_2 - \tau_0) = l_u(\tau_2 - \tau_1) + l_u(\tau_1 - \tau_0)$ .

If we now *assume* that point  $A$  and point  $B$  are at rest to each other and the space distance between them does not change in the time, then

$$l_u(\tau_2 - \tau_1) = l_u(\tau_1 - \tau_0) \quad (72)$$

and

$$l_u(\tau_2 - \tau_0) = 2l_u(\tau_1 - \tau_0) = 2A'B'(\tau_1) = 2AB(\tau_0). \quad (73)$$

Therefore, the space distance between point  $A$  and point  $B$  (at any time, if both the points are at rest to each other) is

$$AB = \frac{1}{2}l_u(\tau_2 - \tau_0), \quad (74)$$

where  $\Delta\tau = \tau_2 - \tau_0$  is the time interval for the propagation of a signal from point  $A$  to point  $B$  and from point  $B$  back to point  $A$ .

3. *Indirect measurement by receiving signals from a fixed point of space without sending a signal to it* (Fig. 3).

2. *Direct measurements by sending signals from a basic point to a fixed point of space and detecting at the basic point the reflected by the fixed point signal* (Fig. 2).

The space distance between two points  $A$  and  $B$  in a space could be measured by sending a signal with velocity with absolute value  $l_u \neq 0$ . Then,  $AB$  of the curve  $x^i(\tau = \tau_0, r)$  through point  $B$  is the distance  $\Delta r$  at the time  $\tau(A) = \tau_0$  and  $\tau(B) = \tau_0$ .

$A'B'$  of the curve  $x^i(\tau = \tau_0 + \Delta\tau, r)$  is the space distance  $\Delta r'$  at the time  $\tau(A') = \tau_1$ . At this time the signal is received at point  $B'$  which is point  $B$  at the time  $\tau_1$ , i.e.,  $\tau(B') = \tau_1$ .  $B'A'$  is the space distance between  $B$  and  $A$  at

If the space distance between point  $A$  and point  $B$  is changing in the time and at point  $B$  there is an emitter, then the frequency of the emitter could change in the time related to the centrifugal (centripetal) or Coriolis' velocities and accelerations between both the points  $A$  and  $B$ . Therefore, a criteria for no relative motion between two (space) points (points with one and the same proper time) could be the lack of change of the frequency of the signals emitted from the second point  $B$  to the basic point  $A$ .

(But there could be motions of an emitter which could change its frequency so that the changes compensate each other and the observer at the basic point  $A$  could come to the conclusion that there is no motions between points  $A$  and  $B$ .)

If an emitter at point  $B(\tau_0)$  emits a signal with velocity  $\bar{u}$  and frequency  $\bar{\omega}$ , then this signal will be received (detected) at the point  $A'(\tau_1)$  after a time interval  $AA' = \Delta\tau = \tau_1 - \tau_0$  by an observer (detector) moving in the time interval  $\Delta\tau$  from point  $A(\tau_0)$  to point  $A'(\tau_1)$  on his world line  $x^i(\tau)$ . If the emitter is moving relatively to point  $A$  with relative velocity  ${}_{\text{rel}}v$  and/or with relative acceleration  ${}_{\text{rel}}a$ , then the detected at the point  $A'$  frequency  $\omega$  will differ from the emitted frequency  $\bar{\omega}$ . If both the points  $A$  and  $B$  are at rest to each other, then  $\bar{\omega} = \omega$ .

C. The question arises: *How can we find the space distance between two points  $A$  and  $B$  lying in such a way in the space that only signals emitted from the one point (point  $B$ ) could be detected at the basic point (point  $A$ ), where an observer detects the signal from point  $B$ ?* First of all, if we knew the propagation velocity  $l_u$  of a signal and the difference  $\bar{\omega} - \omega$  between the emitted frequency  $\bar{\omega}$  and the detected frequency  $\omega$ , we can try to find out the relative velocity (and acceleration) between the emitter (at a point  $B$ ) and the observer (at a point  $A$ ). For doing that we will need relations between the difference  $\bar{\omega} - \omega$  and the relative velocity (and acceleration) between both the points. Such relations could be found on the basis of the structures of the relative velocity and the relative acceleration and their decompositions in centrifugal (centripetal) relative velocity and relative acceleration and Coriolis relative velocity and relative acceleration [13].

### 3. KINEMATIC EFFECTS RELATED TO THE RELATIVE VELOCITY AND TO THE RELATIVE ACCELERATION

1. Let us now consider the change of a null vector field  $\tilde{k}$  under the influence of the relative velocity and of the relative acceleration on the corresponding emitter and its frequency with respect to an observer detecting the emitted radiation by the emitter.

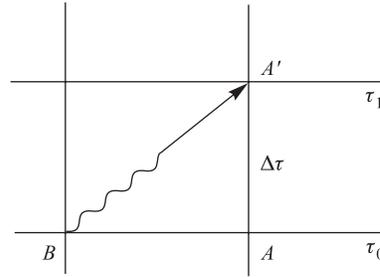


Fig. 3. Indirect measurement by receiving signals from a fixed point in space

Let  $\bar{k}_\perp$  be the orthogonal to  $u$  part of the null vector field  $\bar{k}$  corresponding to the null vector field  $\tilde{k}$  after the influence of the relative velocity  ${}_{\text{rel}}v$  and/or the relative acceleration  ${}_{\text{rel}}a$

$$\begin{aligned}\bar{k} &= \tilde{k} + {}_{\text{rel}}k, & \bar{k} &= \bar{k}_\parallel + \bar{k}_\perp, & \tilde{k} &= k_\parallel + k_\perp, & {}_{\text{rel}}k &= {}_{\text{rel}}k_\parallel + {}_{\text{rel}}k_\perp, \\ \bar{k}_\perp &= k_\perp + {}_{\text{rel}}k_\perp,\end{aligned}\quad (75)$$

where  ${}_{\text{rel}}k$  could depend on the relative velocity  ${}_{\text{rel}}v$  [4, Ch. 10] and on the relative acceleration  ${}_{\text{rel}}a$  [4, Ch. 11, 12].

If  $\bar{k} = k + {}_{\text{rel}}k$ , then

$$\begin{aligned}g(\bar{k}, \bar{k}) &= 0 = g({}_{\text{rel}}k, {}_{\text{rel}}k) + 2g(\tilde{k}, {}_{\text{rel}}k), \\ g({}_{\text{rel}}k, {}_{\text{rel}}k) &= -2g(\tilde{k}, {}_{\text{rel}}k).\end{aligned}\quad (76)$$

In the previous sections we have considered the representation of  $\tilde{k}$  as  $\tilde{k} = k_\parallel + k_\perp$ , where

$$k_\parallel = \pm l_{k_\parallel} n_\parallel = \pm \frac{\omega}{l_u} n_\parallel, \quad l_{k_\parallel} = \frac{\omega}{l_u}, \quad (77)$$

$$k_\perp = \mp l_{k_\perp} \tilde{n}_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp, \quad l_{k_\perp} = \frac{\omega}{l_u} = l_{k_\parallel}. \quad (78)$$

The unit vector  $\tilde{n}_\perp$  is orthogonal to the vector  $u$ , i.e.,  $g(u, \tilde{n}_\perp) = 0$  because of  $g(u, k_\perp) = \mp l_{k_\perp} g(u, \tilde{n}_\perp) = 0$ ,  $l_{k_\perp} \neq 0$ ,  $l_u \neq 0$ . Therefore,  $g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1 = \mp l_{\tilde{n}_\perp}^2$ ,  $l_{\tilde{n}_\perp} \neq 0$ .

We can represent the unit vector  $\tilde{n}_\perp$  (orthogonal to  $u$ ) in two parts: one part  $n_\perp$  collinear to the vector field  $\xi_\perp$  (orthogonal to  $u$ ) and one part  $m_\perp$  orthogonal to the vectors  $u$  and  $\xi_\perp$ , i.e.,

$$\tilde{n}_\perp = \alpha n_\perp + \beta m_\perp, \quad (79)$$

$$g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1 = \mp l_{\tilde{n}_\perp}^2, \quad l_{\tilde{n}_\perp} > 0, \quad l_{\tilde{n}_\perp} = 1, \quad (80)$$

$$\eta_\perp := l_{\xi_\perp} m_\perp, \quad m_\perp = \frac{\eta_\perp}{l_{\xi_\perp}}, \quad g(m_\perp, m_\perp) = \mp 1, \quad (81)$$

$$g(\eta_\perp, \eta_\perp) = l_{\xi_\perp}^2 g(m_\perp, m_\perp) = \mp l_{\xi_\perp}^2 = g(\xi_\perp, \xi_\perp), \quad (82)$$

$$g(\eta_\perp, \xi_\perp) = 0, \quad g(m_\perp, n_\perp) = 0, \quad (83)$$

where

$$n_\perp = \frac{\xi_\perp}{l_{\xi_\perp}}, \quad g(n_\perp, u) = 0, \quad g(n_\perp, n_\perp) = \mp 1 = \mp l_{n_\perp}^2, \quad (84)$$

$$l_{n_\perp} > 0, \quad l_{n_\perp} = 1, \quad (85)$$

$$m_\perp = \frac{\eta_\perp}{l_{\xi_\perp}} = \mp \frac{v_c}{l_{v_c}}, \quad g(m_\perp, u) = 0, \quad g(m_\perp, \xi_\perp) = 0. \quad (86)$$

The vector field  $v_c$  is the Coriolis velocity vector field [13] orthogonal to  $u$  and to the centrifugal (centripetal) velocity  $v_z$  collinear to  $\xi_\perp$ . Since

$$v_c = \mp l_{v_c} m_\perp, \quad g(v_c, v_c) = \mp l_{v_c}^2, \quad (87)$$

we also have

$$g(m_\perp, m_\perp) = \mp 1 = \mp l_{m_\perp}^2, \quad l_{m_\perp} > 0, \quad tl_{m_\perp} = 1. \quad (88)$$

The Coriolis velocity  $v_c$  is related to the change of the vector  $\xi_\perp$  in direction orthogonal to  $u$  and  $\xi_\perp$ .

The functions  $\alpha$  and  $\beta$  appear as direction cosines of  $n_\perp$  and  $m_\perp$  with respect to the unit vector  $\tilde{n}_\perp$  (Appendix 2).

2. For the contravariant null vector field  $\bar{k}$  we have analogous relations as for the contravariant null vector field  $\tilde{k}$  (just changing  $\tilde{k}$  with  $\bar{k}$ ,  $\omega$  with  $\bar{\omega}$ , and  $\tilde{n}_\perp$  with  $\tilde{n}'_\perp$ ) (Appendix 2).

From  $\bar{k} = \tilde{k} + {}_{\text{rel}}k$ ,  $\bar{k}_\perp = \tilde{k}_\perp + {}_{\text{rel}}k_\perp$ , and

$$\mp \frac{\bar{\omega}}{l_u} \tilde{n}'_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp + {}_{\text{rel}}k_\perp, \quad (89)$$

it follows that (Appendix 2)

$$\mp \frac{\bar{\omega}}{l_u} g(\tilde{n}'_\perp, \tilde{n}_\perp) = \frac{\omega}{l_u} + g({}_{\text{rel}}k_\perp, \tilde{n}_\perp).$$

The vector  $\tilde{n}'_\perp$  could be represented by the use of the vectors  $n_\perp$  and  $m_\perp$  in the form

$$\tilde{n}'_\perp = \alpha' n_\perp + \beta' m_\perp. \quad (90)$$

The functions  $\alpha'$  and  $\beta'$  appear as direction cosines of  $n_\perp$  and  $m_\perp$  with respect to the unit vector  $\tilde{n}'_\perp$  (Appendix 2).

If we denote the angle  $(\tilde{n}_\perp, n_\perp)$  between the vectors  $\tilde{n}_\perp$  and  $n_\perp$  as  $\theta = (\tilde{n}_\perp, n_\perp)$  and the angle  $(\tilde{n}'_\perp, n_\perp)$  between the vectors  $\tilde{n}'_\perp$  and  $n_\perp$  as  $\theta' = (\tilde{n}'_\perp, n_\perp)$ , then by the use of the expressions for  $\bar{k}_\perp$ ,  $\tilde{k}_\perp$ , and  ${}_{\text{rel}}k_\perp$  (Appendix 2) the relations follow

$$\frac{\bar{\omega}}{\omega} \cos \theta' = \cos \theta + \frac{1}{\omega} l_u g({}_{\text{rel}}k_\perp, n_\perp) = \cos \theta + \frac{g({}_{\text{rel}}k_\perp, n_\perp)}{g(\tilde{n}_\perp, k_\perp)}, \quad (91)$$

$$\frac{\bar{\omega}}{\omega} \sin \theta' = \sin \theta + \frac{1}{\omega} l_u g({}_{\text{rel}}k_\perp, m_\perp) = \sin \theta + \frac{g({}_{\text{rel}}k_\perp, m_\perp)}{g(\tilde{n}_\perp, k_\perp)}. \quad (92)$$

From the last (above) two relations, it follows for  $\text{tg } \theta'$

$$\text{tg } \theta' = \frac{\sin \theta'}{\cos \theta'} = \frac{\sin \theta + \frac{g({}_{\text{rel}}k_\perp, m_\perp)}{g(\tilde{n}_\perp, k_\perp)}}{\cos \theta + \frac{g({}_{\text{rel}}k_\perp, n_\perp)}{g(\tilde{n}_\perp, k_\perp)}}. \quad (93)$$

If we introduce the abbreviations

$$\overline{S} := \frac{g(\text{rel}k_{\perp}, m_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})}, \quad (94)$$

$$\overline{C} := \frac{g(\text{rel}k_{\perp}, n_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})}, \quad (95)$$

the expression for  $\text{tg } \theta'$  could be written in the form

$$\text{tg } \theta' = \frac{\sin \theta'}{\cos \theta'} = \frac{\sin \theta + \overline{S}}{\cos \theta + \overline{C}}. \quad (96)$$

The relations for  $\sin \theta'$  and  $\cos \theta'$  will have the forms, respectively:

$$\frac{\overline{\omega}}{\omega} \sin \theta' = \sin \theta + \overline{S}, \quad (97)$$

$$\frac{\overline{\omega}}{\omega} \cos \theta' = \cos \theta + \overline{C}. \quad (98)$$

The angle  $\theta'$  describes the deviation of the direction of the vector  $\tilde{k}_{\perp}$  with respect to the vector  $k_{\perp}$ . This type of deviation is usually related to the *aberration* of the wave vector  $\tilde{k}$  during its motion in a time interval. As we will see below, the aberration depends on the relative velocity and relative acceleration included implicitly in the terms  $\overline{S}$  and  $\overline{C}$ .

From the expressions for  $\sin \theta'$  and  $\cos \theta'$  the relation between the emitted frequency  $\overline{\omega}$  and the detected frequency  $\omega$  follows in the forms

$$\frac{\overline{\omega}^2}{\omega^2} (\sin^2 \theta' + \cos^2 \theta') = \frac{\overline{\omega}^2}{\omega^2} = (\sin \theta + \overline{S})^2 + (\cos \theta + \overline{C})^2, \quad (99)$$

$$\overline{\omega} = [(\sin \theta + \overline{S})^2 + (\cos \theta + \overline{C})^2]^{1/2} \omega. \quad (100)$$

*Special case:*  $\overline{S} := 0$ ,  $\sin \theta = 0$ ,  $\cos \theta = 1$ ,  $k_{\perp} = \mp l_{k_{\perp}} \tilde{n}_{\perp}$ .

$$\overline{\omega} = (1 + \overline{C})\omega, \quad \overline{C} = \frac{\overline{\omega} - \omega}{\omega} = z = \frac{1}{l_{k_{\perp}}} g(\text{rel}k_{\perp}, n_{\perp}), \quad (101)$$

$$g(\text{rel}k_{\perp}, n_{\perp}) = l_{k_{\perp}} z. \quad (102)$$

*Special case:*  $\overline{C} := 0$ ,  $\cos \theta = 0$ ,  $\sin \theta = 1$ ,  $k_{\perp} = \mp l_{k_{\perp}} \tilde{n}_{\perp}$ .

$$\overline{\omega} = (1 + \overline{S})\omega, \quad \overline{S} = \frac{\overline{\omega} - \omega}{\omega} = z_c = \frac{1}{l_{k_{\perp}}} g(\text{rel}k_{\perp}, m_{\perp}), \quad (103)$$

$$g(\text{rel}k_{\perp}, m_{\perp}) = l_{k_{\perp}} z_c. \quad (104)$$

The above relations describe the change of the frequency of the emitter during the motion of the signal from the emitter to the receiver (detector, observer) and is related to the description of the Doppler effect and the Hubble effect in spaces with affine connections and metrics considered as models of space-time. It is assumed that the emitter is at rest at a given time with respect to the frame of reference of an observer moving in space-time and receiving signals from an emitter. If at rest with respect to an observer, the emitter sends a signal with frequency  $\bar{\omega}$  detected after a time interval as a frequency  $\omega$  measured by the observer in his proper frame of reference.

The task is now to find out the explicit form for  ${}_{\text{rel}}k_{\perp}$ . For this purpose we would consider the change of a vector field  $\xi_{\perp}$ , orthogonal to the vector field  $u$ , i.e.,  $g(u, \xi_{\perp}) = 0$ , transported along the vector field  $u$ .

**3.1. Change of a Nonisotropic Vector Field  $\xi_{\perp}$  Along a Nonisotropic Vector Field  $u$ , when  $g(u, \xi_{\perp}) = 0$ .** Let us now consider the change of the vector field  $\xi_{\perp}$ ,  $g(u, \xi_{\perp}) = 0$ , along the world line  $x^i(\tau, \lambda_0)$  of an observer with tangent vector  $u$ . The vector  $\xi_{\perp}$  could be expressed at a point  $A(\tau_0 - d\tau, \lambda_0)$  by means of the vector field  $\xi_{\perp}$  at the point  $A'(\tau_0, \lambda_0)$  by the use of the covariant exponential operator [4, p.82–85] up to the second order of  $d\tau$

$$\begin{aligned} \xi_{\perp(A)} &:= \xi_{\perp(\tau_0 - d\tau, \lambda_0)} := \bar{\xi}_{(\tau_0, \lambda_0)} = \\ &= \xi_{\perp(\tau_0, \lambda_0)} - d\tau \nabla_u \xi_{\perp}|_{(\tau_0, \lambda_0)} + \frac{1}{2} d\tau^2 \nabla_u \nabla_u \xi_{\perp}|_{(\tau_0, \lambda_0)}. \end{aligned} \quad (105)$$

The vector  $\bar{\xi}_{\perp(\tau_0, \lambda_0)}$  could not be, in general, collinear to  $\xi_{\perp(\tau_0, \lambda_0)}$  and orthogonal to  $u_{(\tau_0, \lambda_0)}$ . If we, further, consider the part of  $\bar{\xi}_{(\tau_0, \lambda_0)}$ , orthogonal to  $u_{(\tau_0, \lambda_0)}$  at the point  $A'(\tau_0, \lambda_0)$ , we obtain

$$\begin{aligned} \bar{\xi}_{\perp(\tau_0, \lambda_0)} &:= \bar{g}[h_u(\bar{\xi}_{(\tau_0, \lambda_0)})] = \bar{g}[h_u(\xi_{\perp})]_{(\tau_0, \lambda_0)} - \\ &- d\tau \bar{g}[h_u(\nabla_u \xi_{\perp})]_{(\tau_0, \lambda_0)} + \frac{1}{2} d\tau^2 \bar{g}[h_u(\nabla_u \nabla_u \xi_{\perp})]_{(\tau_0, \lambda_0)}. \end{aligned} \quad (106)$$

Since,

$$\bar{g}[h_u(\xi_{\perp})] = \xi_{\perp}, \quad (107)$$

$$\bar{g}[h_u(\nabla_u \xi_{\perp})] = {}_{\text{rel}}v, \quad (108)$$

$$\bar{g}[h_u(\nabla_u \nabla_u \xi_{\perp})] = {}_{\text{rel}}a, \quad (109)$$

we have the relation

$$\bar{\xi}_{\perp(\tau_0, \lambda_0)} = \xi_{\perp(\tau_0, \lambda_0)} - d\tau {}_{\text{rel}}v_{(\tau_0, \lambda_0)} + \frac{1}{2} d\tau^2 {}_{\text{rel}}a_{(\tau_0, \lambda_0)}. \quad (110)$$

Therefore, the vector  $\xi_{\perp(\tau_0 - d\tau, \lambda_0)}$  with  $g(\xi_{\perp}, u)_{(\tau_0 - d\tau, \lambda_0)} = 0$  could be considered as the vector  $\bar{\xi}_{\perp(\tau_0, \lambda_0)}$  with  $g(\bar{\xi}_{\perp}, u)_{(\tau_0, \lambda_0)} = 0$  if transported from

the point  $A(\tau_0 - d\tau, \lambda_0) \equiv A_{(\tau_0 - d\tau, \lambda_0)}$  to the point  $A'(\tau_0, \lambda_0) \equiv A_{(\tau_0, \lambda_0)}$  at the world line with parameter  $\tau$ .

*Remark.* Analogous considerations could be made with the transport of the vector  $\xi_{\perp}$  from the point  $A''(\tau_0 + d\tau, \lambda_0)$  to the point  $A(\tau_0, \lambda_0)$ :

$$\xi_{\perp(\tau_0 + d\tau, \lambda_0)} := \bar{\bar{\xi}}_{\perp(\tau_0, \lambda_0)} = \xi_{\perp(\tau_0, \lambda_0)} + d\tau \text{rel}v_{(\tau_0, \lambda_0)} + \frac{1}{2}d\tau^2 \text{rel}a_{(\tau_0, \lambda_0)}. \quad (111)$$

If we summarize the expressions for  $\bar{\xi}_{\perp(\tau_0, \lambda_0)}$  and  $\bar{\bar{\xi}}_{\perp(\tau_0, \lambda_0)}$ , we can find a relation between the vectors  $\bar{\xi}_{\perp(\tau_0, \lambda_0)}$ ,  $\bar{\bar{\xi}}_{\perp(\tau_0, \lambda_0)}$ ,  $\xi_{\perp(\tau_0, \lambda_0)}$ , and the relative acceleration  $\text{rel}a_{(\tau_0, \lambda_0)}$

$$\bar{\bar{\xi}}_{\perp(\tau_0, \lambda_0)} + \bar{\xi}_{\perp(\tau_0, \lambda_0)} = 2\xi_{\perp(\tau_0, \lambda_0)} + d\tau^2 \text{rel}a_{(\tau_0, \lambda_0)}, \quad (112)$$

or

$$d\tau^2 \text{rel}a_{(\tau_0, \lambda_0)} = \bar{\bar{\xi}}_{\perp(\tau_0, \lambda_0)} + \bar{\xi}_{\perp(\tau_0, \lambda_0)} - 2\xi_{\perp(\tau_0, \lambda_0)}. \quad (113)$$

If the proper time interval  $d\tau$  is expressed by the use of the relations between the proper time  $\tau$  and the space distance  $dl$ , covered by a signal propagating with a velocity with absolute value  $l_u$  (from the point of view of the observer with the world line  $x^i(\tau, \lambda_0)$ ), as

$$d\tau = \pm \frac{l_{\xi_{\perp}}}{l_u} d\lambda = \frac{dl}{l_u}, \quad (114)$$

then the relation between  $\bar{\xi}_{\perp(\tau_0, \lambda_0)}$  and  $\xi_{\perp(\tau_0, \lambda_0)}$  could be written in the form

$$\bar{\xi}_{\perp(\tau_0, \lambda_0)} = \xi_{\perp(\tau_0, \lambda_0)} - \frac{dl}{l_u} \text{rel}v_{(\tau_0, \lambda_0)} + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \text{rel}a_{(\tau_0, \lambda_0)}. \quad (115)$$

For every point  $P'(\tau, \lambda_0)$  of the world line of the observer  $x^i(\tau, \lambda_0)$  the above relation between  $P'(\tau, \lambda_0)$  and another point  $P(\tau - d\tau, \lambda_0)$  is valid. In the further consideration we will omit the indications for the corresponding points:

$$\bar{\xi}_{\perp} = \xi_{\perp} - \frac{dl}{l_u} \text{rel}v + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \text{rel}a. \quad (116)$$

The vector  $\bar{\xi}_{\perp}$  is, in general, not collinear to the vector  $\xi_{\perp}$ . It could be represented by the use of the unit vectors  $n_{\perp}$  and  $m_{\perp}$  in the form

$$\bar{\xi}_{\perp} = l_{\bar{\xi}_{\perp}} \tilde{n}'_{\perp} = l_{\bar{\xi}_{\perp}} (\alpha' n_{\perp} + \beta' m_{\perp}) = \mp l_{\bar{\xi}_{\perp}} [\cos \theta' n_{\perp} + \sin \theta' m_{\perp}]. \quad (117)$$

On the other side, the vector  $\xi_{\perp}$  has the form  $\xi_{\perp} = l_{\xi_{\perp}} n_{\perp}$ , where  $g(\xi_{\perp}, m_{\perp}) = l_{\xi_{\perp}} g(n_{\perp}, m_{\perp}) = 0$ . In the same way, we can express the relative velocity  $\text{rel}v$  and the relative acceleration  $\text{rel}a$  by means of their structures, related to the vectors  $n_{\perp}$  and  $m_{\perp}$ .

If we project the expression for  $\bar{\xi}_\perp$  along the unit vectors  $n_\perp$  and  $m_\perp$  correspondingly, we obtain

$$\begin{aligned} g(\bar{\xi}_\perp, n_\perp) &= l_{\bar{\xi}_\perp} \cos \theta' = g(\xi_\perp, n_\perp) - \frac{dl}{l_u} g(\text{rel}v, n_\perp) + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 g(\text{rel}a, n_\perp) = \\ &= \mp l_{\xi_\perp} - \frac{dl}{l_u} g(\text{rel}v, n_\perp) + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 g(\text{rel}a, n_\perp), \end{aligned} \quad (118)$$

$$\begin{aligned} g(\bar{\xi}_\perp, m_\perp) &= l_{\bar{\xi}_\perp} \sin \theta' = g(\xi_\perp, m_\perp) - \frac{dl}{l_u} g(\text{rel}v, m_\perp) + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 g(\text{rel}a, m_\perp) = \\ &= -\frac{dl}{l_u} g(\text{rel}v, m_\perp) + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 g(\text{rel}a, m_\perp). \end{aligned} \quad (119)$$

We can find now the explicit forms of  $g(\text{rel}v, n_\perp)$ ,  $g(\text{rel}v, m_\perp)$ ,  $g(\text{rel}a, n_\perp)$ , and  $g(\text{rel}a, m_\perp)$ .

*3.1.1. Explicit Form of  $l_{\bar{\xi}_\perp} \cos \theta'$  and  $l_{\bar{\xi}_\perp} \sin \theta'$ .* Let us recall the explicit form of  $\text{rel}v$  and  $\text{rel}a$  with respect to their decompositions in centrifugal (centripetal) and Coriolis velocities and accelerations, respectively [13].

The relative velocity  $\text{rel}v$  could be represented in the form

$$\text{rel}v = v_z + v_c, \quad (120)$$

where

$$v_z = \mp l_{v_z} n_\perp = H l_{\xi_\perp} n_\perp = H \xi_\perp, \quad n_\perp = \frac{\xi_\perp}{l_{\xi_\perp}}, \quad (121)$$

$$v_c = \mp l_{v_c} m_\perp = H_c l_{\xi_\perp} m_\perp = H_c \eta_\perp, \quad m_\perp = \frac{\eta_\perp}{l_{\xi_\perp}} = \frac{v_c}{l_{v_c}}. \quad (122)$$

The relative acceleration  $\text{rel}a$  could be represented in the form

$$\text{rel}a = a_z + a_c, \quad (123)$$

where

$$a_z = \mp l_{a_z} n_\perp = \bar{q} l_{\xi_\perp} n_\perp = \bar{q} \xi_\perp, \quad (124)$$

$$a_c = \mp l_{a_c} m_\perp = \bar{q}_c l_{\xi_\perp} m_\perp = \bar{q}_c \eta_\perp. \quad (125)$$

For the expressions  $g(\text{rel}v, n_\perp)$  and  $g(\text{rel}v, m_\perp)$  we obtain, respectively,

$$g(\text{rel}v, n_\perp) = l_{v_z} = \mp H l_{\xi_\perp}, \quad (126)$$

$$g(\text{rel}v, m_\perp) = l_{v_c} = \mp H_c l_{\xi_\perp}. \quad (127)$$

For the expressions  $g(\text{rel}a, n_\perp)$  and  $g(\text{rel}a, m_\perp)$  we obtain, respectively,

$$g(\text{rel}a, n_\perp) = l_{a_z} = \mp \bar{q} l_{\xi_\perp}, \quad (128)$$

$$g(\text{rel}a, m_\perp) = l_{a_c} = \mp \bar{q}_c l_{\xi_\perp}. \quad (129)$$

By means of the above relations, it follows for  $l_{\bar{\xi}_\perp} \cos \theta'$  and  $l_{\bar{\xi}_\perp} \sin \theta'$ , respectively,

$$\begin{aligned} l_{\bar{\xi}_\perp} \cos \theta' &= \mp l_{\xi_\perp} - \frac{dl}{l_u} l_{v_z} + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 l_{a_z} = \mp l_{\xi_\perp} \pm \frac{dl}{l_u} H l_{\xi_\perp} \mp \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \bar{q} l_{\xi_\perp} = \\ &= \mp l_{\xi_\perp} \left[ 1 - \frac{dl}{l_u} H + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \bar{q} \right]. \end{aligned} \quad (130)$$

$$\begin{aligned} l_{\bar{\xi}_\perp} \sin \theta' &= -\frac{dl}{l_u} l_{v_c} + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 l_{a_c} = \pm \frac{dl}{l_u} H_c l_{\xi_\perp} \mp \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \bar{q}_c l_{\xi_\perp} = \\ &= \pm l_{\xi_\perp} \left[ \frac{dl}{l_u} H_c - \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \bar{q}_c \right] = \mp l_{\xi_\perp} \left[ -\frac{dl}{l_u} H_c + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \bar{q}_c \right]. \end{aligned} \quad (131)$$

If we introduce the abbreviations

$$C = -\frac{dl}{l_u} H + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \bar{q}, \quad (132)$$

$$S = -\frac{dl}{l_u} H_c + \frac{1}{2} \left( \frac{dl}{l_u} \right)^2 \bar{q}_c, \quad (133)$$

the expressions for  $l_{\bar{\xi}_\perp} \sin \theta'$  and  $l_{\bar{\xi}_\perp} \cos \theta'$  could also be written in the forms

$$l_{\bar{\xi}_\perp} \sin \theta' = \mp l_{\xi_\perp} S, \quad (134)$$

$$l_{\bar{\xi}_\perp} \cos \theta' = \mp l_{\xi_\perp} (1 + C). \quad (135)$$

The change of the direction of the vector  $\xi_\perp$  in the time interval  $d\tau$  of the proper time  $\tau$  of the observer on his world line can now be represented as

$$\text{tg } \theta' = \frac{S}{1 + C}. \quad (136)$$

The change of the length of the vector  $\xi_\perp$  in the time interval  $d\tau$  could be found in the form

$$l_{\bar{\xi}_\perp}^2 = l_{\xi_\perp}^2 [(1 + C)^2 + S^2], \quad (137)$$

$$l_{\bar{\xi}_\perp} = [(1 + C)^2 + S^2]^{1/2} l_{\xi_\perp}, \quad (138)$$

$$l_{\bar{\xi}_\perp} > 0, \quad l_{\xi_\perp} > 0. \quad (139)$$

*Special case:* The vector  $\bar{\xi}_\perp$  is collinear to the vector  $\xi_\perp$ . Then  $\tilde{n}'_\perp = n_\perp$ ,  $\sin \theta' = 0$ ,  $\cos \theta' = \mp 1$ ,  $S = 0$ , and

$$l_{\bar{\xi}_\perp} = (1 + C)l_{\xi_\perp}. \tag{140}$$

*Special case:* The vector  $\bar{\xi}_\perp$  is orthogonal to the vector  $\xi_\perp$ . Then  $\tilde{n}'_\perp = m_\perp$ ,  $\sin \theta' = \mp 1$ ,  $\cos \theta' = 0$ ,  $C = -1$ , and

$$l_{\bar{\xi}_\perp} = Sl_{\xi_\perp}. \tag{141}$$

The change of the length of the vector field  $\xi_\perp$  along the world line of an observer shows the role of the relative velocity and the relative acceleration for the deformation of the vector field  $\xi_\perp$  along the world line. This deformation is depending on the corresponding Hubble functions  $H$  and  $H_c$  and acceleration parameters  $\bar{q}$  and  $\bar{q}_c$ .

Since the deformation of the contravariant nonisotropic vector field  $k_\perp$  (orthogonal to the vector field  $u$ ) characterizes uniquely the deformation of the wave vector  $\tilde{k}$ , we could consider the change of  $k_\perp$  along the world line of the observer in analogous way as it has been done for the vector field  $\xi_\perp$ .

**3.2. Change of the Vector  $k_\perp$  Along the World Line of an Observer.**

Let us now consider the change of the vector field  $k_\perp$ ,  $g(u, k_\perp) = 0$ , along the world line  $x^i(\tau, \lambda_0)$  of an observer with the tangent vector  $u$ . For this aim we will consider first of all the wave vector field  $\tilde{k}$  and then  $\tilde{k}$  will be projected to the direction orthogonal to the vector field  $u$ .

The vector field  $\tilde{k}$  could be expressed at a point  $A(\tau_0 - d\tau, \lambda_0)$  by means of the vector field  $\tilde{k}$  at the point  $A'(\tau_0, \lambda_0)$  by the use of the covariant exponential operator ([4, p. 82–85]) up to the second order of  $d\tau$  (Fig. 4)

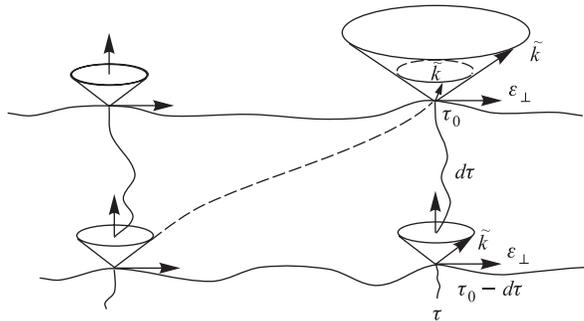


Fig. 4. Signals receiving from a fixed point in space and their comparing with the signals sent by an emitter at rest with respect to the basic trajectory of the observer

$$\begin{aligned} \tilde{k}_{(A)} &:= \tilde{k}_{(\tau_0-d\tau, \lambda_0)} := \bar{k}_{(\tau_0, \lambda_0)} = \\ &= \tilde{k}_{(\tau_0, \lambda_0)} - d\tau \nabla_u \tilde{k}|_{(\tau_0, \lambda_0)} + \frac{1}{2} d\tau^2 \nabla_u \nabla_u \tilde{k}|_{(\tau_0, \lambda_0)}. \end{aligned} \quad (142)$$

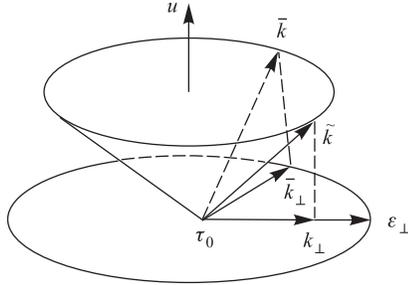


Fig. 5. Projections of the null vectors  $\bar{k}$  and  $\tilde{k}$  to a subspace orthogonal to the vector  $u$

The orthogonal to  $u$  parts of  $\tilde{k}$  and  $\bar{k}$  at the point  $A'(\tau_0, \lambda_0)$  could be found after projection of both the vectors by means of the projection metric  $h_u$  (Fig. 5)

$$\begin{aligned} \bar{g}[h_u(\tilde{k})]_{(\tau_0-d\tau, \lambda_0)} &:= \\ &= \bar{g}[h_u(\bar{k})]_{(\tau_0, \lambda_0)} = \bar{g}[h_u(\tilde{k})]_{(\tau_0, \lambda_0)} - \\ &\quad - d\tau \bar{g}[h_u(\nabla_u \tilde{k})]_{(\tau_0, \lambda_0)} + \\ &\quad + \frac{1}{2} d\tau^2 \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}. \end{aligned} \quad (143)$$

The vectors  $\bar{g}[h_u(\nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}$  and  $\bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}$  could be represented by the use of the kinematic characteristics of the relative velocity and relative acceleration [4] in the corresponding forms (Appendix 3).

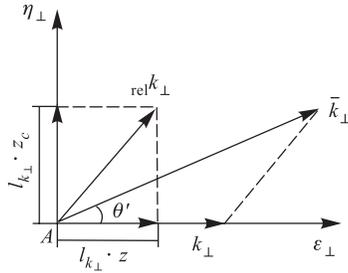


Fig. 6. Interpretation of the projections of  $rel k_{\perp}$  along and orthogonal to  $\xi_{\perp}$ . The quantity  $z$  is the longitudinal Hubble shift parameter and  $z_c$  is the transversal Hubble shift parameter

#### 4. ABERRATION

*Aberration* is the deviation of the direction of the vector field  $\bar{k}_{\perp}$  from the direction of the vector field  $k_{\perp}$ . If  $\bar{k}_{\perp} = \mp l_{k_{\perp}} \tilde{n}'_{\perp}$  and  $k_{\perp} = \mp l_{k_{\perp}} \tilde{n}_{\perp}$ , then the difference between the angles  $\theta'$  for  $\bar{k}_{\perp}$  and  $\theta$  for  $k_{\perp}$  with respect to the direction of the vector field  $\xi_{\perp}$  is given by the relations

$$\frac{\bar{\omega}}{\omega} \cos \theta' = \cos \theta + \frac{1}{l_{k_{\perp}}} g_{(rel k_{\perp}, n_{\perp})} = \cos \theta + \bar{C}, \quad (144)$$

$$\frac{\bar{\omega}}{\omega} \sin \theta' = \sin \theta + \frac{1}{l_{k_{\perp}}} g_{(rel k_{\perp}, m_{\perp})} = \sin \theta + \bar{S}. \quad (145)$$

From the last (above) two relations, it follows for  $\text{tg } \theta'$

$$\text{tg } \theta' = \frac{\sin \theta'}{\cos \theta'} = \frac{\sin \theta + (1/l_{k_{\perp}})g(\text{rel}k_{\perp}, m_{\perp})}{\cos \theta + (1/l_{k_{\perp}})g(\text{rel}k_{\perp}, n_{\perp})} = \frac{\sin \theta + \overline{S}}{\cos \theta + \overline{C}}. \quad (146)$$

If there is no relative velocities and no relative accelerations between the emitter and the observer, then  $\overline{C} = 0$  and  $\overline{S} = 0$ . Then

$$\frac{\overline{\omega}}{\omega} \cos \theta' = \cos \theta, \quad (147)$$

$$\frac{\overline{\omega}}{\omega} \sin \theta' = \sin \theta, \quad (148)$$

$$\frac{\overline{\omega}^2}{\omega^2} = 1, \quad \overline{\omega} = \omega. \quad (149)$$

The frequency of the signal, emitted by the emitter, and the frequency of the signal, detected by the observer, are the same and, at the same time, no aberration occurs.

*Special case:* Autoparallel motion ( $\nabla_u u = a = 0$ ) of an observer detected a signal with emitted frequency  $\overline{\omega}$ . Then

$$\overline{C} = \pm \frac{1}{l_u} [(\alpha \overline{l}_{v_z} + \beta \overline{l}_{v_{\eta z}})] \mp \frac{1}{2} \frac{dl}{l_u^2} [(\alpha \overline{l}_{a_z} + \beta \overline{l}_{a_{\eta z}})], \quad (150)$$

$$\overline{S} = \pm \frac{1}{l_u} [(\alpha \overline{l}_{v_c} + \beta \overline{l}_{v_{\eta c}})] \mp \frac{1}{2} \frac{dl}{l_u^2} [(\alpha \overline{l}_{a_c} + \beta \overline{l}_{a_{\eta c}})], \quad (151)$$

$$\overline{\omega} \sin \theta' = \left\{ \sin \theta \pm \frac{1}{l_u} [(\alpha \overline{l}_{v_c} + \beta \overline{l}_{v_{\eta c}})] \mp \frac{1}{2} \frac{dl}{l_u^2} [(\alpha \overline{l}_{a_c} + \beta \overline{l}_{a_{\eta c}})] \right\} \omega, \quad (152)$$

$$\overline{\omega} \cos \theta' = \left\{ \cos \theta \pm \frac{1}{l_u} [(\alpha \overline{l}_{v_z} + \beta \overline{l}_{v_{\eta z}})] \mp \frac{1}{2} \frac{dl}{l_u^2} [(\alpha \overline{l}_{a_z} + \beta \overline{l}_{a_{\eta z}})] \right\} \omega, \quad (153)$$

$$\alpha = \mp \cos \theta, \quad \beta = \mp \sin \theta. \quad (154)$$

## 5. DOPPLER EFFECT

The *Doppler effect* (Doppler shift) is the shift of the signal's frequency caused by the relative motion between the emitter and the observer.

1. Usually, in classical mechanics, and especially in acoustics, there is a difference between the shift of the frequency when the observer is moving to or out of the emitter and the shift of the frequency when the emitter is moving to or out of the observer. In the first case, the signal is propagating in a medium at rest with respect to the emitter, and in the second case, the signal is propagating

in a medium moving with respect to the emitter. It is assumed that the signal is propagating in a continuous media used as a carrier of the signal.

In relativistic physics, and especially in electrodynamics, there is no difference between the shifts of the frequency of a signal when the observer is moving to or out of the emitter and when the emitter is moving to or out of the observer. The relative motion is the only reason for the shift frequency.

2. If the shift of the frequency of a signal is considered from the point of view of an observer, then only the relative motions with respect to the observer could be taken into account. The observer detects the signals in his proper frame of reference (laboratory) and could make a comparison between the signals sent by the emitter at rest with respect to his proper frame of reference and by the emitter moving relatively to the observer's proper frame of reference.

3. The observer could move in space-time where the space could be filled with a continuous media or with classical fields with physical interpretation. Since every classical field theory could be considered as a theory of a continuous media [2, 3, 5], both types of theories could be used for dynamical description of propagation of signals in space-time. An observer is interested in finding out how signals are propagating, how the emitters are moving with respect to the observer, and how the signals are generated by an emitter. Only the first two questions are subjects of consideration by the use of kinematic characteristics of the relative velocity, the relative acceleration, and the properties of null (isotropic) vector fields. The last question is a matter of consideration of the corresponding dynamical theory.

4. The Doppler effect could be described in spaces with affine connections and metrics as models of space or space-time on the basis of the relations between the emitted frequency  $\bar{\omega}$  and detected frequency  $\omega$  of signals propagating in space or space-time. The same relations are used for consideration of the aberration of signals. As corollary of them, a relation between  $\bar{\omega}$  and  $\omega$  follows in the form

$$\bar{\omega} = [(\sin \theta + \bar{S})^2 + (\cos \theta + \bar{C})^2]^{1/2} \omega \quad (155)$$

appearing as a general formula for the generalized Doppler effect in spaces with affine connections and metrics.

### 5.1. Standard (Longitudinal) Doppler Effect (Doppler Shift)

1. The standard (longitudinal) Doppler effect appears when all Coriolis velocities and Coriolis accelerations compensate each other or do not exist in the relative motion between emitter and detector (observer), i.e., if

$$\bar{S} = 0, \quad (156)$$

then

$$\bar{\omega} = [\sin^2 \theta + (\cos \theta + \bar{C})^2]^{1/2} \omega = [\sin^2 \theta + \cos^2 \theta + 2\bar{C} \cos \theta + \bar{C}^2]^{1/2} \omega, \quad (157)$$

$$\bar{\omega} = [1 + 2\bar{C} \cos \theta + \bar{C}^2]^{1/2} \omega, \quad (158)$$

where

$$\begin{aligned} \bar{C} &= \pm \frac{1}{l_u} \left[ (\mp \cos \theta \bar{l}_{v_z} \mp \sin \theta \bar{l}_{v_{\eta z}}) - \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})_z} \right] \mp \\ &\quad \mp \frac{1}{2} \frac{dl}{l_u^2} \left[ (\mp \cos \theta \bar{l}_{a_z} \mp \sin \theta \bar{l}_{a_{\eta z}}) - \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp z}} \right], \\ \bar{C} &= \frac{1}{l_u} \left[ -(\cos \theta \bar{l}_{v_z} + \sin \theta \bar{l}_{v_{\eta z}}) \mp \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})_z} \right] + \\ &\quad + \frac{1}{2} \frac{dl}{l_u^2} \left[ (\cos \theta \bar{l}_{a_z} + \sin \theta \bar{l}_{a_{\eta z}}) \pm \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp z}} \right]. \end{aligned} \quad (159)$$

2. If the vector field  $k_{\perp}$  is collinear to the vector field  $\xi_{\perp}$  determining the proper frame of reference, i.e., if

$$k_{\perp} = \mp l_{k_{\perp}} n_{\perp}, \quad \cos \theta = \pm 1, \quad \sin \theta = 0, \quad (160)$$

then

$$\bar{C} = \frac{1}{l_u} \left( \mp \bar{l}_{v_z} \mp \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})_z} \right) + \frac{1}{2} \frac{dl}{l_u^2} \left( \pm \bar{l}_{a_z} \pm \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp z}} \right), \quad (161)$$

$$\bar{C} = \mp \left[ \frac{1}{l_u} \left( \bar{l}_{v_z} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})_z} \right) - \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_z} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp z}} \right) \right],$$

$$\bar{\omega} = [1 \pm 2\bar{C} + \bar{C}^2]^{1/2} \omega = [(1 \pm \bar{C})^2]^{1/2} \omega = (1 \pm \bar{C}) \omega, \quad (162)$$

$$\frac{\bar{\omega} - \omega}{\omega} = \pm \bar{C} = z, \quad (163)$$

$$\begin{aligned} \bar{\omega} &= \omega \left[ 1 - \frac{1}{l_u} \left( \bar{l}_{v_z} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})_z} \right) + \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_z} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp z}} \right) \right] = \\ &= \omega \left[ 1 - \frac{\bar{l}_{v_z}}{l_u} - \frac{l_{\xi_{\perp}}}{l_u^2} \bar{l}_{(a_{\perp})_z} + \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_z} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp z}} \right) \right]. \end{aligned} \quad (164)$$

*Special case:* Autoparallel motion of the observer:  $\nabla_u u = a = 0$ :  $\bar{l}_{(a_{\perp})_z} = 0$ ,  $\bar{l}_{(\nabla_u a)_{\perp z}} = 0$ ,

$$\begin{aligned} \bar{\omega} &= \omega \left[ 1 - \frac{\bar{l}_{v_z}}{l_u} + \frac{1}{2} \frac{dl}{l_u^2} \bar{l}_{a_z} \right], \\ \bar{l}_{v_z} &\leq 0, \quad \bar{l}_{a_z} \leq 0. \end{aligned} \quad (165)$$

If the world line of an observer is an autoparallel trajectory and  $k_\perp$  is collinear to  $\xi_\perp$ , then the change of the frequency  $\bar{\omega}$  of the emitter depends on the centrifugal (centripetal) velocity  $\bar{l}_{v_z}$  and the centrifugal (centripetal) acceleration  $\bar{l}_{a_z}$ .

*Special case:*  $\nabla_u u = a = 0$ ,  $k_\perp = \mp l_{k_\perp} n_\perp$ ,  $\bar{l}_{a_z} = 0$ :

$$\bar{\omega} = \omega \left( 1 - \frac{\bar{l}_{v_z}}{l_u} \right), \quad \bar{l}_{v_z} \begin{matrix} \leq \\ \geq \end{matrix} 0. \quad (166)$$

Therefore, if the world line of an observer is an autoparallel trajectory,  $k_\perp$  is collinear to  $\xi_\perp$ , and no centrifugal (centripetal) acceleration  $\bar{l}_{a_z}$  exists between the emitter and observer, then the above expression has the well-known form for description of the standard Doppler effect in classical mechanics in 3-dimensional Euclidean space. Here, the relation is valid in every  $(\bar{L}_n, g)$  space considered as a model of a space or a space-time under the given preconditions.

### 5.2. Transversal Doppler Effect

1. The transversal Doppler effect appears when all centrifugal (centripetal) velocities and centrifugal (centripetal) accelerations compensate each other or do not exist in the relative motion between emitter and detector (observer), i.e., if

$$\bar{C} = 0, \quad (167)$$

then

$$\begin{aligned} \bar{\omega} &= [(\sin \theta + \bar{S})^2 + \cos^2 \theta]^{1/2} \omega = [\sin^2 \theta + 2\bar{S} \sin \theta + \bar{S}^2 + \cos^2 \theta]^{1/2} \omega = \\ &= [1 + 2\bar{S} \sin \theta + \bar{S}^2]^{1/2} \omega, \end{aligned} \quad (168)$$

where

$$\begin{aligned} \bar{S} &:= \pm \frac{1}{l_u} \left[ \frac{dl}{l_{\xi_\perp}} (\alpha l_{v_c} + \beta l_{v_{\eta_c}}) - \frac{dl}{l_u} l_{(a_\perp)_c} \right] \mp \\ &\quad \mp \frac{1}{2} \frac{dl^2}{l_u^2} \left[ \frac{1}{l_{\xi_\perp}} (\alpha l_{a_c} + \beta l_{a_{\eta_c}}) - \frac{1}{l_u} l_{(\nabla_u a)_\perp c} \right], \end{aligned} \quad (169)$$

$$\alpha = \mp \cos \theta, \quad \beta = \mp \sin \theta, \quad (170)$$

$$\begin{aligned} \bar{S} &:= \pm \frac{1}{l_u} \left[ \frac{dl}{l_{\xi_\perp}} (\mp \cos \theta l_{v_c} \mp \sin \theta l_{v_{\eta_c}}) - \frac{dl}{l_u} l_{(a_\perp)_c} \right] \mp \\ &\quad \mp \frac{1}{2} \frac{dl^2}{l_u^2} \left[ \frac{1}{l_{\xi_\perp}} (\mp \cos \theta l_{a_c} \mp \sin \theta l_{a_{\eta_c}}) - \frac{1}{l_u} l_{(\nabla_u a)_\perp c} \right], \end{aligned} \quad (171)$$

$$\begin{aligned} \bar{S} := & \frac{1}{l_u} \left[ -\frac{dl}{l_{\xi_{\perp}}} (\cos \theta l_{v_c} + \sin \theta l_{v_{\eta c}}) \mp \frac{dl}{l_u} l_{(a_{\perp})c} \right] + \\ & + \frac{1}{2} \frac{dl^2}{l_u^2} \left[ \frac{1}{l_{\xi_{\perp}}} (\cos \theta l_{a_c} + \sin \theta l_{a_{\eta c}}) \pm \frac{1}{l_u} l_{(\nabla_u a)_{\perp c}} \right]. \end{aligned} \quad (172)$$

2. If the vector field  $k_{\perp}$  is orthogonal to the vector field  $\xi_{\perp}$  determining the proper frame of reference, i.e., if

$$k_{\perp} = \mp l_{k_{\perp}} m_{\perp}, \quad \sin \theta = \pm 1, \quad \cos \theta = 0, \quad (173)$$

then

$$\bar{\omega} = [1 + 2\bar{S} \sin \theta + \bar{S}^2]^{1/2} \omega = [1 \pm 2\bar{S} + \bar{S}^2]^{1/2} \omega = (1 \pm \bar{S}) \omega, \quad (174)$$

$$\frac{\bar{\omega} - \omega}{\omega} = \pm \bar{S} = z_c, \quad (175)$$

$$\begin{aligned} \bar{S} := & \frac{1}{l_u} \left[ -\frac{dl}{l_{\xi_{\perp}}} (\pm l_{v_{\eta c}}) \mp \frac{dl}{l_u} l_{(a_{\perp})c} \right] + \frac{1}{2} \frac{dl^2}{l_u^2} \left( \frac{1}{l_{\xi_{\perp}}} (\pm l_{a_{\eta c}}) \pm \frac{1}{l_u} l_{(\nabla_u a)_{\perp c}} \right), \\ \bar{S} := & \frac{1}{l_u} \left( \mp \frac{dl}{l_{\xi_{\perp}}} l_{v_{\eta c}} \mp \frac{dl}{l_u} l_{(a_{\perp})c} \right) + \frac{1}{2} \frac{dl^2}{l_u^2} \left( \pm \frac{1}{l_{\xi_{\perp}}} l_{a_{\eta c}} \pm \frac{1}{l_u} l_{(\nabla_u a)_{\perp c}} \right), \end{aligned} \quad (176)$$

$$\begin{aligned} \bar{S} := & \mp \frac{1}{l_u} \left( \frac{dl}{l_{\xi_{\perp}}} l_{v_{\eta c}} + \frac{dl}{l_u} l_{(a_{\perp})c} \right) \pm \frac{1}{2} \frac{dl^2}{l_u^2} \left( \frac{1}{l_{\xi_{\perp}}} l_{a_{\eta c}} + \frac{1}{l_u} l_{(\nabla_u a)_{\perp c}} \right), \\ \bar{\omega} = & \omega \left\{ 1 - \frac{1}{l_u} \left( \frac{dl}{l_{\xi_{\perp}}} l_{v_{\eta c}} + \frac{dl}{l_u} l_{(a_{\perp})c} \right) + \frac{1}{2} \frac{dl^2}{l_u^2} \left( \frac{1}{l_{\xi_{\perp}}} l_{a_{\eta c}} + \frac{1}{l_u} l_{(\nabla_u a)_{\perp c}} \right) \right\}. \end{aligned} \quad (177)$$

If we introduce the abbreviations

$$\bar{l}_{v_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} l_{v_{\eta c}}, \quad \bar{l}_{(a_{\perp})c} = \frac{dl}{l_{\xi_{\perp}}} l_{(a_{\perp})c}, \quad (178)$$

$$\bar{l}_{a_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} l_{a_{\eta c}}, \quad \bar{l}_{(\nabla_u a)_{\perp c}} = \frac{dl}{l_{\xi_{\perp}}} l_{(\nabla_u a)_{\perp c}}, \quad \bar{l}_{\diamond} \leq 0, \quad (179)$$

then the expressions for  $\bar{S}$  and  $\bar{\omega}$  will have the forms (Appendix 3)

$$\begin{aligned} \bar{S} = & \mp \frac{1}{l_u} \left( \bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})c} \right) \pm \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp c}} \right) = \\ = & \mp \left[ \frac{1}{l_u} \left( \bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})c} \right) - \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp c}} \right) \right], \\ \pm \bar{S} = & - \left[ \frac{1}{l_u} \left( \bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})c} \right) - \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp c}} \right) \right] = \\ = & - \frac{1}{l_u} \left( \bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})c} \right) + \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp c}} \right), \end{aligned} \quad (180)$$

$$\begin{aligned}\bar{\omega} &= \omega \left[ 1 - \frac{1}{l_u} \left( \bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})_c} \right) + \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp c}} \right) \right] = \\ &= \omega \left[ 1 - \frac{1}{l_u} \bar{l}_{v_{\eta c}} - \frac{l_{\xi_{\perp}}}{l_u^2} \bar{l}_{(a_{\perp})_c} + \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp c}} \right) \right]. \quad (181)\end{aligned}$$

*Remark.* The expression for  $\bar{S}$  has the same form as the expression for  $\bar{C}$  under the change of  $\bar{l}_{v_z}$  with  $\bar{l}_{v_{\eta c}}$ ,  $\bar{l}_{(a_{\perp})_z}$  with  $\bar{l}_{(a_{\perp})_c}$ , and  $\bar{l}_{(\nabla_u a)_{\perp z}}$  with  $\bar{l}_{(\nabla_u a)_{\perp c}}$ .

*Special case:* Autoparallel motion of the observer:  $\nabla_u u = a = 0$ :  $\bar{l}_{(a_{\perp})_c} = 0$ ,  $\bar{l}_{(\nabla_u a)_{\perp c}} = 0$ ,

$$\bar{\omega} = \omega \left[ 1 - \frac{1}{l_u} \bar{l}_{v_{\eta c}} + \frac{1}{2} \frac{dl}{l_u^2} \bar{l}_{a_{\eta c}} \right]. \quad (182)$$

If the world line of an observer is an autoparallel trajectory and  $k_{\perp}$  is orthogonal to  $\xi_{\perp}$ , then the change of the frequency  $\bar{\omega}$  of the emitter depends on the Coriolis velocity  $\bar{l}_{v_{\eta c}}$  and the Coriolis acceleration  $\bar{l}_{a_{\eta c}}$ .

*Special case:*  $\nabla_u u = a = 0$ ,  $k_{\perp} = \mp l_{k_{\perp}} m_{\perp}$ ,  $\bar{l}_{a_{\eta c}} = 0$ :

$$\bar{\omega} = \omega \left( 1 - \frac{\bar{l}_{v_{\eta c}}}{l_u} \right), \quad \bar{l}_{v_{\eta c}} \begin{matrix} \leq \\ \geq \end{matrix} 0. \quad (183)$$

Therefore, if the world line of an observer is an autoparallel trajectory,  $k_{\perp}$  is orthogonal to  $\xi_{\perp}$ , and no Coriolis acceleration  $\bar{l}_{a_{\eta c}}$  exists between emitter and observer, then the above expression has analogous form for description of the transversal Doppler effect as the standard (longitudinal) Doppler effect in classical mechanics in 3-dimensional Euclidean space. Here, the relation for the transversal Doppler effect is valid in every  $(\bar{L}_n, g)$  space considered as a model of a space or a space-time under the given preconditions.

## 6. HUBBLE EFFECT

The *Hubble effect* (Hubble shift) is the Doppler shift (Doppler effect) of the signal's frequency caused by the relative motion between the emitter and the observer when the explicit form of the relative velocities and of the relative accelerations are given. The *Hubble law (law of redshift)* is defined as the linear dependence of the distances to galaxies on their red shift. In more general sense, the Hubble law is the statement that the relative velocity between an observer and a particle (from the point of view of the proper frame of reference of the observer) is proportional to the distance between the observer and the particle. Usually, the Hubble effect is defined as the change of the frequency  $\bar{\omega}$  under the motion of an emitter with centrifugal (centripetal) velocity  $v_z$  relatively to an observer.

Usually, the Hubble effect is related only to the centrifugal velocity of an emitter with respect to an observer on the basis of the Hubble distance-redshift relation [14, 15] discovered in 1929 and interpreted as a result of the expansion of the Universe. The explicit form of the kinematic characteristics of the centrifugal (centripetal) and Coriolis velocities and accelerations determine uniquely the Hubble effect [13].

**6.1. Explicit Forms of the Centrifugal (Centripetal) and Coriolis Velocities and Accelerations.** Let us now consider the explicit forms of the relative velocities and relative accelerations determining a Doppler shift (Doppler effect).

The vector fields generating a Doppler effect could be represented in two groups with respect to their lengths  $\bar{l}_\diamond$ :

(a) Vector fields generating a standard (longitudinal) Doppler effect

• centrifugal (centripetal) part  $\bar{l}_{v_z}$  of the relative centrifugal (centripetal) velocity  $\bar{l}_v$ ,

• centrifugal (centripetal) part  $\bar{l}_{v_{\eta z}}$  of the relative velocity  $\bar{l}_{\text{rel}v_\eta}$ ,

• centrifugal (centripetal) part  $\bar{l}_{(a_\perp)_z}$  of the acceleration  $\bar{l}_{a_\perp}$ ,

• centrifugal (centripetal) part  $\bar{l}_{a_z}$  of the relative acceleration  $\bar{l}_{\text{rel}a}$ ,

• centrifugal (centripetal) part  $\bar{l}_{a_{\eta z}}$  of the relative acceleration  $\bar{l}_{\text{rel}a_\eta}$ ,

• centrifugal (centripetal) part  $\bar{l}_{(\nabla_{u a})_\perp z}$  of the change of the acceleration  $\bar{l}_{a_\perp}$ .

(b) Vector fields generating a transversal Doppler effect

• Coriolis part  $\bar{l}_{v_c}$  of the relative velocity  $\bar{l}_v$ ,

• Coriolis part  $\bar{l}_{v_{\eta c}}$  of the relative velocity  $\bar{l}_{\text{rel}v_\eta}$ ,

• Coriolis part  $\bar{l}_{(a_\perp)_c}$  of the acceleration  $\bar{l}_a$ ,

• Coriolis part  $\bar{l}_{a_c}$  of the relative acceleration  $\bar{l}_{\text{rel}a}$ ,

• Coriolis part  $\bar{l}_{a_{\eta c}}$  of the relative acceleration  $\bar{l}_{\text{rel}a_\eta}$ ,

• Coriolis part  $\bar{l}_{(\nabla_{u a})_\perp c}$  of the change  $\bar{l}_{(\nabla_{u a})_\perp}$  of the acceleration  $\bar{l}_a$ .

*6.1.1. Explicit Form of the Relative Velocities and Accelerations Generating a Standard (Longitudinal) Doppler Effect.* The relative velocities have two essential components:  $\text{rel}v = v_z + v_c$ ,  $\text{rel}v_\eta = v_{\eta z} + v_{\eta c}$ .

(a) Relative centrifugal (centripetal) velocity  $v_z$

$$v_z = \frac{g(\text{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \xi_\perp, \quad (184)$$

could be represented in its explicit form as

$$v_z = \left[ \frac{1}{n-1} \theta \mp \sigma(n_\perp, n_\perp) \right] \xi_\perp, \quad (185)$$

$$v_z = \mp l_{v_z} n_\perp = H l_{\xi_\perp} n_\perp = H \xi_\perp, \quad (186)$$

where

$$H = \frac{1}{n-1} \theta \mp \sigma(n_{\perp}, n_{\perp}), \quad (187)$$

$$l_{v_z} = \mp H l_{\xi_{\perp}}, \quad \bar{l}_{v_z} = \frac{dl}{l_{\xi_{\perp}}} l_{v_z} = \mp H dl. \quad (188)$$

The last (above) expressions for  $\bar{l}_{v_z}$  and for  $\bar{v}_z$

$$\bar{l}_{v_z} = \mp H dl, \quad \bar{v}_z = \frac{dl}{l_{\xi_{\perp}}} v_z = H dl n_{\perp} \quad (189)$$

are the well-known relations called standard (longitudinal) Hubble law: the centrifugal (centripetal) relative velocity  $\bar{v}_z$  is proportional to the distance  $dl$  between an emitter and a detector (observer). This form of the Hubble law is a very special form of the law for the case when only the centrifugal (centripetal) relative velocity is taken into account.

(b) Centrifugal (centripetal) part  $\bar{l}_{v_{\eta z}}$  of the relative velocity  $\bar{l}_{\text{rel}v_{\eta}}$

$$v_{\eta z} = \mp g(\text{rel}v_{\eta}, n_{\perp}) n_{\perp} = \mp l_{\xi_{\perp}} d(n_{\perp}, m_{\perp}) n_{\perp} = \mp l_{v_{\eta z}} n_{\perp}, \quad (190)$$

$$\begin{aligned} d(n_{\perp}, m_{\perp}) &= \sigma(n_{\perp}, m_{\perp}) + \omega(n_{\perp}, m_{\perp}) + \frac{1}{n-1} \theta h_u(n_{\perp}, m_{\perp}) = \\ &= \sigma(n_{\perp}, m_{\perp}) + \omega(n_{\perp}, m_{\perp}), \end{aligned} \quad (191)$$

$$\begin{aligned} h_u(n_{\perp}, m_{\perp}) &= g(n_{\perp}, m_{\perp}) - \frac{1}{\pm l_u^2} g(u, n_{\perp}) g(u, m_{\perp}) = \\ &= g(n_{\perp}, m_{\perp}) = 0, \end{aligned} \quad (192)$$

$$v_{\eta z} = \mp l_{v_{\eta z}} n_{\perp} = H_{\eta z} l_{\xi_{\perp}} n_{\perp} = \mp l_{\xi_{\perp}} d(n_{\perp}, m_{\perp}) n_{\perp}, \quad (193)$$

$$H_{\eta z} = \mp d(n_{\perp}, m_{\perp}) = \mp [\sigma(n_{\perp}, m_{\perp}) + \omega(n_{\perp}, m_{\perp})] = H_c, \quad (194)$$

$$H_{\eta z} = H_c, \quad (195)$$

could be represented in its explicit form as

$$l_{v_{\eta z}} = \mp H_c l_{\xi_{\perp}}, \quad \bar{l}_{v_{\eta z}} = \frac{dl}{l_{\xi_{\perp}}} l_{v_{\eta z}} = \mp H_c dl, \quad (196)$$

$$\bar{l}_{v_{\eta z}} = \mp H_c dl, \quad \bar{v}_{\eta z} = \frac{dl}{l_{\xi_{\perp}}} v_{\eta z} = H_c dl n_{\perp}. \quad (197)$$

(c) Centrifugal part (centripetal)  $\bar{l}_{(a_\perp)_z}$  of the acceleration  $\bar{l}_{a_\perp}$

$$a_\perp = \mp g(a_\perp, n_\perp) n_\perp + \bar{g}[h_{\xi_\perp}(a_\perp)], \quad (198)$$

$$(a_\perp)_z = \mp l_{(a_\perp)_z} n_\perp, \quad (199)$$

$$\begin{aligned} g(a_\perp, n_\perp) &= l_{(a_\perp)_z} = g(\bar{g}[h_u(a)], n_\perp) = g_{ij}^{-1} g^{ik} h_{kl}^{-1} a^l n_\perp^j = g_j^k h_{kl}^{-1} a^l n_\perp^j = \\ &= h_{jl}^{-1} a^l n_\perp^j = h_u(n_\perp, a) = g(n_\perp, a), \end{aligned} \quad (200)$$

$$l_{(a_\perp)_z} = g(a_\perp, n_\perp), \quad \bar{l}_{(a_\perp)_z} = \frac{dl}{l_{\xi_\perp}} l_{(a_\perp)_z} = \frac{dl}{l_{\xi_\perp}} g(a_\perp, n_\perp). \quad (201)$$

(d) Centrifugal (centripetal) part  $\bar{l}_{a_z}$  of the relative acceleration  $\bar{l}_{rel a}$

$$a_z = \mp l_{\xi_\perp} A(n_\perp, n_\perp) n_\perp = \mp A(n_\perp, n_\perp) \xi_\perp = \mp l_{a_z} n_\perp = \bar{q} \xi_\perp = \bar{q} l_{\xi_\perp} n_\perp, \quad (202)$$

$$g(a_z, n_\perp) = l_{a_z} = l_{\xi_\perp} A(n_\perp, n_\perp) = \mp \bar{q} l_{\xi_\perp}, \quad (203)$$

$$A(n_\perp, n_\perp) = \mp \bar{q}, \quad (204)$$

$$\bar{q} = \frac{1}{n-1} U \mp_s D(n_\perp, n_\perp), \quad (205)$$

$$a_z = \bar{q} l_{\xi_\perp} n_\perp = \mp l_{a_z} n_\perp, \quad (206)$$

$$l_{a_z} = \mp \bar{q} l_{\xi_\perp}, \quad (207)$$

$$\bar{l}_{a_z} = \mp \bar{q} \frac{dl}{l_{\xi_\perp}} l_{\xi_\perp} = \mp \bar{q} dl, \quad (208)$$

$$\bar{a}_z = \frac{dl}{l_{\xi_\perp}} a_z = \mp \bar{l}_{a_z} n_\perp = \bar{q} dl n_\perp. \quad (209)$$

The last (above) two relations for  $\bar{a}_z$  and  $\bar{l}_{a_z}$  represent the part of the Hubble effect generated by the centrifugal (centripetal) part  $\bar{l}_{a_z}$  of the acceleration  $\bar{l}_{rel a}$ . The centrifugal (centripetal) acceleration is proportional to the distance  $dl$  between an emitter and a detector (observer).

(e) Centrifugal (centripetal) part  $\bar{l}_{a_{\eta z}}$  of the relative acceleration  $\bar{l}_{rel a_\eta}$

$$\begin{aligned} a_{\eta z} &= \mp g_{(rel a_\eta, n_\perp)} n_\perp = \mp l_{\xi_\perp} A(m_\perp, n_\perp) n_\perp = \\ &= \mp l_{a_{\eta z}} n_\perp = \bar{q}_\eta l_{\xi_\perp} n_\perp = \bar{q}_\eta \xi_\perp, \end{aligned} \quad (210)$$

$$g(a_{\eta z}, n_\perp) = l_{a_{\eta z}} = l_{\xi_\perp} A(m_\perp, n_\perp) = \mp \bar{q}_\eta l_{\xi_\perp}, \quad (211)$$

$$A(n_\perp, m_\perp) = \mp \bar{q}_\eta, \quad (212)$$

$$\bar{q}_\eta = \mp A(m_\perp, n_\perp) = \mp [{}_s D(m_\perp, n_\perp) + W(m_\perp, n_\perp)] = \bar{q}_c, \quad (213)$$

$$\bar{l}_{a_{\eta z}} = \mp \bar{q}_c \frac{dl}{l_{\xi_{\perp}}} l_{\xi_{\perp}} = \mp \bar{q}_c dl, \quad (214)$$

$$\bar{a}_{\eta z} = \frac{dl}{l_{\xi_{\perp}}} a_{\eta z} = \mp \bar{l}_{a_{\eta z}} n_{\perp} = \bar{q}_c dl n_{\perp}. \quad (215)$$

(f) Centrifugal (centripetal) part  $\bar{l}_{(\nabla_u a)_{\perp z}}$  of the change of the acceleration  $\bar{l}_{a_{\perp}}$

$$(\nabla_u a)_{\perp z} = \mp g((\nabla_u a)_{\perp}, n_{\perp}) n_{\perp} = \mp l_{(\nabla_u a)_{\perp z}} n_{\perp}, \quad (216)$$

$$l_{(\nabla_u a)_{\perp z}} = g((\nabla_u a)_{\perp}, n_{\perp}) = g(\bar{g}[h_u(\nabla_u a)], n_{\perp}) = h_u(n_{\perp}, \nabla_u a), \quad (217)$$

$$\bar{l}_{(\nabla_u a)_{\perp z}} = \frac{dl}{l_{\xi_{\perp}}} l_{(\nabla_u a)_{\perp z}} = \frac{dl}{l_{\xi_{\perp}}} g(\nabla_u a_{\perp}, n_{\perp}). \quad (218)$$

6.1.2. *Explicit Form of the Relative Velocities and Accelerations Generating a Transversal Doppler Effect*

(a) Coriolis part  $\bar{l}_{v_c}$  of relative Coriolis velocity  $\bar{l}_v$

$$v_c = \bar{g}[h_{\xi_{\perp}}(\text{rel}v)] \quad (219)$$

could be represented in its explicit form as

$$\begin{aligned} v_c = \bar{g}[\sigma(\xi_{\perp})] - \frac{\sigma(\xi_{\perp}, \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp} &= \mp l_{v_c} m_{\perp} = l_{\xi_{\perp}} \bar{g}[\sigma(n_{\perp})] \pm \\ &\pm \sigma(n_{\perp}, n_{\perp}) l_{\xi_{\perp}} n_{\perp} + l_{\xi_{\perp}} \bar{g}[\omega(n_{\perp})] = H_c l_{\xi_{\perp}} m_{\perp}. \end{aligned} \quad (220)$$

By the use of the relations

$$g(v_c, m_{\perp}) = \mp l_{v_c} g(m_{\perp}, m_{\perp}) = l_{v_c}, \quad (221)$$

$$g(v_c, m_{\perp}) = l_{\xi_{\perp}} g(\bar{g}[\sigma(n_{\perp})], m_{\perp}) + l_{\xi_{\perp}} g(\bar{g}[\omega(n_{\perp})], m_{\perp}) = l_{v_c}, \quad (222)$$

$$\begin{aligned} g(\bar{g}[\sigma(n_{\perp})], m_{\perp}) &= g_{\bar{i}\bar{j}} g^{ik} \sigma_{kl} n_{\perp}^{\bar{i}} m_{\perp}^{\bar{j}} = \\ &= g_j^k \sigma_{kl} n_{\perp}^{\bar{i}} m_{\perp}^{\bar{j}} = \sigma_{jl} m_{\perp}^{\bar{j}} n_{\perp}^{\bar{i}} = \sigma(m_{\perp}, n_{\perp}), \end{aligned} \quad (223)$$

$$g(\bar{g}[\omega(n_{\perp})], m_{\perp}) = \omega(m_{\perp}, n_{\perp}), \quad (224)$$

we obtain the expressions for  $l_{v_c}$  and  $v_c$ , respectively

$$l_{v_c} = [\sigma(m_{\perp}, n_{\perp}) + \omega(m_{\perp}, n_{\perp})] l_{\xi_{\perp}}, \quad (225)$$

$$v_c = \mp l_{v_c} m_{\perp} = H_c l_{\xi_{\perp}} m_{\perp} = \mp [\sigma(m_{\perp}, n_{\perp}) + \omega(m_{\perp}, n_{\perp})] l_{\xi_{\perp}} m_{\perp}, \quad (226)$$

where

$$H_c = \mp [\sigma(m_{\perp}, n_{\perp}) + \omega(m_{\perp}, n_{\perp})]. \quad (227)$$

Then

$$\bar{l}_{v_c} = \frac{dl}{l_{\xi_{\perp}}} l_{v_c} = \mp H_c dl, \quad (228)$$

$$\bar{l}_{v_c} = \mp H_c dl. \quad (229)$$

The last (above) expressions for  $\bar{l}_{v_c}$  and for  $\bar{v}_c$

$$\bar{l}_{v_c} = \mp H_c dl, \quad \bar{v}_c = \frac{dl}{l_{\xi_{\perp}}} v_c = H_c dl m_{\perp}, \quad (230)$$

are the relations describing the transversal Hubble law: the Coriolis relative velocity  $\bar{v}_c$  is proportional to the distance  $dl$  between an emitter and a detector (observer). This form of the Hubble law is a very special form of the law for the case when only the Coriolis relative velocity is taken into account.

(b) Coriolis part  $\bar{l}_{v_{\eta c}}$  of the relative velocity  $\bar{l}_{rel v_{\eta}}$

$$v_{\eta c} = l_{\xi_{\perp}} h^{n_{\perp}}[d(m_{\perp})] = \mp l_{v_{\eta c}} m_{\perp} = \bar{H}_c l_{\xi_{\perp}} m_{\perp}, \quad (231)$$

could be represented in its explicit form by the use of the relations

$$\begin{aligned} h^{n_{\perp}}[d(m_{\perp})] &= \left( \bar{g} - \frac{1}{g(n_{\perp}, n_{\perp})} n_{\perp} \otimes n_{\perp} \right) [d(m_{\perp})] = \\ &= \bar{g}[d(m_{\perp})] - \frac{1}{\mp l} (n_{\perp}) [d(m_{\perp})] n_{\perp} = \\ &= \bar{g}[d(m_{\perp})] \pm (n_{\perp}) [d(m_{\perp})] n_{\perp} = \bar{g}[d(m_{\perp})] \pm d(n_{\perp}, m_{\perp}) n_{\perp}, \end{aligned} \quad (232)$$

$$\begin{aligned} g(v_{\eta c}, m_{\perp}) &= l_{v_{\eta c}} = l_{\xi_{\perp}} g(\bar{g}[d(m_{\perp})], m_{\perp}) = l_{\xi_{\perp}} g_{\bar{r}\bar{j}} g^{ik} d_{\bar{k}\bar{l}} m_{\perp}^l m_{\perp}^j = \\ &= l_{\xi_{\perp}} g_j^k d_{\bar{k}\bar{l}} m_{\perp}^l m_{\perp}^j = l_{\xi_{\perp}} d_{\bar{j}\bar{l}} m_{\perp}^l m_{\perp}^j = l_{\xi_{\perp}} d(m_{\perp}, m_{\perp}) = \\ &= l_{\xi_{\perp}} \left\{ \left[ \sigma + \omega + \frac{1}{n-1} \theta h_u \right] (m_{\perp}) \right\} (m_{\perp}) = \\ &= l_{\xi_{\perp}} \left[ \sigma(m_{\perp}, m_{\perp}) \mp \frac{1}{n-1} \theta \right] = \mp l_{\xi_{\perp}} \left[ \frac{1}{n-1} \theta \mp \sigma(m_{\perp}, m_{\perp}) \right], \end{aligned} \quad (233)$$

$$l_{v_{\eta c}} = \mp \left[ \frac{1}{n-1} \theta \mp \sigma(m_{\perp}, m_{\perp}) \right] l_{\xi_{\perp}} = \mp \bar{H}_c l_{\xi_{\perp}}, \quad (234)$$

$$\bar{H}_c = \frac{1}{n-1} \theta \mp \sigma(m_{\perp}, m_{\perp}), \quad (235)$$

$$v_{\eta c} = \mp l_{v_{\eta c}} m_{\perp} = \bar{H}_c l_{\xi_{\perp}} m_{\perp}, \quad (236)$$

$$\bar{l}_{v_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} l_{v_{\eta c}} = \mp \bar{H}_c dl, \quad \bar{v}_{\eta c} = \frac{dl}{l_{\xi_{\perp}}} v_{\eta c} = \bar{H}_c dl m_{\perp}. \quad (237)$$

(c) Coriolis part  $\bar{l}_{(a_{\perp})_c}$  of acceleration  $\bar{l}_a$ . From the relations

$$(a_{\perp})_c = \bar{g}[h_{\xi_{\perp}}(a_{\perp})], \quad (238)$$

$$g(\xi_{\perp}, (a_{\perp})_c) = 0, \quad (a_{\perp})_c = \mp l_{(a_{\perp})_c} m_{\perp}, \quad (239)$$

$$\begin{aligned} l_{(a_{\perp})_c} &= g((a_{\perp})_c, m_{\perp}) = g(\bar{g}[h_{\xi_{\perp}}(a_{\perp})], m_{\perp}) = g_{ij} g^{ik} (h_{\xi_{\perp}})_{kl} a_{\perp}^l m_{\perp}^j = \\ &= h_{\xi_{\perp}}(m_{\perp}, a_{\perp}) = h_{n_{\perp}}(m_{\perp}, a_{\perp}) = g(m_{\perp}, a_{\perp}), \end{aligned} \quad (240)$$

$$\begin{aligned} h_{n_{\perp}}(m_{\perp}, a_{\perp}) &= [g \pm g(n_{\perp}) \otimes g(n_{\perp})](m_{\perp}, a_{\perp}) = \\ &= g(m_{\perp}, a_{\perp}) \pm g(n_{\perp}, m_{\perp})g(n_{\perp}, a_{\perp}) = g(m_{\perp}, a_{\perp}), \end{aligned} \quad (241)$$

it follows the form of  $\bar{l}_{(a_{\perp})_c}$

$$l_{(a_{\perp})_c} = g(m_{\perp}, a_{\perp}), \quad (242)$$

$$(a_{\perp})_c = \mp l_{(a_{\perp})_c} m_{\perp} = \mp g(m_{\perp}, a_{\perp}) m_{\perp}, \quad (243)$$

$$\bar{l}_{(a_{\perp})_c} = \frac{dl}{l_{\xi_{\perp}}} l_{(a_{\perp})_c}, \quad (244)$$

$$(\bar{a}_{\perp})_c = \frac{dl}{l_{\xi_{\perp}}} (a_{\perp})_c = \mp \bar{l}_{(a_{\perp})_c} m_{\perp}. \quad (245)$$

(d) Coriolis part  $\bar{l}_{a_c}$  of the relative acceleration  $\bar{l}_{rel a}$ . By means of the expressions

$$\begin{aligned} a_c &= \bar{g}[h_{\xi_{\perp}}(rel a)] = \bar{g}[h_{n_{\perp}}(rel a)] = l_{\xi_{\perp}} \bar{g}(h_{\xi_{\perp}})(\bar{g})[A(n_{\perp})] = \\ &= l_{\xi_{\perp}} h^{n_{\perp}}[A(n_{\perp})] = \mp l_{a_c} m_{\perp} = \bar{q}_c l_{\xi_{\perp}} m_{\perp}, \end{aligned}$$

$$a_c = l_{\xi_{\perp}} \{ \bar{g}[{}_s D(n_{\perp})] \mp {}_s D(n_{\perp}, n_{\perp}) n_{\perp} + \bar{g}[W(n_{\perp})] \} = \mp l_{a_c} m_{\perp}, \quad (246)$$

$$g(a_c, m_{\perp}) = l_{a_c} = l_{\xi_{\perp}} \{ g(\bar{g}[{}_s D(n_{\perp})], m_{\perp}) + g(\bar{g}[W(n_{\perp})], m_{\perp}) \},$$

$$g(\bar{g}[{}_s D(n_{\perp})], m_{\perp}) = g_{ij} g^{ik} {}_s D_{kl} n_{\perp}^l m_{\perp}^j = {}_s D_{jt} m_{\perp}^j n_{\perp}^t = {}_s D(m_{\perp}, n_{\perp}), \quad (247)$$

$$g(\bar{g}[W(n_{\perp})], m_{\perp}) = W(m_{\perp}, n_{\perp}), \quad (248)$$

the explicit form of  $\bar{l}_{a_c}$  follows

$$l_{a_c} = l_{\xi_{\perp}} [{}_s D(m_{\perp}, n_{\perp}) + W(m_{\perp}, n_{\perp})] = \mp l_{\xi_{\perp}} \bar{q}_c, \quad (249)$$

$$a_c = \mp l_{a_c} m_{\perp} = \bar{q}_c l_{\xi_{\perp}} m_{\perp}, \quad (250)$$

$$\bar{q}_c = \mp [{}_s D(m_{\perp}, n_{\perp}) + W(m_{\perp}, n_{\perp})], \quad (251)$$

$$\bar{l}_{a_c} = \frac{dl}{l_{\xi_{\perp}}} l_{a_c} = \mp \bar{q}_c dl, \quad \bar{a}_c = \frac{dl}{l_{\xi_{\perp}}} a_c = \bar{q}_c dl m_{\perp}. \quad (252)$$

(e) Coriolis part  $\bar{l}_{a_{\eta c}}$  of the relative acceleration  $\bar{l}_{\text{rel}a_{\eta}}$ . By the use of the relations

$$a_{\eta c} = \bar{g}[h_{\xi_{\perp}}(\text{rel}a_{\eta})] = \bar{g}[h_{n_{\perp}}(\text{rel}a_{\eta})] = l_{\xi_{\perp}} h^{n_{\perp}}[A(m_{\perp})] = \mp l_{a_{\eta c}} m_{\perp}, \quad (253)$$

$$a_{\eta c} = l_{\xi_{\perp}} h^{n_{\perp}}[A(m_{\perp})] = \mp l_{a_{\eta c}} m_{\perp}, \quad (254)$$

$$\begin{aligned} h^{n_{\perp}}[A(m_{\perp})] &= (\bar{g} \pm n_{\perp} \otimes n_{\perp}) \left[ {}_s D(m_{\perp}) + W(m_{\perp}) + \frac{1}{n-1} U h_u(m_{\perp}) \right] = \\ &= \bar{g} [{}_s D(m_{\perp})] + \bar{g} [W(m_{\perp})] \pm (n_{\perp}) {}_s D(m_{\perp}) n_{\perp} \pm (n_{\perp}) [W(m_{\perp})] n_{\perp}, \end{aligned} \quad (255)$$

$$\begin{aligned} g(a_{\eta c}, m_{\perp}) &= l_{a_{\eta c}} = l_{\xi_{\perp}} g(\bar{g} [{}_s D(m_{\perp})], m_{\perp}) + g(\bar{g} [W(m_{\perp})], m_{\perp}) = \\ &= l_{\xi_{\perp}} [{}_s D(m_{\perp}, m_{\perp}) + W(m_{\perp}, m_{\perp})] = l_{\xi_{\perp}} {}_s D(m_{\perp}, m_{\perp}), \end{aligned} \quad (256)$$

$$a_{\eta c} = \mp l_{a_{\eta c}} m_{\perp} = \bar{q}_{\eta c} l_{\xi_{\perp}} m_{\perp}, \quad (257)$$

$$\bar{q}_{\eta c} = \mp {}_s D(m_{\perp}, m_{\perp}), \quad l_{a_{\eta c}} = \mp \bar{q}_{\eta c} l_{\xi_{\perp}}, \quad (258)$$

the form of  $\bar{l}_{a_{\eta c}}$  follows

$$\bar{l}_{a_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} l_{a_{\eta c}} = \mp \bar{q}_{\eta c} dl, \quad \bar{a}_{\eta c} = \frac{dl}{l_{\xi_{\perp}}} a_{\eta c} = \bar{q}_{\eta c} dl m_{\perp}. \quad (259)$$

(f) Coriolis part  $\bar{l}_{(\nabla_u a)_{\perp c}}$  of the change  $\bar{l}_{(\nabla_u a)_{\perp}}$  of the acceleration  $\bar{l}_a$ . By means of the expressions

$$(\nabla_u a)_{\perp c} = \bar{g}[h_{\xi_{\perp}}(\nabla_u a)_{\perp}] = \mp l_{(\nabla_u a)_{\perp c}} m_{\perp}, \quad (260)$$

$$g(\xi_{\perp}, (\nabla_u a)_{\perp c}) = 0, \quad (261)$$

$$\bar{g}[h_{\xi_{\perp}}(\nabla_u a)_{\perp}] = \bar{g}[h_{n_{\perp}}(\nabla_u a)_{\perp}] = \mp l_{(\nabla_u a)_{\perp c}} m_{\perp}, \quad (262)$$

$$\begin{aligned} g((\nabla_u a)_{\perp c}, m_{\perp}) &= l_{(\nabla_u a)_{\perp c}} = g(\bar{g}[h_{n_{\perp}}(\nabla_u a)_{\perp}], m_{\perp}) = \\ &= g(\bar{g}[h_{n_{\perp}} \bar{g}[h_u(\nabla_u a)_{\perp}]], m_{\perp}) = g(\bar{g}[h_{n_{\perp}} \bar{g}[h_{n_{\parallel}}(\nabla_u a)_{\perp}]], m_{\perp}) = \\ &= g_{\bar{i}\bar{j}} g^{ik} (h_{n_{\perp}})_{\bar{k}\bar{l}} g^{lm} (h_{n_{\parallel}})_{\bar{m}\bar{n}} a^n ;_r u^r m_{\perp}^{\bar{j}} = (h_{n_{\perp}})_{\bar{j}\bar{l}} m_{\perp}^{\bar{j}} g^{lm} (h_{n_{\parallel}})_{\bar{m}\bar{n}} a^n ;_r u^r = \\ &= g_{\bar{j}\bar{l}} g^{lm} m_{\perp}^{\bar{j}} (h_{n_{\parallel}})_{\bar{m}\bar{n}} a^n ;_r u^r = (h_{n_{\parallel}})_{\bar{j}\bar{n}} m_{\perp}^{\bar{j}} a^n ;_r u^r = \\ &= h_{n_{\parallel}}(m_{\perp}, \nabla_u a) = h_u(m_{\perp}, \nabla_u a) = g(m_{\perp}, \nabla_u a), \end{aligned} \quad (263)$$

$$l_{(\nabla_u a)_{\perp c}} = h_{n_{\perp}}(m_{\perp}, \nabla_u a) = g(m_{\perp}, \nabla_u a), \quad (264)$$

$$(\nabla_u a)_{\perp c} = \mp l_{(\nabla_u a)_{\perp c}} m_{\perp} = \mp h_{n_{\perp}}(m_{\perp}, \nabla_u a) m_{\perp} = \mp g(m_{\perp}, \nabla_u a), \quad (265)$$

the form of  $\bar{l}_{(\nabla_u a)_{\perp c}}$  follows

$$\bar{l}_{(\nabla_u a)_{\perp c}} = \frac{dl}{l_{\xi_{\perp}}} l_{(\nabla_u a)_{\perp c}}, \quad (\nabla_u \bar{a})_{\perp} = \frac{dl}{l_{\xi_{\perp}}} (\nabla_u a)_{\perp c}. \quad (266)$$

**6.2. Standard (Longitudinal) Hubble Effect (Hubble Shift).** The standard (longitudinal) Hubble effect (Hubble shift) corresponds to the standard (longitudinal) Doppler effect (Doppler shift). Only the different types of velocities and accelerations generating the standard Doppler effect are given in their explicit form by means of the corresponding Hubble functions and acceleration parameters.

If the vector field  $k_\perp$  is collinear to the vector field  $\xi_\perp$  determining the proper frame of reference, i.e., if

$$k_\perp = \mp l_{k_\perp} n_\perp, \quad \cos \theta = \pm 1 \quad \sin \theta = 0, \quad (267)$$

the frequency of the emitter  $\bar{\omega}$  and the frequency  $\omega$  detected by the observer are related to each other by the expression

$$\bar{\omega} = \omega \left[ 1 - \frac{\bar{l}_{v_z}}{l_u} - \frac{l_{\xi_\perp}}{l_u^2} \bar{l}_{(a_\perp)_z} + \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_z} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(\nabla_u a)_\perp z} \right) \right]. \quad (268)$$

If we now replace the velocity  $\bar{l}_{v_z}$  and the accelerations  $\bar{l}_{(a_\perp)_z}$ ,  $\bar{l}_{a_z}$ , and  $\bar{l}_{(\nabla_u a)_\perp z}$  with their corresponding explicit forms

$$\bar{l}_{v_z} = \mp H dl, \quad (269)$$

$$\bar{l}_{a_z} = \mp \bar{q} dl, \quad (270)$$

$$\bar{l}_{(a_\perp)_z} = \frac{dl}{l_{\xi_\perp}} g(a_\perp, n_\perp), \quad (271)$$

$$\bar{l}_{(\nabla_u a)_\perp z} = \frac{dl}{l_{\xi_\perp}} g(\nabla_u a_\perp, n_\perp), \quad (272)$$

then we obtain the relation between  $\bar{\omega}$  and  $\omega$  in the form

$$\begin{aligned} \bar{\omega} &= \omega \left[ 1 - \frac{\bar{l}_{v_z}}{l_u} - \frac{l_{\xi_\perp}}{l_u^2} \bar{l}_{(a_\perp)_z} + \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_z} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(\nabla_u a)_\perp z} \right) \right] = \\ &= \omega \left\{ 1 - \frac{1}{l_u} \left[ \bar{l}_{v_z} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(a_\perp)_z} - \frac{1}{2} \frac{dl}{l_u} \left( \bar{l}_{a_z} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(\nabla_u a)_\perp z} \right) \right] \right\} = \\ &= \omega \left\{ 1 - \frac{1}{l_u} \left[ \mp H dl + \frac{l_{\xi_\perp}}{l_u} \frac{dl}{l_{\xi_\perp}} g(a_\perp, n_\perp) - \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \frac{dl}{l_u} \left( \mp \bar{q} dl + \frac{l_{\xi_\perp}}{l_u} \frac{dl}{l_{\xi_\perp}} g(\nabla_u a_\perp, n_\perp) \right) \right] \right\}, \quad (273) \end{aligned}$$

$$\begin{aligned} \bar{\omega} &= \omega \left\{ 1 - \frac{1}{l_u} \left[ \mp H dl + \frac{dl}{l_u} g(a_\perp, n_\perp) - \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \frac{dl}{l_u} \left( \mp \bar{q} dl + \frac{dl}{l_u} g(\nabla_u a_\perp, n_\perp) \right) \right] \right\}, \quad (274) \end{aligned}$$

$$\bar{\omega} = \omega \left\{ 1 + \frac{1}{l_u} \left[ \pm H dl - \frac{dl}{l_u} g(a_{\perp}, n_{\perp}) - \frac{1}{2} \frac{dl}{l_u} \left( \pm \bar{q} dl - \frac{dl}{l_u} g(\nabla_u a_{\perp}, n_{\perp}) \right) \right] \right\}, \quad (275)$$

$$\bar{\omega} = \omega \left\{ 1 \pm \frac{1}{l_u} \left[ H dl \mp \frac{dl}{l_u} g(a_{\perp}, n_{\perp}) - \frac{1}{2} \frac{dl}{l_u} \left( \bar{q} dl \mp \frac{dl}{l_u} g(\nabla_u a_{\perp}, n_{\perp}) \right) \right] \right\}. \quad (276)$$

*Special case:* Autoparallel motion of the observer:  $\nabla_u u = a = 0$ :  $\bar{l}_{(a_{\perp})z} = 0$ ,  $\bar{l}_{(\nabla_u a)_{\perp z}} = 0$ ,

$$\begin{aligned} \bar{\omega} &= \omega \left[ 1 - \frac{\bar{l}_{vz}}{l_u} + \frac{1}{2} \frac{dl}{l_u^2} \bar{l}_{az} \right], \\ \bar{\omega} &= \omega \left[ 1 \pm \frac{1}{l_u} \left( H dl - \frac{1}{2} \frac{dl}{l_u} \bar{q} dl \right) \right] = \omega \left[ 1 \pm \frac{1}{l_u} \left( H - \frac{1}{2} \frac{dl}{l_u} \bar{q} \right) dl \right]. \end{aligned} \quad (277)$$

If the world line of an observer is an autoparallel trajectory and  $k_{\perp}$  is collinear to  $\xi_{\perp}$ , then the change of the frequency of the emitter  $\bar{\omega}$  depends on the Hubble function  $H$  and the acceleration parameter  $\bar{q}$ .

*Special case:*  $\nabla_u u = a = 0$ ,  $k_{\perp} = \mp l_{k_{\perp}} n_{\perp}$ ,  $\bar{l}_{az} = 0$ :

$$\begin{aligned} \bar{\omega} &= \omega \left( 1 - \frac{\bar{l}_{vz}}{l_u} \right), \quad \bar{l}_{vz} \leq 0, \\ \bar{\omega} &= \omega \left( 1 \pm \frac{1}{l_u} H dl \right). \end{aligned} \quad (278)$$

Therefore, if the world line of an observer is an autoparallel trajectory,  $k_{\perp}$  is collinear to  $\xi_{\perp}$ , and no centrifugal (centripetal) acceleration  $\bar{l}_{az}$  exists between emitter and observer, then the above expression has the well-known form for description of the standard Hubble effect in relativistic astrophysics. Here, this relation is valid in every  $(\bar{\mathcal{T}}_n, g)$  space considered as a model of a space or a space-time under the given preconditions.

*6.2.1. Standard (Longitudinal) Hubble Shift Frequency Parameter  $z$ .* The relative difference between both the frequencies (emitted  $\bar{\omega}$  and detected  $\omega$ )

$$\frac{\bar{\omega} - \omega}{\omega} := z = \pm \bar{C}, \quad (279)$$

under the condition  $k_{\perp} = \mp l_{k_{\perp}} n_{\perp}$  appears in the form

$$\frac{\bar{\omega} - \omega}{\omega} := z = \pm \bar{C} = \pm \frac{1}{l_u} \left[ H dl \mp \frac{dl}{l_u} g(a_{\perp}, n_{\perp}) - \frac{1}{2} \frac{dl}{l_u} \left( \bar{q} dl \mp \frac{dl}{l_u} g(\nabla_u a_{\perp}, n_{\perp}) \right) \right], \quad (280)$$

where

$$\bar{\omega} = (1 + z)\omega, \quad (281)$$

$$z = \pm \frac{1}{l_u} \left[ H dl \mp \frac{dl}{l_u} g(a_{\perp}, n_{\perp}) - \frac{1}{2} \frac{dl}{l_u} \left( \bar{q} dl \mp \frac{dl}{l_u} g(\nabla_u a_{\perp}, n_{\perp}) \right) \right]. \quad (282)$$

The quantity  $z$  could be denoted as an *observed standard (longitudinal) Hubble shift frequency parameter*. If  $z = 0$ , then there will be no difference between the emitted and the detected frequencies, i.e.,  $\bar{\omega} = \omega$ . This will be the case when the emitter and observer (detector) are at rest to each other, i.e., when no relative velocities and relative accelerations occur, when centrifugal (centripetal) relative velocities and accelerations do not exist, or when the centripetal (centrifugal) velocities and accelerations compensate each other under the condition

$$H \mp \frac{1}{l_u} g(a_{\perp}, n_{\perp}) - \frac{1}{2} \frac{1}{l_u} \left( \bar{q} \mp \frac{1}{l_u} g(\nabla_u a_{\perp}, n_{\perp}) \right) dl = 0. \quad (283)$$

If  $z > 0$ , the observed Hubble shift frequency parameter is called *longitudinal Hubble red shift*. If  $z < 0$ , the observed Hubble shift frequency parameter is called *longitudinal Hubble blue shift*. If  $\bar{\omega}$  and  $\omega$  are known, the observed Hubble shift frequency parameter  $z$  could be found. If  $\omega$  and  $z$  are given, then the corresponding  $\bar{\omega}$  could be estimated.

On the other side, from the explicit form of  $z$

$$z = \pm \frac{1}{l_u} \left[ H dl \mp \frac{dl}{l_u} g(a_{\perp}, n_{\perp}) - \frac{1}{2} \frac{dl}{l_u} \left( \bar{q} dl \mp \frac{dl}{l_u} g(\nabla_u a_{\perp}, n_{\perp}) \right) \right], \quad (284)$$

if we consider the explicit form of the Hubble function  $H$  and of the acceleration parameter  $\bar{q}$ , we could find the relation between the observed shift frequency parameter  $z$  and the kinematic characteristics of the relative velocity such as expansion and shear velocities and accelerations.

*Special case:* Autoparallel motion of the observer:  $\nabla_u u = a = 0$ :  $\bar{l}_{(a_{\perp})z} = 0$ ,  $\bar{l}_{(\nabla_u a)_{\perp z}} = 0$ ,

$$\bar{\omega} = \omega \left[ 1 - \frac{\bar{l}_{v_z}}{l_u} + \frac{1}{2} \frac{dl}{l_u^2} \bar{l}_{a_z} \right], \quad (285)$$

$$z = \pm \frac{1}{l_u} \left( H dl - \frac{1}{2} \frac{dl}{l_u} \bar{q} dl \right) = \pm \frac{1}{l_u} \left( H - \frac{1}{2} \frac{dl}{l_u} \bar{q} \right) dl.$$

If the world line of an observer is an autoparallel trajectory and  $k_{\perp}$  is collinear to  $\xi_{\perp}$ , then the observed Hubble shift frequency parameter  $z$  depends on the Hubble function  $H$  and on the acceleration parameter  $\bar{q}$  as well as on the absolute value  $l_u$  of the velocity of the signal and on the space distance  $dl$  propagated by the signal.

*Special case:*  $\nabla_u u = a = 0$ ,  $k_{\perp} = \mp l_{k_{\perp}} n_{\perp}$ ,  $\bar{l}_{a_z} = 0$ :

$$\begin{aligned}\bar{\omega} &= \omega \left( 1 - \frac{\bar{l}_{v_z}}{l_u} \right), \quad \bar{l}_{v_z} \leq 0, \\ z &= \pm \frac{1}{l_u} H dl.\end{aligned}\tag{286}$$

*Remark.* In relativistic physics ( $l_u = c, 1$ ) the last (above) relation is also called Hubble law.

If we express in this special case the observed longitudinal Hubble shift frequency parameter  $z$  in its infinitesimal form

$$z = \frac{\bar{\omega} - \omega}{\omega} = \frac{d\omega}{\omega} = \pm \frac{1}{l_u} H dl,\tag{287}$$

then we can find the change of the frequency  $\omega$  for a global distance  $l$  propagated by a signal for finite proper time interval of an observer

$$\begin{aligned}\int \frac{d\omega}{\omega} &= \pm \int \frac{1}{l_u} H dl, \\ \log \omega &= \pm \int \frac{1}{l_u} H dl + \text{const}, \\ \omega &= \omega_0 \exp \left( \pm \int \frac{1}{l_u} H dl \right), \quad \omega_0 = \text{const}.\end{aligned}\tag{288}$$

In the relativistic astrophysics, it is assumed that  $l_u = \text{const} = c$  or  $1$ . The Hubble function is also assumed to be a constant function  $H = H_0 = \text{const}$ .

Then

$$\bar{\omega} = \omega_0 \exp \left( \pm \frac{1}{l_u} H_0 l \right) = \omega_0 \exp \left( \pm H_0 \frac{l}{l_u} \right) = \omega_0 \exp (\pm H_0 \tau),\tag{289}$$

where  $l_u$  is the absolute value of the velocity of a signal;  $\tau$  is the proper time interval of the observer for which a signal propagates from the emitter to the observer, and  $l$  is the space distance covered by the signal from the emitter to the observer.

Therefore, if the world line of an observer is an autoparallel trajectory,  $k_\perp$  is collinear to  $\xi_\perp$ , and no centrifugal (centripetal) acceleration  $\bar{l}_{a_z}$  exists between an emitter and an observer, then the above expression has the well-known form for description of the standard Hubble effect in relativistic astrophysics. Here, the relation is valid in every  $(\bar{L}_n, g)$  space considered as a model of a space or a space-time under the given preconditions.

**6.3. Transversal Hubble Effect (Hubble Shift).** The transversal Hubble effect (Hubble shift) corresponds to the transversal Doppler effect (Doppler shift). Only the different types of velocities and accelerations generating the transversal Doppler effect are given in their explicit form by means of the corresponding Hubble functions and acceleration parameters.

If the vector field  $k_\perp$  is collinear to the vector field  $\xi_\perp$  determining the proper frame of reference, i.e., if

$$k_\perp = \mp l_{k_\perp} n_\perp, \quad \cos \theta = \pm 1, \quad \sin \theta = 0, \quad (290)$$

the frequency of the emitter  $\bar{\omega}$  and the frequency  $\omega$  detected by the observer are related to each other by the expression

$$\begin{aligned} \bar{\omega} &= \omega \left[ 1 - \frac{1}{l_u} \bar{l}_{v_{\eta c}} - \frac{l_{\xi_\perp}}{l_u^2} \bar{l}_{(a_\perp)_c} + \frac{1}{2} \frac{dl}{l_u^2} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(\nabla_u a)_\perp c} \right) \right] = \\ &= \omega \left\{ 1 - \frac{1}{l_u} \left[ \bar{l}_{v_{\eta c}} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(a_\perp)_c} - \frac{1}{2} \frac{dl}{l_u} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(\nabla_u a)_\perp c} \right) \right] \right\}. \quad (291) \end{aligned}$$

If we now replace the velocity  $\bar{l}_{v_{\eta c}}$  and the accelerations  $\bar{l}_{(a_\perp)_c}$ ,  $\bar{l}_{a_{\eta c}}$ , and  $\bar{l}_{(\nabla_u a)_\perp c}$  with their corresponding explicit forms

$$\bar{l}_{v_{\eta c}} = \mp \bar{H}_c dl, \quad (292)$$

$$\bar{l}_{(a_\perp)_c} = \frac{dl}{l_{\xi_\perp}} l_{(a_\perp)_c} = \frac{dl}{l_{\xi_\perp}} g(m_\perp, a_\perp), \quad (293)$$

$$\bar{l}_{a_{\eta c}} = \mp \bar{q}_{\eta c} dl, \quad (294)$$

$$\bar{l}_{(\nabla_u a)_\perp c} = \frac{dl}{l_{\xi_\perp}} l_{(\nabla_u a)_\perp c} = \frac{dl}{l_{\xi_\perp}} g(m_\perp, \nabla_u a), \quad (295)$$

then we obtain the relation between  $\bar{\omega}$  and  $\omega$  in the form

$$\begin{aligned} \bar{\omega} &= \omega \left\{ 1 - \frac{1}{l_u} \left[ \bar{l}_{v_{\eta c}} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(a_\perp)_c} - \frac{1}{2} \frac{dl}{l_u} \left( \bar{l}_{a_{\eta c}} + \frac{l_{\xi_\perp}}{l_u} \bar{l}_{(\nabla_u a)_\perp c} \right) \right] \right\} = \\ &= \omega \left\{ 1 - \frac{1}{l_u} \left[ \mp \bar{H}_c dl + \frac{l_{\xi_\perp}}{l_u} \frac{dl}{l_{\xi_\perp}} g(m_\perp, a_\perp) \right] - \right. \\ &\quad \left. - \frac{1}{2} \frac{dl}{l_u} \left[ \mp \bar{q}_{\eta c} dl + \frac{l_{\xi_\perp}}{l_u} \frac{dl}{l_{\xi_\perp}} g(m_\perp, \nabla_u a) \right] \right\}, \quad (296) \end{aligned}$$

$$\bar{\omega} = \omega \left\{ 1 \pm \frac{1}{l_u} \left[ \bar{H}_c dl \mp \frac{dl}{l_u} g(m_\perp, a_\perp) - \frac{1}{2} \frac{dl}{l_u} \left( \bar{q}_{\eta c} dl \mp \frac{dl}{l_u} g(m_\perp, \nabla_u a) \right) \right] \right\}, \quad (297)$$

compared with the case of the standard (longitudinal) Hubble effect

$$\bar{\omega} = \omega \left\{ 1 \pm \frac{1}{l_u} \left[ H dl \mp \frac{dl}{l_u} g(a_\perp, n_\perp) - \frac{1}{2} \frac{dl}{l_u} \left( \bar{q} dl \mp \frac{dl}{l_u} g(\nabla_u a_\perp, n_\perp) \right) \right] \right\}.$$

*Special case:* Autoparallel motion of the observer:  $\nabla_u u = a = 0$ :  $\bar{l}_{(a_\perp)_c} = 0$ ,  $\bar{l}_{(\nabla_u a)_\perp c} = 0$ ,

$$\bar{\omega} = \omega \left[ 1 - \frac{1}{l_u} \bar{l}_{v_{\eta c}} + \frac{1}{2} \frac{dl}{l_u^2} \bar{l}_{a_{\eta c}} \right], \quad (298)$$

$$\bar{\omega} = \omega \left[ 1 \pm \frac{1}{l_u} \left( \bar{H}_c dl - \frac{1}{2} \frac{dl}{l_u} \bar{q}_{\eta c} dl \right) \right] = \omega \left[ 1 \pm \frac{1}{l_u} \left( \bar{H}_c - \frac{1}{2} \frac{dl}{l_u} \bar{q}_{\eta c} \right) dl \right].$$

If the world line of an observer is an autoparallel trajectory and  $k_\perp$  is orthogonal to  $\xi_\perp$ , then the change of the frequency of the emitter  $\bar{\omega}$  depends on the Coriolis velocity  $\bar{l}_{v_{\eta c}}$  and the Coriolis acceleration  $\bar{l}_{a_{\eta c}}$ .

*Special case:*  $\nabla_u u = a = 0$ ,  $k_\perp = \mp l_{k_\perp} m_\perp$ ,  $\bar{l}_{a_{\eta c}} = 0$ :

$$\bar{\omega} = \omega \left( 1 - \frac{\bar{l}_{v_{\eta c}}}{l_u} \right), \quad \bar{l}_{v_{\eta z}} \begin{matrix} \leq \\ \geq \end{matrix} 0, \quad (299)$$

$$\bar{\omega} = \omega \left( 1 \pm \frac{1}{l_u} \bar{H}_c dl \right).$$

Therefore, if the world line of an observer is an autoparallel trajectory,  $k_\perp$  is orthogonal to  $\xi_\perp$ , and no Coriolis acceleration  $\bar{l}_{a_{\eta c}}$  exists between emitter and observer, then the above expression has analogous form for description of the transversal Hubble effect as the standard (longitudinal) Hubble effect in relativistic astrophysics. Here, this relation for the transversal Hubble effect is valid in every  $(\bar{L}_n, g)$  space considered as a model of a space or a space-time under the given preconditions.

*6.3.1. Transversal Hubble Shift Frequency Parameter  $z_c$ .* The relative difference between both the frequencies (emitted  $\bar{\omega}$  and detected  $\omega$ ) when a transversal Doppler effect and a transversal Hubble effect correspondingly occur could be written as

$$\frac{\bar{\omega} - \omega}{\omega} := z_c = \pm \bar{S} \quad (300)$$

and under the condition  $k_{\perp} = \mp l_{k_{\perp}} n_{\perp}$  appears in the form

$$\frac{\bar{\omega} - \omega}{\omega} := z_c = \pm \frac{1}{l_u} \left[ \bar{H}_c dl \mp \frac{dl}{l_u} g(m_{\perp}, a_{\perp}) - \frac{1}{2} \frac{dl}{l_u} \bar{q}_{\eta c} dl \mp \frac{dl}{l_u} g(m_{\perp}, \nabla_u a) \right], \quad (301)$$

where

$$\bar{\omega} = (1 + z_c)\omega, \quad (302)$$

$$z_c = \pm \frac{1}{l_u} \left[ \bar{H}_c dl \mp \frac{dl}{l_u} g(m_{\perp}, a_{\perp}) - \frac{1}{2} \frac{dl}{l_u} \left( \bar{q}_{\eta c} dl \mp \frac{dl}{l_u} g(m_{\perp}, \nabla_u a) \right) \right]. \quad (303)$$

The quantity  $z_c$  could be denoted as an *observed transversal Hubble shift frequency parameter*. If  $z_c = 0$ , then there will be no difference between the emitted and the detected frequencies, i.e.,  $\bar{\omega} = \omega$ . This will be the case when the emitter and observer (detector) are at rest to each other, i.e., when no relative velocities and relative accelerations occur, when Coriolis relative velocities and Coriolis relative accelerations do not exist, or when the Coriolis centripetal velocities and accelerations compensate each other under the condition

$$\bar{H}_c \mp \frac{1}{l_u} g(m_{\perp}, a_{\perp}) - \frac{1}{2} \frac{1}{l_u} \left( \bar{q}_{\eta c} \mp \frac{1}{l_u} g(m_{\perp}, \nabla_u a) \right) dl = 0. \quad (304)$$

If  $z_c > 0$ , the observed transversal Hubble shift frequency parameter is called *transversal Hubble red shift*. If  $z_c < 0$ , the observed transversal Hubble shift frequency parameter is called *transversal Hubble's blue shift*. If  $\bar{\omega}$  and  $\omega$  are known the observed transversal Hubble shift frequency parameter  $z_c$  could be found. If  $\omega$  and  $z$  are given, then the corresponding  $\bar{\omega}$  could be estimated.

On the other side, from the explicit form of  $z_c$

$$z_c = \pm \frac{1}{l_u} \left[ \bar{H}_c dl \mp \frac{dl}{l_u} g(m_{\perp}, a_{\perp}) - \frac{1}{2} \frac{dl}{l_u} \left( \bar{q}_{\eta c} dl \mp \frac{dl}{l_u} g(m_{\perp}, \nabla_u a) \right) \right] \quad (305)$$

if we consider the explicit form of the Hubble function  $\bar{H}_c$  and of the acceleration parameter  $\bar{q}_{\eta c}$ , we could find the relation between the observed shift frequency parameter  $z_c$  and the kinematic characteristics of the relative velocity such as expansion and shear velocities and accelerations.

*Special case:* Autoparallel motion of the observer:  $\nabla_u u = a = 0$ :  $\bar{l}_{(a_{\perp})z} = 0$ ,  $\bar{l}_{(\nabla_u a)_{\perp}z} = 0$ ,

$$\bar{\omega} = \omega \left[ 1 - \frac{1}{l_u} \bar{l}_{v_{\eta c}} + \frac{1}{2} \frac{dl}{l_u^2} \bar{l}_{a_{\eta c}} \right], \quad (306)$$

$$z_c = \pm \frac{1}{l_u} \left( \bar{H}_c - \frac{1}{2} \frac{dl}{l_u} \bar{q}_{\eta c} \right) dl.$$

If the world line of an observer is an autoparallel trajectory and  $k_{\perp}$  is collinear to  $\xi_{\perp}$ , then the observed transversal Hubble shift frequency parameter  $z_c$  depends on the Hubble function  $\overline{H}_c$  and of the acceleration parameter  $\overline{q}_{\eta c}$  as well as on the absolute value  $l_u$  of the velocity of the signal and on the space distance  $dl$  propagated by the signal.

*Special case:*  $\nabla_u u = a = 0$ ,  $k_{\perp} = \mp l_{k_{\perp}} n_{\perp}$ ,  $\overline{l}_{a_z} = 0$ :

$$\begin{aligned} \overline{\omega} &= \omega \left( 1 - \frac{\overline{l}_{v_{\eta c}}}{l_u} \right), \quad \overline{l}_{v_{\eta z}} \begin{matrix} \leq \\ \geq \end{matrix} 0, \\ z_c &= \pm \frac{1}{l_u} \overline{H}_c dl. \end{aligned} \tag{307}$$

If we express in this special case the observed transversal Hubble shift frequency parameter  $z_c$  in its infinitesimal form

$$z_c = \frac{\overline{\omega} - \omega}{\omega} = \frac{d\omega}{\omega} = \pm \frac{1}{l_u} \overline{H}_c dl, \tag{308}$$

then we can find the change of the frequency  $\omega$  for a global distance  $l$  propagated by a signal for finite proper time interval of an observer

$$\begin{aligned} \int \frac{d\omega}{\omega} &= \pm \int \frac{1}{l_u} \overline{H}_c dl, \\ \log \omega &= \pm \int \frac{1}{l_u} \overline{H}_c dl + \text{const}, \\ \omega &= \omega_0 \exp \left( \pm \int \frac{1}{l_u} \overline{H}_c dl \right), \quad \omega_0 = \text{const}. \end{aligned} \tag{309}$$

In the relativistic astrophysics, it is assumed that  $l_u = \text{const} = c$  or 1; if the Hubble function is also assumed to be a constant function  $\overline{H}_c = \overline{H}_{c0} = \text{const}$ , then

$$\overline{\omega} = \omega_0 \exp \left( \pm \frac{1}{l_u} \overline{H}_{c0} l \right) = \omega_0 \exp \left( \pm \overline{H}_{c0} \frac{l}{l_u} \right) = \omega_0 \exp (\pm \overline{H}_{c0} \tau), \tag{310}$$

where  $l_u$  is the absolute value of the velocity of a signal;  $\tau$  is the proper time interval of the observer for which a signal propagates from the emitter to the observer, and  $l$  is the space distance covered by the signal from the emitter to the observer. If  $l = \text{const}$ , then the  $\overline{\omega}$  will change by a constant quantity

$$\overline{\omega} = K_0 \omega, \quad K_0 = \exp \left( \pm \overline{H}_{c0} \frac{l}{l_u} \right) = \exp (\pm \overline{H}_{c0} \tau) = \text{const}, \tag{311}$$

and we will observe a constant shift of the emitted frequency with respect to the observer (detector) during a time interval. If we further write  $K_0$  in the form

$$K_0 = 1 \pm \bar{K}_0, \quad \bar{K}_0 = \text{const},$$

the change of the frequency  $\bar{\omega}$  could be represented in the forms

$$\bar{\omega} = (1 \pm \bar{K}_0)\omega, \quad (312)$$

$$z_c = \pm \bar{K}_0 = \text{const}. \quad (313)$$

If the space distance between an emitter and an observer does not change but there is a Coriolis velocity  $\bar{l}_{v_{\eta c}}$  between them, then there will be a constant difference  $z_c = \pm \bar{K}_0 = \text{const}$  between the emitted frequency  $\bar{\omega}$  and the detected frequency  $\omega$ . This could be the case when an emitter sends signals to a detector and rotates along the detector at a constant space distance from it. At the same time the detector moves at an autoparallel world line.

Therefore, if the world line of an observer is an autoparallel trajectory,  $k_{\perp}$  is collinear to  $\xi_{\perp}$ , and no Coriolis acceleration  $\bar{l}_{a_{\eta c}}$  exists between emitter and observer, then the above expression could be used for description of the transversal Hubble effect in relativistic astrophysics. Here, the relation is valid in every  $(\bar{L}_n, g)$  space considered as a model of a space or of a space-time under the given preconditions.

## 7. COMMENTS

1. It has been shown that the propagation of signals in a space or in a space-time is in close relations to the kinematic characteristics of the velocity of the signals, considered as the tangent vector of the world line of an observer detecting the signals. The absolute value of this velocity is exactly equal to the absolute value of the length of the tangent vector to the world line of the observer. A null (isotropic) vector (called also wave vector) describes the propagation of signals and characterizes the direction and the frequency of the signals. The velocity of the signals is usually assumed to be preliminary given for an observer. The wave vector does not determine the velocity of a signal because it contains the notion of velocity in its definition.

2. The relative velocity and the relative acceleration between the emitter and the observer are the reasons for the change of the wave vector determining the direction of the propagation of the signals and their frequencies. A wave vector field is identified in a space-time as a null (isotropic) vector field. The null vector field could be represented by the use of  $(n - 1) + 1$  projective formalism into two parts: a time-like part and a space-like part. The space-like part describes

uniquely a null vector field because the properties of a null vector field are reduced to these of its space-like part. On these grounds, the change of the wave vector in a time period is considered from the point of view of an observer. The observer detects the signal at his world line and compares it with a signal emitted by an emitter who is at rest with respect to the proper frame of reference of the observer.

3. The observations of the change of the wave vector and the frequency of a signal have begun at a time point as a starting point where the emitter and the observer are at rest to each other in the proper frame of reference of the observer. Furthermore, the process of the propagation of the signal is considered by the observer in his proper frame of reference. The propagation of the signal in a time interval is identified with the propagation of the signal in space. The covered space distance is then found as the product of the velocity of the signal and the time interval passed for covering the space distance (both measured in the proper frame of reference of the observer).

4. The relative velocity and the relative acceleration are related to the motion between a point lying at the world line of an observer and points of a subspace orthogonal to the world line and having one and the same proper time measured by the observer for the point at his world line, i.e., the relative velocity and the relative acceleration are related to the motion between a point at the world line of an observer and the corresponding points lying at a cross section orthogonal to the world line. As kinematic characteristics of a relative motion between particles (mass elements) the notions of relative velocity and of relative acceleration could be applied to description of relative motions of particles (mass elements) in a continuous media or to description of relative motions between single particles moving in space-time. The considerations are made from the point of view of the one particle identified as an observer (detector). The representation of the relative velocity and the relative acceleration in their corresponding centrifugal (centripetal) and Coriolis parts allow an observer to investigate the changes of the direction of a wave vector and of the corresponding frequency during a time interval in his proper frame of reference. On this basis the observer could consider the change of the direction and of the frequency of a signal emitted by an emitter moving relatively to the observer. The results of the considerations are related to the well-known notions of aberration, Doppler effect, and Hubble effect. All notions are proved to be closely related to each other and to the kinematic characteristics of the relative motion, such as shear velocity and acceleration, expansion velocity and acceleration, and rotation velocity and acceleration.

5. The theoretical considerations and results leading to the existence of two types (longitudinal and transversal) of the Doppler effect and their corresponding two types (longitudinal and transversal) of the Hubble effect are a well defined ground for an experimental check-up of the applications of these effects for description of propagation of signals in spaces with affine connections and metrics considered as models of space or of space-time.

### CONCLUSION

In the present paper we have considered the notion of null (isotropic) vector field in spaces with affine connections and metrics for describing effects caused by the relative motion between emitters and detectors in spaces with affine connections and metrics used as models of space or of space-time. On the basis of the notions of centrifugal (centripetal) and Coriolis velocities and accelerations the notions of aberration, standard (longitudinal) and transversal Doppler effects, and standard and transversal Hubble effects are introduced and considered. It is shown that the reasons for aberration, the Doppler effect, and Hubble effect could be not only relative velocities between an emitter and a detector but also relative accelerations between them. It is shown that the Hubble effect is nothing more than the Doppler effect with explicitly given structures of the relative velocities and relative accelerations. By the use of the Hubble law, leading to the introduction of the Hubble effect, some connections between the kinematic characteristics of the relative velocity and the relative acceleration, on the one side, and the Doppler effects, the Hubble effects, and the aberration, on the other side, are investigated.

The aberration, the Doppler effects, and the Hubble effects are considered on the grounds of purely kinematic considerations. It should be stressed that the Hubble functions  $H$  and  $\overline{H}_c$  are introduced on a purely kinematic basis related to the notions of relative centrifugal (centripetal) velocity and to the notions of Coriolis velocities, respectively. Their dynamic interpretations in a theory of gravitation depend on the structures of the theory and the relations between the field equations and on both the functions. In this paper it is shown that notions the specialists use to apply in theories of gravitation and cosmological models could have a good kinematic grounds independent of any concrete classical field theory. The aberration, the Doppler and Hubble effects could be used in mechanics of continuous media and in other classical field theories in the same way as the standard Doppler effect is used in classical and relativistic mechanics.

### APPENDIX 1

The change of the contravariant vector field  $d$  under the change  $d\tau$  of the parameter  $\tau$  could be expressed in the form

$$\frac{d}{d\tau} = \frac{dx^i}{d\tau} \partial_i = u + l_v \bar{\xi} = \bar{u}^i \partial_i = \bar{u}, \quad \bar{u}^i = \frac{dx^i}{d\tau}, \quad (314)$$

where the relations are valid

$$\begin{aligned} g(\bar{u}, u) &= g(u, u) + l_v g(\bar{\xi}, u), \\ g(\bar{u}, \bar{\xi}) &= g(u, \bar{\xi}) + l_v g(\bar{\xi}, \bar{\xi}), \\ g(\bar{u}, \bar{u}) &= g(u + l_v \bar{\xi}, u + l_v \bar{\xi}) = g(u, u) + 2l_v g(u, \bar{\xi}) + l_v^2 g(\bar{\xi}, \bar{\xi}). \end{aligned} \quad (315)$$

The contravariant vector field  $\bar{u} = \bar{u}^i \partial_i$  is usually interpreted as the velocity of an observer moving in a space-time described by a  $(\bar{L}_n, g)$  space as its model. The contravariant vector  $u$  is a tangent vector field to the curve  $x^i(\tau, r(\tau, \lambda) = r_0 = \text{const}) = x^i(\tau, \lambda = \lambda_0 = \text{const})$

$$u = u^i \partial_i = \frac{\partial x^i}{\partial \tau} \partial_i, \quad (316)$$

$$\bar{u} = \frac{1}{g(u, u)} g(\bar{u}, u) u + \bar{g}[h_u(\bar{u})].$$

The contravariant vector  $\bar{\xi}$  is a collinear vector to the tangent vector  $\xi$  to the curve  $x^i(\tau = \tau_0 = \text{const}, r(\tau_0, \lambda)) = x^i(\tau = \tau_0 = \text{const}, \lambda)$ . This is so because of the relations

$$\bar{\xi} = \bar{\xi}^i \partial_i = \frac{\partial x^i}{\partial r} \partial_i, \quad (317)$$

$$\frac{\partial x^i}{\partial r} = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} = \frac{\partial x^i(\tau, \lambda(r, \tau))}{\partial r} = \frac{\partial x^i}{\partial \lambda} \frac{\partial \lambda}{\partial r} = \xi^i \frac{\partial \lambda}{\partial r} = \bar{\xi}^i, \quad (318)$$

where

$$r = r(\lambda, \tau), \quad \lambda = \lambda(\tau, r), \quad (319)$$

$$\bar{\xi} = \bar{\xi}^i \partial_i = \frac{\partial \lambda}{\partial r} \xi^i \partial_i = \frac{\partial \lambda}{\partial r} \xi, \quad \xi = \frac{\partial x^i}{\partial \lambda} \partial_i.$$

3. Further, since we wish to consider the vector field  $u$  as the velocity vector field of an observer moving at the curve  $x^i(\tau, \lambda = \lambda_0 = \text{const})$ , interpreted as his world line, the vector field  $\xi$  (and  $\bar{\xi}$ , respectively) could be chosen to lie in the subspace orthogonal to  $u$ , i.e.,  $u$  and  $\bar{\xi}$  could obey the condition  $g(u, \bar{\xi}) = 0$  and, therefore,  $g(u, \xi) = 0$ ,  $\xi = \xi_{\perp} = \bar{g}[h_u(\xi)]$ , and  $\bar{\xi} = \bar{\xi}_{\perp}$ .

4. In the next step, we could consider the vector field  $\bar{\xi}$  as a unit vector field in direction of the vector field  $\xi$ , i.e.,

$$\bar{\xi}_{\perp} = n_{\perp} = \frac{\xi_{\perp}}{l_{\xi_{\perp}}}, \quad g(u, n_{\perp}) = 0, \quad (320)$$

$$g(\bar{\xi}_{\perp}, \bar{\xi}_{\perp}) = g(n_{\perp}, n_{\perp}) = \frac{1}{l_{\xi_{\perp}}^2} g(\xi_{\perp}, \xi_{\perp}) = \mp \frac{1}{l_{\xi_{\perp}}^2} l_{\xi_{\perp}}^2 = \mp 1, \quad (321)$$

$$g(\bar{\xi}_{\perp}, \bar{\xi}_{\perp}) = g\left(\frac{\partial \lambda}{\partial r} \xi_{\perp}, \frac{\partial \lambda}{\partial r} \xi_{\perp}\right) = \left(\frac{\partial \lambda}{\partial r}\right)^2 g(\xi_{\perp}, \xi_{\perp}) = \mp \left(\frac{\partial \lambda}{\partial r}\right)^2 l_{\xi_{\perp}}^2 = \mp 1, \quad (322)$$

$$\left(\frac{\partial \lambda}{\partial r}\right)^2 l_{\xi_{\perp}}^2 = 1, \quad l_{\xi_{\perp}}^2 = \left(\frac{\partial \lambda}{\partial r}\right)^{-2}, \quad l_{\xi_{\perp}} = \pm \left(\frac{\partial \lambda}{\partial r}\right)^{-1}. \quad (323)$$

After all above considerations for  $\xi_{\perp}$  and  $\bar{\xi}_{\perp}$  we obtain the relations

$$\begin{aligned} g(\bar{u}, u) &= g(u, u), \\ g(\bar{u}, \bar{\xi}_{\perp}) &= l_v g(\bar{\xi}_{\perp}, \bar{\xi}_{\perp}) = l_v g(n_{\perp}, n_{\perp}) = \mp l_v, \\ g(\bar{u}, \bar{u}) &= g(u, u) + l_v^2 g(n_{\perp}, n_{\perp}) = \pm l_u^2 \mp l_v^2 = \frac{ds^2}{d\tau^2}, \end{aligned} \quad (324)$$

$$\frac{ds^2}{d\tau^2} = g\left(\frac{d}{d\tau}, \frac{d}{d\tau}\right) = \pm l_u^2 \mp l_v^2 = \pm l_u^2 \left(1 - \frac{l_v^2}{l_u^2}\right). \quad (325)$$

Moreover,

$$dx^i = (u^i + l_v n_{\perp}^i) d\tau + \frac{\partial r}{\partial \lambda} d\lambda n_{\perp}^i, \quad (326)$$

$$\begin{aligned} d &= d\tau(u + l_v n_{\perp}) + \frac{\partial r}{\partial \lambda} d\lambda n_{\perp} = \\ &= d\tau u + \left(d\tau l_v + \frac{\partial r}{\partial \lambda} d\lambda\right) n_{\perp} = d\tau u + dr n_{\perp}, \end{aligned} \quad (327)$$

$$dr = d\tau l_v + \frac{\partial r}{\partial \lambda} d\lambda, \quad (328)$$

$$\frac{dr(\tau, \lambda)}{d\tau} = l_v, \quad \frac{d\lambda}{d\tau} = 0, \quad (329)$$

$$\bar{u} = u + l_v n_{\perp}, \quad g(\bar{u}, u) = g(u, u), \quad (330)$$

$$n_{\perp} = \bar{g}[h_u(n_{\perp})]. \quad (331)$$

*Remark.* If  $l_v \neq 0$ , then the vector  $\bar{u}$  would have a part  $l_v n_{\perp}$  orthogonal to  $u$ .

$$\begin{aligned} ds^2 &= g(d, d) = d\tau^2 g(u, u) + \left(d\tau l_v + \frac{\partial r}{\partial \lambda} d\lambda\right)^2 g(n_{\perp}, n_{\perp}) = \\ &= \pm d\tau^2 l_u^2 \mp \left(d\tau l_v + \frac{\partial r}{\partial \lambda} d\lambda\right)^2 = \pm d\tau^2 l_u^2 \mp d\tau^2 l_v^2 = \\ &= \pm d\tau^2 l_u^2 \mp dr^2 = \pm(l_u^2 d\tau^2 - dr^2) = \pm d\tau^2 \left(l_u^2 - \frac{dr^2}{d\tau^2}\right) = \pm d\tau^2 (l_u^2 - l_v^2). \end{aligned}$$

Therefore,

$$ds^2 = g(d, d) = \pm d\tau^2 (l_u^2 - l_v^2), \quad (332)$$

$$\frac{ds^2}{d\tau^2} = g\left(\frac{d}{d\tau}, \frac{d}{d\tau}\right) = g(\bar{u}, \bar{u}) = \pm l_u^2 \left(1 - \frac{l_v^2}{l_u^2}\right), \quad (333)$$

$$ds^2 = \pm l_u^2 d\tau^2 \left(1 - \frac{l_v^2}{l_u^2}\right). \quad (334)$$

**APPENDIX 2**

Since  $\tilde{n}_\perp$  is a unit vector as well as the vectors  $n_\perp$  and  $m_\perp$ , and, further,  $g(n_\perp, m_\perp) = 0$ , we obtain

$$\begin{aligned} g(\tilde{n}_\perp, \tilde{n}_\perp) &= \mp 1 = g(\alpha n_\perp + \beta m_\perp, \alpha n_\perp + \beta m_\perp) = \\ &= \alpha^2 g(n_\perp, n_\perp) + \beta^2 g(m_\perp, m_\perp) = \mp \alpha^2 \mp \beta^2. \end{aligned} \quad (335)$$

Therefore,

$$\alpha^2 + \beta^2 = 1. \quad (336)$$

On the other side,

$$g(\tilde{n}_\perp, n_\perp) = g(\alpha n_\perp + \beta m_\perp, n_\perp) = \alpha g(n_\perp, n_\perp) = \mp \alpha, \quad (337)$$

$$g(\tilde{n}_\perp, m_\perp) = g(\alpha n_\perp + \beta m_\perp, m_\perp) = \beta g(n_\perp, m_\perp) = \mp \beta, \quad (338)$$

i.e.,

$$\alpha = \mp g(\tilde{n}_\perp, n_\perp) = \mp l_{\tilde{n}_\perp} l_{n_\perp} \cos(\tilde{n}_\perp, n_\perp) = \mp \cos(\tilde{n}_\perp, n_\perp), \quad (339)$$

$$\beta = \mp g(\tilde{n}_\perp, m_\perp) = \mp l_{\tilde{n}_\perp} l_{m_\perp} \cos(\tilde{n}_\perp, m_\perp) = \mp \cos(\tilde{n}_\perp, m_\perp). \quad (340)$$

Therefore,  $\alpha$  and  $\beta$  appear as direction cosines of  $n_\perp$  and  $m_\perp$  with respect to the unit vector  $\tilde{n}_\perp$ . Since

$$\cos^2(\tilde{n}_\perp, n_\perp) + \cos^2(\tilde{n}_\perp, m_\perp) = 1, \quad (341)$$

it follows that

$$\cos^2(\tilde{n}_\perp, m_\perp) = 1 - \cos^2(\tilde{n}_\perp, n_\perp) = 1 - \sin^2(\tilde{n}_\perp, m_\perp) = \sin^2(\tilde{n}_\perp, n_\perp),$$

$$\sin^2(\tilde{n}_\perp, m_\perp) = \cos^2(\tilde{n}_\perp, n_\perp),$$

$$\cos(\tilde{n}_\perp, m_\perp) = \pm \sin(\tilde{n}_\perp, n_\perp),$$

$$\alpha = \mp \cos(\tilde{n}_\perp, n_\perp), \quad (342)$$

$$\beta = \mp \sin(\tilde{n}_\perp, n_\perp), \quad (343)$$

$$\tilde{n}_\perp = \alpha n_\perp + \beta m_\perp = \mp [\cos(\tilde{n}_\perp, n_\perp) n_\perp + \sin(\tilde{n}_\perp, n_\perp) m_\perp]. \quad (344)$$

If we denote the angle  $(\tilde{n}_\perp, n_\perp)$  between the vectors  $\tilde{n}_\perp$  and  $n_\perp$  as  $\theta = (\tilde{n}_\perp, n_\perp)$ , then the above relations could be written in the form

$$\cos^2(\tilde{n}_\perp, m_\perp) = \sin^2 \theta,$$

$$\sin^2(\tilde{n}_\perp, m_\perp) = \cos^2 \theta,$$

$$\alpha = \mp \cos \theta, \quad (345)$$

$$\beta = \mp \sin \theta, \quad (346)$$

$$\tilde{n}_\perp = \alpha n_\perp + \beta m_\perp = \mp[\cos \theta n_\perp + \sin \theta m_\perp]. \quad (347)$$

Further, since  $k_\perp = \mp l_{k_\perp} \tilde{n}_\perp$ , then (see above)

$$g(k_\perp, k_\perp) = l_{k_\perp}^2 g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp l_{k_\perp}^2, \quad g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1, \quad (348)$$

$$k_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp, \quad k_\parallel = \pm \frac{\omega}{l_u} n_\parallel, \quad l_{k_\perp} = \frac{\omega}{l_u} = l_{k_\parallel}, \quad (349)$$

$$g(\tilde{n}_\perp, k_\perp) = \mp l_{k_\perp} g(\tilde{n}_\perp, \tilde{n}_\perp) = \frac{\omega}{l_u} = l_{k_\parallel} = l_{k_\perp}. \quad (350)$$

2. For the contravariant null vector field  $\bar{k}$  we have analogous relations as for the contravariant null vector field  $\tilde{k}$  (just changing  $\tilde{k}$  with  $\bar{k}$ ,  $\omega$  with  $\bar{\omega}$ , and  $\tilde{n}_\perp$  with  $\tilde{n}'_\perp$ )

$$\bar{k} = \bar{k}_\parallel + \bar{k}_\perp, \quad \bar{\omega} = g(u, \bar{k}), \quad (351)$$

$$\bar{k}_\parallel = \pm \frac{\bar{\omega}}{l_u} n_\parallel, \quad l_{\bar{k}_\parallel} = \frac{\bar{\omega}}{l_u}, \quad (352)$$

$$\bar{k}_\perp = \mp \frac{\bar{\omega}}{l_u} \tilde{n}'_\perp, \quad l_{\bar{k}_\perp} = \frac{\bar{\omega}}{l_u} = l_{\bar{k}_\parallel}, \quad (353)$$

$$g(\bar{k}_\perp, \bar{k}_\perp) = \frac{\bar{\omega}^2}{l_u^2} g(\tilde{n}'_\perp, \tilde{n}'_\perp) = \mp l_{\bar{k}_\perp}^2 = \mp \frac{\bar{\omega}^2}{l_u^2}, \quad (354)$$

$$g(\tilde{n}'_\perp, \tilde{n}'_\perp) = \mp 1, \quad (355)$$

$$g(\tilde{n}'_\perp, \bar{k}_\perp) = \mp l_{\bar{k}_\perp} g(\tilde{n}'_\perp, \tilde{n}'_\perp) = \frac{\bar{\omega}}{l_u} = l_{\bar{k}_\parallel} = l_{\bar{k}_\perp}. \quad (356)$$

From  $\bar{k} = \tilde{k} + \text{rel}k$ ,  $\bar{k}_\perp = \tilde{k}_\perp + \text{rel}k_\perp$ , and

$$\mp \frac{\bar{\omega}}{l_u} \tilde{n}'_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp + \text{rel}k_\perp, \quad (357)$$

it follows that

$$g(\tilde{n}_\perp, \bar{k}) = g(\tilde{n}_\perp, k) + g(\tilde{n}_\perp, \text{rel}k),$$

$$g(\tilde{n}_\perp, \bar{k}_\perp) = g(\tilde{n}_\perp, k_\perp) + g(\tilde{n}_\perp, \text{rel}k_\perp), \quad (358)$$

$$\mp \frac{\bar{\omega}}{l_u} g(\tilde{n}'_\perp, \tilde{n}_\perp) = \frac{\omega}{l_u} + g(\text{rel}k_\perp, \tilde{n}_\perp). \quad (359)$$

The vector  $\tilde{n}'_\perp$  could be represented by the use of the vectors  $n_\perp$  and  $m_\perp$  in the form

$$\tilde{n}'_\perp = \alpha' n_\perp + \beta' m_\perp, \quad (360)$$

where

$$g(\tilde{n}'_\perp, \tilde{n}'_\perp) = \mp 1 = g(\alpha' n_\perp + \beta' m_\perp, \alpha' n_\perp + \beta' m_\perp) =$$

$$= \alpha'^2 g(n_\perp, n_\perp) + \beta'^2 g(m_\perp, m_\perp) = \mp \alpha'^2 \mp \beta'^2. \quad (361)$$

Therefore,

$$\alpha'^2 + \beta'^2 = 1. \quad (362)$$

On the other side, we have analogous relations as in the case of the vector  $\tilde{n}_\perp$ :

$$g(\tilde{n}'_\perp, n_\perp) = g(\alpha' n_\perp + \beta' m_\perp, n_\perp) = \alpha' g(n_\perp, n_\perp) = \mp \alpha', \quad (363)$$

$$g(\tilde{n}'_\perp, m_\perp) = g(\alpha' n_\perp + \beta' m_\perp, m_\perp) = \beta' g(n_\perp, m_\perp) = \mp \beta'. \quad (364)$$

i.e.,

$$\alpha' = \mp g(\tilde{n}'_\perp, n_\perp) = \mp l_{\tilde{n}'_\perp} l_{n_\perp} \cos(\tilde{n}'_\perp, n_\perp) = \mp \cos(\tilde{n}'_\perp, n_\perp), \quad (365)$$

$$\beta' = \mp g(\tilde{n}'_\perp, m_\perp) = \mp l_{\tilde{n}'_\perp} l_{m_\perp} \cos(\tilde{n}'_\perp, m_\perp) = \mp \cos(\tilde{n}'_\perp, m_\perp). \quad (366)$$

Therefore,  $\alpha'$  and  $\beta'$  appear as direction cosines of  $n_\perp$  and  $m_\perp$  with respect to the unit vector  $\tilde{n}'_\perp$ . Since

$$\cos^2(\tilde{n}'_\perp, n_\perp) + \cos^2(\tilde{n}'_\perp, m_\perp) = 1, \quad (367)$$

it follows that

$$\cos^2(\tilde{n}'_\perp, m_\perp) = 1 - \cos^2(\tilde{n}'_\perp, n_\perp) = 1 - \sin^2(\tilde{n}'_\perp, m_\perp) = \sin^2(\tilde{n}'_\perp, n_\perp),$$

$$\sin^2(\tilde{n}'_\perp, m_\perp) = \cos^2(\tilde{n}'_\perp, n_\perp),$$

$$\cos(\tilde{n}'_\perp, m_\perp) = \pm \sin(\tilde{n}'_\perp, n_\perp),$$

$$\alpha' = \mp \cos(\tilde{n}'_\perp, n_\perp), \quad (368)$$

$$\beta' = \mp \sin(\tilde{n}'_\perp, n_\perp), \quad (369)$$

$$\tilde{n}'_\perp = \alpha' n_\perp + \beta' m_\perp = \mp [\cos(\tilde{n}'_\perp, n_\perp) n_\perp + \sin(\tilde{n}'_\perp, n_\perp) m_\perp]. \quad (370)$$

If we denote the angle  $(\tilde{n}'_\perp, n_\perp)$  between the vectors  $\tilde{n}'_\perp$  and  $n_\perp$  as  $\theta' = (\tilde{n}'_\perp, n_\perp)$ , then the above relations could be written in the form

$$\cos^2(\tilde{n}'_\perp, m_\perp) = \sin^2 \theta',$$

$$\sin^2(\tilde{n}'_\perp, m_\perp) = \cos^2 \theta',$$

$$\alpha' = \mp \cos \theta', \quad (371)$$

$$\beta' = \mp \sin \theta', \quad (372)$$

$$\tilde{n}_\perp = \alpha n_\perp + \beta m_\perp = \mp [\cos \theta n_\perp + \sin \theta m_\perp]. \quad (373)$$

From the relation

$$\mp \frac{\bar{\omega}}{l_u} \tilde{n}'_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp + \text{rel} k_\perp \quad (374)$$

the relations for  ${}_{\text{rel}}k_{\perp}$  follow

$$\mp \frac{\bar{\omega}}{l_u} g(\tilde{n}'_{\perp}, n_{\perp}) = \mp \frac{\omega}{l_u} g(\tilde{n}_{\perp}, n_{\perp}) + g({}_{\text{rel}}k_{\perp}, n_{\perp}), \quad (375)$$

$$\mp \alpha' \frac{\bar{\omega}}{l_u} = \mp \alpha \frac{\omega}{l_u} + g({}_{\text{rel}}k_{\perp}, n_{\perp}), \quad (376)$$

$$\frac{\bar{\omega}}{l_u} \cos \theta' = \frac{\omega}{l_u} \cos \theta + g({}_{\text{rel}}k_{\perp}, n_{\perp}), \quad (377)$$

$$\bar{\omega} \cos \theta' = \omega \cos \theta + l_u g({}_{\text{rel}}k_{\perp}, n_{\perp}), \quad (378)$$

$$\mp \frac{\bar{\omega}}{l_u} g(\tilde{n}'_{\perp}, m_{\perp}) = \mp \frac{\omega}{l_u} g(\tilde{n}_{\perp}, m_{\perp}) + g({}_{\text{rel}}k_{\perp}, m_{\perp}), \quad (379)$$

$$\mp \beta' \frac{\bar{\omega}}{l_u} = \mp \beta \frac{\omega}{l_u} + g({}_{\text{rel}}k_{\perp}, m_{\perp}), \quad (380)$$

$$\frac{\bar{\omega}}{l_u} \sin \theta' = \frac{\omega}{l_u} \sin \theta + g({}_{\text{rel}}k_{\perp}, m_{\perp}), \quad (381)$$

$$\bar{\omega} \sin \theta' = \omega \sin \theta + l_u g({}_{\text{rel}}k_{\perp}, m_{\perp}), \quad (382)$$

$$\omega = l_u g(\tilde{n}_{\perp}, k_{\perp}), \quad (383)$$

$$\frac{\bar{\omega}}{\omega} \cos \theta' = \cos \theta + \frac{1}{\omega} l_u g({}_{\text{rel}}k_{\perp}, n_{\perp}) = \cos \theta + \frac{g({}_{\text{rel}}k_{\perp}, n_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})}, \quad (384)$$

$$\frac{\bar{\omega}}{\omega} \sin \theta' = \sin \theta + \frac{1}{\omega} l_u g({}_{\text{rel}}k_{\perp}, m_{\perp}) = \sin \theta + \frac{g({}_{\text{rel}}k_{\perp}, m_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})}. \quad (385)$$

From the last (above) two relations, it follows for  $\text{tg } \theta'$

$$\text{tg } \theta' = \frac{\sin \theta'}{\cos \theta'} = \frac{\sin \theta + \frac{g({}_{\text{rel}}k_{\perp}, m_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})}}{\cos \theta + \frac{g({}_{\text{rel}}k_{\perp}, n_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})}}. \quad (386)$$

### APPENDIX 3

The orthogonal to  $u$  parts of  $\tilde{k}$  and  $\bar{k}$  at the point  $A'(\tau_0, \lambda_0)$  could be found after projection of both the vectors by means of the projection metric  $h_u$

$$\begin{aligned} \bar{g}[h_u(\tilde{k})]_{(\tau_0-d\tau, \lambda_0)} &:= \bar{g}[h_u(\bar{k})]_{(\tau_0, \lambda_0)} = \\ &= \bar{g}[h_u(\tilde{k})]_{(\tau_0, \lambda_0)} - d\tau \bar{g}[h_u(\nabla_u \tilde{k})]_{(\tau_0, \lambda_0)} + \\ &\quad + \frac{1}{2} d\tau^2 \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}. \end{aligned} \quad (387)$$

The vectors  $\bar{g}[h_u(\nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}$  and  $\bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}$  could be represented by the use of the kinematic characteristics of the relative velocity and relative acceleration [4].

Since,

$$\bar{g}[h_u(\tilde{k})]_{(\tau_0, \lambda_0)} = \bar{k}_\perp(\tau_0, \lambda_0), \quad (388)$$

$$\bar{g}[h_u(\tilde{k})]_{(\tau_0, \lambda_0)} = k_\perp(\tau_0, \lambda_0), \quad (389)$$

$$\bar{g}[h_u(\nabla_u \tilde{k})]_{(\tau_0, \lambda_0)} = (\nabla_u \tilde{k})_\perp(\tau_0, \lambda_0), \quad (390)$$

$$\bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{(\tau_0, \lambda_0)} = (\nabla_u \nabla_u \tilde{k})_\perp(\tau_0, \lambda_0), \quad (391)$$

$$\bar{\omega}_{(\tau_0, \lambda_0)} = g(\tilde{k}, u)_{(\tau_0, \lambda_0)}, \quad \omega_{(\tau_0, \lambda_0)} = g(\tilde{k}, u)_{(\tau_0, \lambda_0)}, \quad (392)$$

we have the relation

$$\begin{aligned} \bar{k}_\perp(\tau_0, \lambda_0) &= k_\perp(\tau_0, \lambda_0) - d\tau \bar{g}[h_u(\nabla_u \tilde{k})]_{(\tau_0, \lambda_0)} + \\ &\quad + \frac{1}{2} d\tau^2 \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}. \end{aligned} \quad (393)$$

The vectors  $\bar{g}[h_u(\nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}$  and  $\bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}$  could be represented by the use of the kinematic characteristics of the relative velocity and relative acceleration [4] in the form

$$\begin{aligned} \bar{g}[h_u(\nabla_u \tilde{k})] &= \bar{g} \left[ h_u \left( \frac{g(u, \tilde{k})}{e} a + \mathcal{L}_u \tilde{k} \right) + d(\tilde{k}) \right] = \\ &= \bar{g} \left[ h_u \left( \pm \frac{\omega}{l_u^2} a + \mathcal{L}_u \tilde{k} \right) + d(\tilde{k}) \right] = \\ &= \bar{g} \left[ \pm \frac{\omega}{l_u^2} h_u(a) + h_u(\mathcal{L}_u \tilde{k}) + d(\tilde{k}) \right], \end{aligned} \quad (394)$$

$$\begin{aligned} \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})] &= \\ &= \bar{g} \left\{ h_u \left[ \frac{g(u, \tilde{k})}{e} \nabla_u a + k(g)(\mathcal{L}_u \tilde{k}) + \nabla_u(\mathcal{L}_u \tilde{k}) \right] + A(\tilde{k}) \right\} = \\ &= \bar{g} \left\{ \pm \frac{\omega}{l_u^2} h_u(\nabla_u a) + h_u[k(g)(\mathcal{L}_u \tilde{k})] + h_u[\nabla_u(\mathcal{L}_u \tilde{k})] + A(\tilde{k}) \right\}. \end{aligned} \quad (395)$$

If we assume that  $\mathcal{L}_u \tilde{k} = 0$ , we obtain the relations

$$\begin{aligned} \bar{g}[h_u(\nabla_u \tilde{k})] &= \bar{g} \left[ \pm \frac{\omega}{l_u^2} h_u(a) + d(\tilde{k}) \right] = \\ &= \pm \frac{\omega}{l_u^2} \bar{g}[h_u(a)] + \bar{g}[d(k_\perp)] = \pm \frac{\omega}{l_u^2} a_\perp + \bar{g}[d(k_\perp)], \end{aligned} \quad (396)$$

$$\begin{aligned}\bar{g}[h_u(\nabla_u \nabla_u \tilde{k})] &= \pm \frac{\omega}{l_u^2} \bar{g}[h_u(\nabla_u a)] + \bar{g}[A(k_\perp)] = \\ &= \pm \frac{\omega}{l_u^2} (\nabla_u a)_\perp + \bar{g}[A(k_\perp)],\end{aligned}\quad (397)$$

where

$$\frac{g(u, \tilde{k})}{e} = \frac{\omega}{\pm l_u^2} = \pm \frac{\omega}{l_u^2}, \quad (398)$$

$$a_\perp = \bar{g}[h_u(a)], \quad (399)$$

$$(\nabla_u a)_\perp = \bar{g}[h_u(\nabla_u a)], \quad (400)$$

$$\bar{g}[d(\tilde{k})] = \bar{g}[d(k_\perp)], \quad (401)$$

$$\bar{g}[A(\tilde{k})] = \bar{g}[A(k_\perp)]. \quad (402)$$

*Remark.* The condition  $\mathcal{L}_u \tilde{k} = 0$  assures the possibility for introducing co-ordinates with tangent vectors  $u$  and  $\tilde{k}$ , respectively.

On the other side, the vector  $k_\perp$  could be represented in its projections along the vectors  $n_\perp$  and  $m_\perp$  in the form

$$k_\perp = \mp l_{k_\perp} \tilde{n}_\perp = \mp \frac{\omega}{l_u} \tilde{n}_\perp, \quad (403)$$

$$\tilde{n}_\perp = \alpha n_\perp + \beta m_\perp = \mp [\cos \theta n_\perp + \sin \theta m_\perp]. \quad (404)$$

Then

$$k_\perp = \frac{\omega}{l_u} [\cos \theta n_\perp + \sin \theta m_\perp]. \quad (405)$$

In analogous way, the vector  $\bar{k}_\perp$  could be represented in the form

$$\bar{k}_\perp = \mp l_{\bar{k}_\perp} \tilde{n}'_\perp = \mp \frac{\bar{\omega}}{l_u} \tilde{n}'_\perp, \quad (406)$$

$$\tilde{n}'_\perp = \alpha' n_\perp + \beta' m_\perp = \mp [\cos \theta' n_\perp + \sin \theta' m_\perp], \quad (407)$$

$$\bar{k}_\perp = \frac{\bar{\omega}}{l_u} [\cos \theta' n_\perp + \sin \theta' m_\perp]. \quad (408)$$

The representation of  $\bar{k}_\perp$  in the form  $\bar{k}_\perp = k_\perp + {}_{\text{rel}}k_\perp$  will be given now in the form

$$\bar{k}_\perp = k_\perp + {}_{\text{rel}}k_\perp = k_\perp - d\tau \bar{g}[h_u(\nabla_u \tilde{k})] + \frac{1}{2} d\tau^2 \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})], \quad (409)$$

where

$$\begin{aligned}{}_{\text{rel}}k_\perp &= -d\tau (\nabla_u \tilde{k})_\perp + \frac{1}{2} d\tau^2 (\nabla_u \nabla_u \tilde{k})_\perp = -d\tau \left\{ \pm \frac{\omega}{l_u^2} a_\perp + \bar{g}[d(k_\perp)] \right\} + \\ &+ \frac{1}{2} d\tau^2 \left\{ \pm \frac{\omega}{l_u^2} (\nabla_u a)_\perp + \bar{g}[A(k_\perp)] \right\},\end{aligned}\quad (410)$$

$$(\nabla_u \tilde{k})_{\perp} = \bar{g}[h_u(\nabla_u \tilde{k})], \quad (411)$$

$$(\nabla_u \nabla_u \tilde{k})_{\perp} = \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]. \quad (412)$$

The terms in  ${}_{\text{rel}}k_{\perp}$  could be further represented by means of the structures of the relative velocity and the relative acceleration corresponding to the centrifugal (centripetal) and Coriolis velocities and accelerations.

**Representation of  ${}_{\text{rel}}k_{\perp}$  by Means of the Centrifugal (Centripetal) and Coriolis Velocities and Accelerations**

1. The orthogonal to  $u$  acceleration  $a_{\perp}$  could be found after projection by the use of the projective metrics

$$h_{\xi_{\perp}} = g - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} g(\xi_{\perp}) \otimes g(\xi_{\perp}), \quad (413)$$

$$h^{\xi_{\perp}} = \bar{g} - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp} \otimes \xi_{\perp}, \quad (414)$$

in parts collinear to the vector field  $\xi_{\perp}$  and orthogonal to  $\xi_{\perp}$ . At the same time both the parts are orthogonal to the vector field  $u$

$$\begin{aligned} a_{\perp} &= \frac{g(a_{\perp}, \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp} + \bar{g}[h_{\xi_{\perp}}(a_{\perp})] = \frac{l_{\xi_{\perp}}^2}{\mp l_{\xi_{\perp}}^2} g(a_{\perp}, n_{\perp}) n_{\perp} + \bar{g}[h_{\xi_{\perp}}(a_{\perp})] = \\ &= \mp g(a_{\perp}, n_{\perp}) n_{\perp} + \bar{g}[h_{\xi_{\perp}}(a_{\perp})] = (a_{\perp})_z + (a_{\perp})_c, \end{aligned} \quad (415)$$

where

$$(a_{\perp})_z = \mp g(a_{\perp}, n_{\perp}) n_{\perp}, \quad (416)$$

$$(a_{\perp})_c = \bar{g}[h_{\xi_{\perp}}(a_{\perp})], \quad (417)$$

$$g(\xi_{\perp}, (a_{\perp})_c) = 0, \quad (418)$$

$$(a_{\perp})_z = \mp l_{(a_{\perp})_z} n_{\perp}, \quad (419)$$

$$(a_{\perp})_c = \mp l_{(a_{\perp})_c} m_{\perp}. \quad (420)$$

2. The orthogonal to  $u$  change  $(\nabla_u a)_{\perp}$  of the acceleration  $a$  along  $u$  could be found after projection by the use of the projective metrics  $h_{\xi_{\perp}}$  and  $h^{\xi_{\perp}}$  in parts collinear to the vector field  $\xi_{\perp}$  and orthogonal to  $\xi_{\perp}$ . At the same time both the parts are orthogonal to the vector field  $u$

$$\begin{aligned} (\nabla_u a)_{\perp} &= \frac{g((\nabla_u a)_{\perp}, \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp} + \bar{g}[h_{\xi_{\perp}}(\nabla_u a)_{\perp}] = \\ &= \mp g((\nabla_u a)_{\perp}, n_{\perp}) n_{\perp} + \bar{g}[h_{\xi_{\perp}}(\nabla_u a)_{\perp}] = (\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}, \end{aligned} \quad (421)$$

where

$$(\nabla_u a)_{\perp z} = \mp g((\nabla_u a)_{\perp}, n_{\perp})n_{\perp}, \quad (422)$$

$$(\nabla_u a)_{\perp c} = \bar{g}[h_{\xi_{\perp}}(\nabla_u a)_{\perp}], \quad (423)$$

$$g(\xi_{\perp}, (\nabla_u a)_{\perp c}) = 0, \quad (424)$$

$$(\nabla_u a)_{\perp z} = \mp l_{(\nabla_u a)_{\perp z}} n_{\perp}, \quad (425)$$

$$(\nabla_u a)_{\perp c} = \mp l_{(\nabla_u a)_{\perp c}} m_{\perp}. \quad (426)$$

3. The orthogonal to  $u$  deformation velocity vector  $\bar{g}[d(k_{\perp})]$  could be found after projection by the use of the projective metrics  $h_{\xi_{\perp}}$  and  $h^{\xi_{\perp}}$  in parts collinear to the vector field  $\xi_{\perp}$  and orthogonal to  $\xi_{\perp}$ . At the same time both the parts are orthogonal to the vector field  $u$ . Since  $k_{\perp} = \mp l_{k_{\perp}} \tilde{n}_{\perp}$ , we have the relations

$$\begin{aligned} d(k_{\perp}) &= d(\mp l_{k_{\perp}} \tilde{n}_{\perp}) = \mp l_{k_{\perp}} d(\tilde{n}_{\perp}) = \mp l_{k_{\perp}} d(\alpha n_{\perp} + \beta m_{\perp}) = \\ &= \mp l_{k_{\perp}} [\alpha d(n_{\perp}) + \beta d(m_{\perp})], \end{aligned} \quad (427)$$

$$\bar{g}[d(n_{\perp})] = \frac{g(\bar{g}[d(n_{\perp})], \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp} + \bar{g}[h_{\xi_{\perp}}(\bar{g}[d(n_{\perp})])], \quad (428)$$

$$\begin{aligned} \bar{g}[d(n_{\perp})] &= \frac{g(\bar{g}[d(n_{\perp})], n_{\perp})}{\mp l_{\xi_{\perp}}^2} l_{\xi_{\perp}}^2 n_{\perp} + \bar{g}[h_{\xi_{\perp}}(\bar{g}[d(n_{\perp})])] = \\ &= \mp g(\bar{g}[d(n_{\perp})], n_{\perp}) n_{\perp} + \bar{g}[h_{\xi_{\perp}}(\bar{g}[d(n_{\perp})])], \end{aligned} \quad (429)$$

$$g(\bar{g}[d(n_{\perp})], n_{\perp}) = g_{ij} g^{ik} d_{kl} n_{\perp}^l n_{\perp}^j = d_{jl} n_{\perp}^j n_{\perp}^l = d(n_{\perp}, n_{\perp}), \quad (430)$$

$$\mp g(\bar{g}[d(n_{\perp})], n_{\perp}) n_{\perp} = \mp d(n_{\perp}, n_{\perp}) n_{\perp}. \quad (431)$$

On the other side, the following relations can be proved:

$$\begin{aligned} h^{\xi_{\perp}} &= \bar{g} - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp} \otimes \xi_{\perp} = \bar{g} - \frac{1}{\mp l_{\xi_{\perp}}^2} l_{\xi_{\perp}} l_{\xi_{\perp}} n_{\perp} \otimes n_{\perp} = \\ &= \bar{g} \pm n_{\perp} \otimes n_{\perp} = \bar{g} - \frac{1}{g(n_{\perp}, n_{\perp})} n_{\perp} \otimes n_{\perp} = h^{n_{\perp}}, \end{aligned} \quad (432)$$

$$\begin{aligned} h^u &= \bar{g} - \frac{1}{g(u, u)} u \otimes u = \bar{g} - \frac{1}{\pm l_u^2} l_u l_u n_{\parallel} \otimes n_{\parallel} = \\ &= \bar{g} \mp n_{\parallel} \otimes n_{\parallel} = \bar{g} - \frac{1}{g(n_{\parallel}, n_{\parallel})} n_{\parallel} \otimes n_{\parallel} = h^{n_{\parallel}}, \end{aligned} \quad (433)$$

$$\begin{aligned}
 h_{\xi_{\perp}} &= g - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} g(\xi_{\perp}) \otimes g(\xi_{\perp}) = g - \frac{1}{\mp l_{\xi_{\perp}}^2} l_{\xi_{\perp}} l_{\xi_{\perp}} g(n_{\perp}) \otimes (n_{\perp}) = \\
 &= g - \frac{1}{g(n_{\perp}, n_{\perp})} g(n_{\perp}) \otimes (n_{\perp}) = h_{n_{\perp}}, \quad (434)
 \end{aligned}$$

$$\begin{aligned}
 h_u &= g - \frac{1}{g(u, u)} g(u) \otimes g(u) = g - \frac{1}{\pm l_u^2} l_u l_u g(n_{\parallel}) \otimes g(n_{\parallel}) = \\
 &= g - \frac{1}{g(n_{\parallel}, n_{\parallel})} g(n_{\parallel}) \otimes g(n_{\parallel}) = h_{n_{\parallel}}. \quad (435)
 \end{aligned}$$

By the use of the above expressions we can find the explicit form of the term  $\bar{g}[h_{\xi_{\perp}}(\bar{g}[d(n_{\perp})])]$ :

$$\begin{aligned}
 \bar{g}[h_{\xi_{\perp}}(\bar{g}[d(n_{\perp})])] &= g^{ij} (h_{\xi_{\perp}})_{\bar{j}k} g^{kl} d_{\bar{l}m} n_{\perp}^m \partial_i = \\
 &= g^{ij} \left( g_{\bar{j}k} - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} g_{\bar{j}l} \xi_{\perp}^l g_{\bar{k}r} \xi_{\perp}^r \right) g^{kl} d_{\bar{l}m} n_{\perp}^m \partial_i = \\
 &= \left( g_k^i - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} g_l^i \xi_{\perp}^l g_{\bar{k}r} \xi_{\perp}^r \right) g^{kl} d_{\bar{l}m} n_{\perp}^m \partial_i = \\
 &= \left( g^{il} - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp}^i g_r^l \xi_{\perp}^r \right) d_{\bar{l}m} n_{\perp}^m \partial_i = \\
 &= \left( g^{il} - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp}^i \xi_{\perp}^l \right) d_{\bar{l}m} n_{\perp}^m \partial_i = h^{\xi_{\perp}}[d(n_{\perp})] = h^{n_{\perp}}[d(n_{\perp})]. \quad (436)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \bar{g}[d(n_{\perp})] &= \mp g(\bar{g}[d(n_{\perp})], n_{\perp}) n_{\perp} + \bar{g}[h_{\xi_{\perp}}(\bar{g}[d(n_{\perp})])] = \\
 &= \mp d(n_{\perp}, n_{\perp}) n_{\perp} + h^{n_{\perp}}[d(n_{\perp})]. \quad (437)
 \end{aligned}$$

On the other side, the term  $\bar{g}[d(m_{\perp})]$  could be projected in an analogous way

$$\bar{g}[d(m_{\perp})] = \mp g(\bar{g}[d(m_{\perp})], n_{\perp}) n_{\perp} + \bar{g}[h_{\xi_{\perp}}(\bar{g}[d(m_{\perp})])]. \quad (438)$$

The terms at the right side of the expression could be found in the corresponding forms by the use of the relations

$$g(\bar{g}[d(m_{\perp})], n_{\perp}) = g_{\bar{i}j} g^{ik} d_{\bar{k}l} m_{\perp}^l n_{\perp}^j = d_{\bar{j}l} m_{\perp}^l n_{\perp}^j = d(n_{\perp}, m_{\perp}), \quad (439)$$

$$\mp g(\bar{g}[d(n_{\perp})], n_{\perp}) n_{\perp} = \mp d(n_{\perp}, m_{\perp}) n_{\perp}, \quad (440)$$

$$\bar{g}[h_{\xi_{\perp}}(\bar{g}[d(m_{\perp})])] = g^{ij} (h_{\xi_{\perp}})_{\bar{j}k} g^{kl} d_{\bar{l}r} m_{\perp}^r \partial_i = \bar{g}(h_{\xi_{\perp}})(\bar{g})[d(m_{\perp})], \quad (441)$$

$$\bar{g}(h_{\xi_{\perp}})(\bar{g}) = h^{\xi_{\perp}} = h^{n_{\perp}}, \quad (442)$$

$$\bar{g}[h_{\xi_{\perp}}(\bar{g}[d(m_{\perp})])] = h^{n_{\perp}}[d(m_{\perp})]. \quad (443)$$

Therefore,

$$\bar{g}[d(m_{\perp})] = \mp d(n_{\perp}, m_{\perp})n_{\perp} + h^{n_{\perp}}[d(m_{\perp})]. \quad (444)$$

For  $\bar{g}[d(\tilde{n}_{\perp})]$  the expressions follow

$$\begin{aligned} \bar{g}[d(\tilde{n}_{\perp})] &= \alpha\bar{g}[d(n_{\perp})] + \beta\bar{g}[d(m_{\perp})] = \mp\alpha d(n_{\perp}, n_{\perp})n_{\perp} + \alpha h^{n_{\perp}}[d(n_{\perp})] \mp \\ &\quad \mp \beta d(n_{\perp}, m_{\perp})n_{\perp} + \beta h^{n_{\perp}}[d(m_{\perp})], \end{aligned} \quad (445)$$

$$\begin{aligned} \bar{g}[d(\tilde{n}_{\perp})] &= \mp[\alpha d(n_{\perp}, n_{\perp}) + \beta d(n_{\perp}, m_{\perp})]n_{\perp} + \\ &\quad + \alpha h^{n_{\perp}}[d(n_{\perp})] + \beta h^{n_{\perp}}[d(m_{\perp})]. \end{aligned} \quad (446)$$

On the other side, the structures of  ${}_{\text{rel}}v = v_z + v_c$  could be represented under the condition  $\mathcal{L}_u \xi_{\perp} = 0$  in the forms

$${}_{\text{rel}}v = l_{\xi_{\perp}} \bar{g}[d(n_{\perp})], \quad (447)$$

$$v_z = \mp l_{\xi_{\perp}} d(n_{\perp}, n_{\perp})n_{\perp}, \quad (448)$$

$$v_c = \bar{g}[h_{\xi_{\perp}}({}_{\text{rel}}v)] = l_{\xi_{\perp}} \bar{g}(h_{\xi_{\perp}})(\bar{g})[d(n_{\perp})] = l_{\xi_{\perp}} h^{n_{\perp}}[d(n_{\perp})], \quad (449)$$

$$g(v_c, n_{\perp}) = 0,$$

$${}_{\text{rel}}v = v_z + v_c = l_{\xi_{\perp}} \bar{g}[d(n_{\perp})] = \mp l_{\xi_{\perp}} d(n_{\perp}, n_{\perp})n_{\perp} + l_{\xi_{\perp}} h^{n_{\perp}}[d(n_{\perp})]. \quad (450)$$

Let us introduce now a vector field  $\eta_{\perp} = l_{\xi_{\perp}} m_{\perp}$ , orthogonal to the vector field  $\xi_{\perp} = l_{\xi_{\perp}} n_{\perp}$  and  $u$ , but with the same length as  $\xi_{\perp}$ , i.e.,

$$g(\eta_{\perp}, \xi_{\perp}) = l_{\xi_{\perp}}^2 g(m_{\perp}, n_{\perp}) = 0, \quad g(\eta_{\perp}, \eta_{\perp}) = \mp l_{\xi_{\perp}}^2. \quad (451)$$

The corresponding to  $\eta_{\perp}$  relative velocity  ${}_{\text{rel}}v_{\eta}$  and relative acceleration  ${}_{\text{rel}}a_{\eta}$  have analogous forms as  ${}_{\text{rel}}v$  and  ${}_{\text{rel}}a$

$${}_{\text{rel}}v_{\eta} = \bar{g}[d(\eta_{\perp})] = l_{\xi_{\perp}} \bar{g}[d(m_{\perp})], \quad (452)$$

$${}_{\text{rel}}a_{\eta} = \bar{g}[A(\eta_{\perp})] = l_{\xi_{\perp}} \bar{g}[A(m_{\perp})]. \quad (453)$$

The decomposition of  ${}_{\text{rel}}v_{\eta}$  has the form

$${}_{\text{rel}}v_{\eta} = v_{\eta z} + v_{\eta c}, \quad (454)$$

$$\begin{aligned} {}_{\text{rel}}v_{\eta} &= \frac{1}{g(\xi_{\perp}, \xi_{\perp})} g({}_{\text{rel}}v_{\eta}, \xi_{\perp}) \xi_{\perp} + \bar{g}[h_{\xi_{\perp}}({}_{\text{rel}}v_{\eta})] = v_{\eta z} + v_{\eta c} = \\ &= \mp g({}_{\text{rel}}v_{\eta}, n_{\perp})n_{\perp} + \bar{g}[h_{\xi_{\perp}}({}_{\text{rel}}v_{\eta})], \end{aligned} \quad (455)$$

where

$$v_{\eta z} = \mp g(\text{rel}v_{\eta}, n_{\perp})n_{\perp}, \quad (456)$$

$$v_{\eta c} = \bar{g}[h_{\xi_{\perp}}(\text{rel}v_{\eta})]. \quad (457)$$

The explicit form of  $v_{\eta z}$  and  $v_{\eta c}$  could be found by the use of the relations under the condition  $\mathcal{L}_u \eta_{\perp} = 0$

$$\begin{aligned} g(\text{rel}v_{\eta}, n_{\perp}) &= g(\bar{g}[d(\eta_{\perp})], n_{\perp}) = g_{ij}g^{ik}d_{\bar{k}l}l_{\xi_{\perp}}^l m_{\perp}^j n_{\perp}^i = \\ &= l_{\xi_{\perp}} d_{\bar{j}l} n_{\perp}^j m_{\perp}^l = l_{\xi_{\perp}} d(n_{\perp}, m_{\perp}), \end{aligned} \quad (458)$$

$$\bar{g}[h_{\xi_{\perp}}(\text{rel}v_{\eta})] = l_{\xi_{\perp}} \bar{g}[h_{\xi_{\perp}}(\bar{g}[d(m_{\perp})])] = l_{\xi_{\perp}} h^{n_{\perp}}[d(m_{\perp})], \quad (459)$$

as

$$v_{\eta z} = \mp g(\text{rel}v_{\eta}, n_{\perp})n_{\perp} = \mp l_{\xi_{\perp}} d(n_{\perp}, m_{\perp})n_{\perp}, \quad (460)$$

$$v_{\eta c} = \bar{g}[h_{\xi_{\perp}}(\text{rel}v_{\eta})] = l_{\xi_{\perp}} h^{n_{\perp}}[d(m_{\perp})]. \quad (461)$$

Now we can find the relations between the relative velocities  $\text{rel}v$ ,  $\text{rel}v_{\eta}$ , and the expression for  $\bar{g}[d(\tilde{n}_{\perp})]$

$$\begin{aligned} \bar{g}[d(\tilde{n}_{\perp})] &= \alpha \bar{g}[d(n_{\perp})] + \beta \bar{g}[d(m_{\perp})] = \mp \alpha d(n_{\perp}, n_{\perp})n_{\perp} + \alpha h^{n_{\perp}}[d(n_{\perp})] \mp \\ &\quad \mp \beta d(n_{\perp}, m_{\perp})n_{\perp} + \beta h^{n_{\perp}}[d(m_{\perp})], \end{aligned} \quad (462)$$

$$\bar{g}[d(\tilde{n}_{\perp})] = \mp \alpha \left( \mp \frac{1}{l_{\xi_{\perp}}} v_z \right) + \alpha \frac{1}{l_{\xi_{\perp}}} v_c \mp \beta \left( \mp \frac{1}{l_{\xi_{\perp}}} v_{\eta z} \right) + \beta \frac{1}{l_{\xi_{\perp}}} v_{\eta c}, \quad (463)$$

$$\bar{g}[d(\tilde{n}_{\perp})] = \alpha \frac{1}{l_{\xi_{\perp}}} v_z + \alpha \frac{1}{l_{\xi_{\perp}}} v_c + \beta \frac{1}{l_{\xi_{\perp}}} v_{\eta z} + \beta \frac{1}{l_{\xi_{\perp}}} v_{\eta c}, \quad (464)$$

$$\bar{g}[d(\tilde{n}_{\perp})] = \frac{1}{l_{\xi_{\perp}}} [\alpha(v_z + v_c) + \beta(v_{\eta z} + v_{\eta c})] = \frac{1}{l_{\xi_{\perp}}} (\alpha \text{rel}v + \beta \text{rel}v_{\eta}). \quad (465)$$

4. The orthogonal to  $u$  deformation acceleration vector  $\bar{g}[A(k_{\perp})]$  could be found after projection by the use of the projective metrics  $h_{\xi_{\perp}}$  and  $h^{\xi_{\perp}}$  in parts collinear to the vector field  $\xi_{\perp}$  and orthogonal to  $\xi_{\perp}$ . At the same time both the parts are orthogonal to the vector field  $u$ . Since  $k_{\perp} = \mp l_{k_{\perp}} \tilde{n}_{\perp}$ , we have the relations

$$\begin{aligned} A(k_{\perp}) &= A(\mp l_{k_{\perp}} \tilde{n}_{\perp}) = \mp l_{k_{\perp}} A(\tilde{n}_{\perp}) = \\ &= \mp l_{k_{\perp}} A(\alpha n_{\perp} + \beta m_{\perp}) = \mp l_{k_{\perp}} [\alpha A(n_{\perp}) + \beta A(m_{\perp})], \end{aligned} \quad (466)$$

$$\bar{g}[A(n_{\perp})] = \frac{g(\bar{g}[A(n_{\perp})], \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \xi_{\perp} + \bar{g}[h_{\xi_{\perp}}(\bar{g}[A(n_{\perp})])], \quad (467)$$

$$\begin{aligned}\bar{g}[A(n_\perp)] &= \frac{g(\bar{g}[A(n_\perp)], n_\perp)}{\mp l_{\xi_\perp}^2} l_{\xi_\perp}^2 n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(n_\perp)])] = \\ &= \mp g(\bar{g}[A(n_\perp)], n_\perp) n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(n_\perp)])],\end{aligned}\quad (468)$$

$$g(\bar{g}[A(n_\perp)], n_\perp) = g_{\bar{i}\bar{j}} g^{ik} A_{\bar{k}\bar{l}} n_\perp^l n_\perp^j = A_{\bar{j}\bar{l}} n_\perp^j n_\perp^l = A(n_\perp, n_\perp),\quad (469)$$

$$\mp g(\bar{g}[A(n_\perp)], n_\perp) n_\perp = \mp A(n_\perp, n_\perp) n_\perp.\quad (470)$$

Therefore,

$$\begin{aligned}\bar{g}[A(n_\perp)] &= \mp g(\bar{g}[A(n_\perp)], n_\perp) n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(n_\perp)])] = \\ &= \mp A(n_\perp, n_\perp) n_\perp + h^{n_\perp}[A(n_\perp)].\end{aligned}\quad (471)$$

On the other side, the term  $\bar{g}[A(m_\perp)]$  could be projected in an analogous way

$$\bar{g}[A(m_\perp)] = \mp g(\bar{g}[A(m_\perp)], n_\perp) n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(m_\perp)])].\quad (472)$$

The terms at the left side of the expression could be found in the corresponding forms by the use of the relations

$$g(\bar{g}[A(m_\perp)], n_\perp) = g_{\bar{i}\bar{j}} g^{ik} A_{\bar{k}\bar{l}} m_\perp^l n_\perp^j = A_{\bar{j}\bar{l}} m_\perp^l n_\perp^j = A(n_\perp, m_\perp),\quad (473)$$

$$\mp g(\bar{g}[A(m_\perp)], n_\perp) n_\perp = \mp A(n_\perp, m_\perp) n_\perp,\quad (474)$$

$$\bar{g}[h_{\xi_\perp}(\bar{g}[A(m_\perp)])] = g^{ij} (h_{\xi_\perp})_{\bar{j}\bar{k}} g^{kl} A_{\bar{l}\bar{r}} m_\perp^r \partial_i = \bar{g}(h_{\xi_\perp})(\bar{g})[A(m_\perp)],\quad (475)$$

$$\bar{g}(h_{\xi_\perp})(\bar{g}) = h^{\xi_\perp} = h^{n_\perp},\quad (476)$$

$$\bar{g}[h_{\xi_\perp}(\bar{g}[A(m_\perp)])] = h^{n_\perp}[A(m_\perp)].\quad (477)$$

Therefore,

$$\bar{g}[A(m_\perp)] = \mp A(n_\perp, m_\perp) n_\perp + h^{n_\perp}[A(m_\perp)].\quad (478)$$

For  $\bar{g}[A(\tilde{n}_\perp)]$  the expressions follow

$$\begin{aligned}\bar{g}[A(\tilde{n}_\perp)] &= \alpha \bar{g}[A(n_\perp)] + \beta \bar{g}[A(m_\perp)] = \mp \alpha A(n_\perp, n_\perp) n_\perp + \alpha h^{n_\perp}[A(n_\perp)] \mp \\ &\quad \mp \beta A(n_\perp, m_\perp) n_\perp + \beta h^{n_\perp}[A(m_\perp)],\end{aligned}\quad (479)$$

$$\begin{aligned}\bar{g}[A(\tilde{n}_\perp)] &= \mp [\alpha A(n_\perp, n_\perp) + \beta A(n_\perp, m_\perp)] n_\perp + \\ &\quad + \alpha h^{n_\perp}[A(n_\perp)] + \beta h^{n_\perp}[A(m_\perp)].\end{aligned}\quad (480)$$

On the other side, the structures of  ${}_{\text{rel}}a = a_z + a_c$  could be represented under the condition  $\mathcal{L}_u \xi_\perp = 0$  in the forms

$${}_{\text{rel}}a = l_{\xi_\perp} \bar{g}[A(n_\perp)],\quad (481)$$

$$a_z = \mp l_{\xi_\perp} A(n_\perp, n_\perp) n_\perp,\quad (482)$$

$$a_c = \bar{g}[h_{\xi_\perp}({}_{\text{rel}}a)] = l_{\xi_\perp} \bar{g}(h_{\xi_\perp})(\bar{g})[A(n_\perp)] = l_{\xi_\perp} h^{n_\perp}[A(n_\perp)],\quad (483)$$

$$g(a_c, n_\perp) = 0,$$

$${}_{\text{rel}}a = a_z + a_c = l_{\xi_{\perp}} \bar{g}[A(n_{\perp})] = \mp l_{\xi_{\perp}} A(n_{\perp}, n_{\perp}) n_{\perp} + l_{\xi_{\perp}} h^{n_{\perp}}[A(n_{\perp})]. \quad (484)$$

Let us introduce now a vector field  $\eta_{\perp} = l_{\xi_{\perp}} m_{\perp}$ , orthogonal to the vector field  $\xi_{\perp} = l_{\xi_{\perp}} n_{\perp}$  and  $u$ , but with the same length as  $\xi_{\perp}$ , i.e.,

$$g(\eta_{\perp}, \xi_{\perp}) = l_{\xi_{\perp}}^2 g(m_{\perp}, n_{\perp}) = 0, \quad g(\eta_{\perp}, \eta_{\perp}) = \mp l_{\xi_{\perp}}^2. \quad (485)$$

The corresponding to  $\eta_{\perp}$  relative velocity  ${}_{\text{rel}}v_{\eta}$  and relative acceleration  ${}_{\text{rel}}a_{\eta}$  have analogous forms as  ${}_{\text{rel}}v$  and  ${}_{\text{rel}}a$

$${}_{\text{rel}}v_{\eta} = \bar{g}[d(\eta_{\perp})] = l_{\xi_{\perp}} \bar{g}[d(m_{\perp})], \quad (486)$$

$${}_{\text{rel}}a_{\eta} = \bar{g}[A(\eta_{\perp})] = l_{\xi_{\perp}} \bar{g}[A(m_{\perp})]. \quad (487)$$

The decomposition of  ${}_{\text{rel}}a_{\eta}$  has the form

$${}_{\text{rel}}a_{\eta} = a_{\eta z} + a_{\eta c}, \quad (488)$$

$$\begin{aligned} {}_{\text{rel}}a_{\eta} &= \frac{1}{g(\xi_{\perp}, \xi_{\perp})} g({}_{\text{rel}}a_{\eta}, \xi_{\perp}) \xi_{\perp} + \bar{g}[h_{\xi_{\perp}}({}_{\text{rel}}a_{\eta})] = a_{\eta z} + a_{\eta c} = \\ &= \mp g({}_{\text{rel}}a_{\eta}, n_{\perp}) n_{\perp} + \bar{g}[h_{\xi_{\perp}}({}_{\text{rel}}a_{\eta})], \end{aligned} \quad (489)$$

where

$$a_{\eta z} = \mp g({}_{\text{rel}}a_{\eta}, n_{\perp}) n_{\perp}, \quad (490)$$

$$a_{\eta c} = \bar{g}[h_{\xi_{\perp}}({}_{\text{rel}}a_{\eta})]. \quad (491)$$

The explicit form of  $a_{\eta z}$  and  $a_{\eta c}$  could be found by the use of the relations under the condition  $\mathcal{L}_u \eta_{\perp} = 0$

$$\begin{aligned} g({}_{\text{rel}}a_{\eta}, n_{\perp}) &= g(\bar{g}[A(\eta_{\perp})], n_{\perp}) = g_{\bar{i}\bar{j}} g^{ik} A_{\bar{k}l} l_{\xi_{\perp}} m_{\perp}^l n_{\perp}^j = \\ &= l_{\xi_{\perp}} A_{\bar{j}l} n_{\perp}^j m_{\perp}^l = l_{\xi_{\perp}} A(n_{\perp}, m_{\perp}), \end{aligned} \quad (492)$$

$$\bar{g}[h_{\xi_{\perp}}({}_{\text{rel}}a_{\eta})] = l_{\xi_{\perp}} \bar{g}[h_{\xi_{\perp}}(\bar{g}[A(m_{\perp})])] = l_{\xi_{\perp}} h^{n_{\perp}}[A(m_{\perp})], \quad (493)$$

as

$$a_{\eta z} = \mp g({}_{\text{rel}}a_{\eta}, n_{\perp}) n_{\perp} = \mp l_{\xi_{\perp}} A(n_{\perp}, m_{\perp}) n_{\perp}, \quad (494)$$

$$a_{\eta c} = \bar{g}[h_{\xi_{\perp}}({}_{\text{rel}}a_{\eta})] = l_{\xi_{\perp}} h^{n_{\perp}}[A(m_{\perp})]. \quad (495)$$

Now we can find the relations between the relative velocities  ${}_{\text{rel}}a$ ,  ${}_{\text{rel}}a_{\eta}$ , and the expression for  $\bar{g}[A(\tilde{n}_{\perp})]$

$$\begin{aligned} \bar{g}[A(\tilde{n}_{\perp})] &= \alpha \bar{g}[A(n_{\perp})] + \beta \bar{g}[A(m_{\perp})] = \mp \alpha A(n_{\perp}, n_{\perp}) n_{\perp} + \alpha h^{n_{\perp}}[A(n_{\perp})] \mp \\ &\mp \beta A(n_{\perp}, m_{\perp}) n_{\perp} + \beta h^{n_{\perp}}[A(m_{\perp})], \end{aligned} \quad (496)$$

$$\bar{g}[A(\tilde{n}_\perp)] = \mp\alpha \left( \mp\frac{1}{l_{\xi_\perp}}a_z \right) + \alpha\frac{1}{l_{\xi_\perp}}a_c \mp\beta \left( \mp\frac{1}{l_{\xi_\perp}}a_{\eta z} \right) + \beta\frac{1}{l_{\xi_\perp}}a_{\eta c}, \quad (497)$$

$$\bar{g}[A(\tilde{n}_\perp)] = \alpha\frac{1}{l_{\xi_\perp}}a_z + \alpha\frac{1}{l_{\xi_\perp}}a_c + \beta\frac{1}{l_{\xi_\perp}}a_{\eta z} + \beta\frac{1}{l_{\xi_\perp}}a_{\eta c}, \quad (498)$$

$$\bar{g}[A(\tilde{n}_\perp)] = \frac{1}{l_{\xi_\perp}}[\alpha(a_z + a_c) + \beta(a_{\eta z} + a_{\eta c})] = \frac{1}{l_{\xi_\perp}}(\alpha_{\text{rel}a} + \beta_{\text{rel}a_\eta}). \quad (499)$$

5. After the consideration and finding out of the explicit forms of the terms in  ${}_{\text{rel}}k_\perp$

$$\begin{aligned} {}_{\text{rel}}k_\perp &= -d\tau(\nabla_u \tilde{k})_\perp + \frac{1}{2}d\tau^2(\nabla_u \nabla_u \tilde{k})_\perp = \\ &= -d\tau \left\{ \pm\frac{\omega}{l_u^2}a_\perp + \bar{g}[d(k_\perp)] \right\} + \frac{1}{2}d\tau^2 \left\{ \pm\frac{\omega}{l_u^2}(\nabla_u a)_\perp + \bar{g}[A(k_\perp)] \right\}, \quad (500) \end{aligned}$$

we can find the explicit forms of the change  ${}_{\text{rel}}k_\perp$  of the vector  $k_\perp$  along the world line of the observer

$$\begin{aligned} {}_{\text{rel}}k_\perp &= -d\tau \left\{ \pm\frac{\omega}{l_u^2}[(a_\perp)_z + (a_\perp)_c] \mp l_{k_\perp} \bar{g}[d(\tilde{n}_\perp)] \right\} + \\ &\quad + \frac{1}{2}d\tau^2 \left\{ \pm\frac{\omega}{l_u^2}[(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] \mp l_{k_\perp} \bar{g}[A(\tilde{n}_\perp)] \right\}, \quad (501) \end{aligned}$$

$$\begin{aligned} {}_{\text{rel}}k_\perp &= -d\tau \left\{ \pm\frac{\omega}{l_u^2}[(a_\perp)_z + (a_\perp)_c] \mp l_{k_\perp} \left[ \frac{1}{l_{\xi_\perp}}(\alpha_{\text{rel}v} + \beta_{\text{rel}v_\eta}) \right] \right\} + \\ &\quad + \frac{1}{2}d\tau^2 \left\{ \pm\frac{\omega}{l_u^2}[(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] \mp l_{k_\perp} \left[ \frac{1}{l_{\xi_\perp}}(\alpha_{\text{rel}a} + \beta_{\text{rel}a_\eta}) \right] \right\}, \quad (502) \end{aligned}$$

$$\begin{aligned} {}_{\text{rel}}k_\perp &= -d\tau \left\{ \pm\frac{\omega}{l_u^2}[(a_\perp)_z + (a_\perp)_c] \mp \frac{l_{k_\perp}}{l_{\xi_\perp}}(\alpha_{\text{rel}v} + \beta_{\text{rel}v_\eta}) \right\} + \\ &\quad + \frac{1}{2}d\tau^2 \left\{ \pm\frac{\omega}{l_u^2}[(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] \mp \frac{l_{k_\perp}}{l_{\xi_\perp}}(\alpha_{\text{rel}a} + \beta_{\text{rel}a_\eta}) \right\}. \quad (503) \end{aligned}$$

Since

$$l_{k_\perp} = \frac{\omega}{l_u} = l_{k_\parallel}, \quad (504)$$

we obtain the final form of  ${}_{\text{rel}}k_\perp$  with respect to the relative velocity and relative acceleration

$$\begin{aligned} {}_{\text{rel}}k_\perp &= \mp d\tau l_{k_\perp} \left\{ \frac{1}{l_u}[(a_\perp)_z + (a_\perp)_c] - \frac{1}{l_{\xi_\perp}}(\alpha_{\text{rel}v} + \beta_{\text{rel}v_\eta}) \right\} \pm \\ &\quad \pm \frac{1}{2}d\tau^2 l_{k_\perp} \left\{ \frac{1}{l_u}[(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] - \frac{1}{l_{\xi_\perp}}(\alpha_{\text{rel}a} + \beta_{\text{rel}a_\eta}) \right\}. \quad (505) \end{aligned}$$

If we, further, express the time interval  $d\tau$  by its equivalent relations

$$d\tau = \mp \frac{l_{\xi_{\perp}} d\lambda}{l_u} = \frac{dl}{l_u} \quad (506)$$

the relation of  ${}_{\text{rel}}k_{\perp}$  to the relative velocity and relative acceleration could also be written in the forms

$$\begin{aligned} {}_{\text{rel}}k_{\perp} &= \frac{l_{k_{\perp}}}{l_u} \left\{ \frac{l_{\xi_{\perp}} d\lambda}{l_u} [(a_{\perp})_z + (a_{\perp})_c] - d\lambda(\alpha_{\text{rel}v} + \beta_{\text{rel}v_{\eta}}) \right\} \pm \\ &\pm \frac{1}{2} \frac{l_{\xi_{\perp}}^2 d\lambda^2}{l_u^2} l_{k_{\perp}} \left\{ \frac{1}{l_u} [(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] - \frac{1}{l_{\xi_{\perp}}} (\alpha_{\text{rel}a} + \beta_{\text{rel}a_{\eta}}) \right\}. \end{aligned} \quad (507)$$

$$\begin{aligned} {}_{\text{rel}}k_{\perp} &= \frac{l_{k_{\perp}}}{l_u} \left\{ \mp \frac{dl}{l_u} [(a_{\perp})_z + (a_{\perp})_c] \pm \frac{dl}{l_{\xi_{\perp}}} (\alpha_{\text{rel}v} + \beta_{\text{rel}v_{\eta}}) \right\} \pm \\ &\pm \frac{1}{2} \frac{dl^2}{l_u^2} l_{k_{\perp}} \left\{ \frac{1}{l_u} [(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] - \frac{1}{l_{\xi_{\perp}}} (\alpha_{\text{rel}a} + \beta_{\text{rel}a_{\eta}}) \right\}. \end{aligned} \quad (508)$$

By the use of the explicit forms of  $\bar{k}_{\perp}$  and  $k_{\perp}$

$$\bar{k}_{\perp} = \mp l_{\bar{k}_{\perp}} \tilde{n}'_{\perp} = \mp \frac{\bar{\omega}}{l_u} \tilde{n}'_{\perp}, \quad (509)$$

$$k_{\perp} = \mp l_{k_{\perp}} \tilde{n}_{\perp} = \mp \frac{\omega}{l_u} \tilde{n}_{\perp}, \quad (510)$$

we can find the explicit forms of the expressions

$$\bar{S} := \frac{g({}_{\text{rel}}k_{\perp}, m_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})} = \frac{1}{l_{k_{\perp}}} g({}_{\text{rel}}k_{\perp}, m_{\perp}), \quad (511)$$

$$\bar{C} := \frac{g({}_{\text{rel}}k_{\perp}, n_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})} = \frac{1}{l_{k_{\perp}}} g({}_{\text{rel}}k_{\perp}, n_{\perp}), \quad (512)$$

because of the relation

$$g(\tilde{n}_{\perp}, k_{\perp}) = g(\tilde{n}_{\perp}, \mp l_{k_{\perp}} \tilde{n}_{\perp}) = \mp l_{k_{\perp}} g(\tilde{n}_{\perp}, \tilde{n}_{\perp}) = l_{k_{\perp}}. \quad (513)$$

6. By the use of the relations

$$g((a_{\perp})_z + (a_{\perp})_c, n_{\perp}) = g((a_{\perp})_z, n_{\perp}), \quad g((a_{\perp})_c, n_{\perp}) = 0, \quad (514)$$

$$g({}_{\text{rel}v}, n_{\perp}) = g(v_z, n_{\perp}), \quad g({}_{\text{rel}v_{\eta}}, n_{\perp}) = g(v_{\eta z}, n_{\perp}), \quad (515)$$

$$g((\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}, n_{\perp}) = g((\nabla_u a)_{\perp z}, n_{\perp}), \quad (516)$$

$$g(\text{rel}a, n_{\perp}) = g(a_z, n_{\perp}), \quad g(\text{rel}a_{\eta}, n_{\perp}) = g(a_{\eta z}, n_{\perp}), \quad (517)$$

$$g(\text{rel}k_{\perp}, n_{\perp}) = \frac{l_{k_{\perp}}}{l_u} \left\{ \mp \frac{dl}{l_u} g((a_{\perp})_z, n_{\perp}) \pm \frac{dl}{l_{\xi_{\perp}}} [\alpha g(v_z, n_{\perp}) + \beta g(v_{\eta z}, n_{\perp})] \right\} \pm \frac{1}{2} \frac{dl^2}{l_u^2} l_{k_{\perp}} \left\{ \frac{1}{l_u} g((\nabla_u a)_{\perp z}, n_{\perp}) - \frac{1}{l_{\xi_{\perp}}} [\alpha g(a_z, n_{\perp}) + \beta g(a_{\eta z}, n_{\perp})] \right\}, \quad (518)$$

we can find the explicit form of the quantity  $\overline{C}$ .

On the other side, we can use the relations

$$(a_{\perp})_z = \mp l_{(a_{\perp})_z} n_{\perp}, \quad (519)$$

$$g((a_{\perp})_z, n_{\perp}) = \mp l_{(a_{\perp})_z} g(n_{\perp}, n_{\perp}) = l_{(a_{\perp})_z}, \quad (520)$$

$$v_z = \mp l_{v_z} n_{\perp}, \quad (521)$$

$$g(v_z, n_{\perp}) = \mp l_{v_z} g(n_{\perp}, n_{\perp}) = l_{v_z}, \quad (522)$$

$$v_{\eta z} = \mp l_{v_{\eta z}} n_{\perp}, \quad (523)$$

$$g(v_{\eta z}, n_{\perp}) = \mp l_{v_{\eta z}} g(n_{\perp}, n_{\perp}) = l_{v_{\eta z}}, \quad (524)$$

$$(\nabla_u a)_{\perp z} = \mp l_{(\nabla_u a)_{\perp z}} n_{\perp}, \quad (525)$$

$$g((\nabla_u a)_{\perp z}, n_{\perp}) = \mp l_{(\nabla_u a)_{\perp z}} g(n_{\perp}, n_{\perp}) = l_{(\nabla_u a)_{\perp z}}, \quad (526)$$

$$g((\nabla_u a)_{\perp z}, m_{\perp}) = \mp l_{(\nabla_u a)_{\perp z}} g(n_{\perp}, m_{\perp}) = 0, \quad (527)$$

$$a_z = \mp l_{a_z} n_{\perp}, \quad (528)$$

$$g(a_z, n_{\perp}) = \mp l_{a_z} g(n_{\perp}, n_{\perp}) = l_{a_z}, \quad (529)$$

$$g(a_z, m_{\perp}) = \mp l_{a_z} g(n_{\perp}, m_{\perp}) = 0, \quad (530)$$

$$a_{\eta z} = \mp l_{a_{\eta z}} n_{\perp}, \quad (531)$$

$$g(a_{\eta z}, n_{\perp}) = \mp l_{a_{\eta z}} g(n_{\perp}, n_{\perp}) = l_{a_{\eta z}}, \quad (532)$$

$$g(a_{\eta z}, m_{\perp}) = \mp l_{a_{\eta z}} g(n_{\perp}, m_{\perp}) = 0 \quad (533)$$

for finding out the explicit form of  $g(\text{rel}k_{\perp}, n_{\perp})$  and  $g(\text{rel}k_{\perp}, m_{\perp})$

$$g(\text{rel}k_{\perp}, n_{\perp}) = \frac{l_{k_{\perp}}}{l_u} \left\{ \mp \frac{dl}{l_u} l_{(a_{\perp})_z} \pm \frac{dl}{l_{\xi_{\perp}}} [\alpha l_{v_z} + \beta l_{v_{\eta z}}] \right\} \pm \frac{1}{2} \frac{dl^2}{l_u^2} l_{k_{\perp}} \left\{ \frac{1}{l_u} l_{(\nabla_u a)_{\perp z}} - \frac{1}{l_{\xi_{\perp}}} [\alpha l_{a_z} + \beta l_{a_{\eta z}}] \right\}, \quad (534)$$

or the form

$$g(\text{rel}k_{\perp}, n_{\perp}) = \frac{l_{k_{\perp}}}{l_u} \left\{ \pm \frac{dl}{l_{\xi_{\perp}}} [\alpha l_{v_z} + \beta l_{v_{\eta z}}] \mp \frac{dl}{l_u} l_{(a_{\perp})_z} \right\} \mp \frac{1}{2} \frac{dl^2}{l_u^2} l_{k_{\perp}} \left\{ \frac{1}{l_{\xi_{\perp}}} [\alpha l_{a_z} + \beta l_{a_{\eta z}}] - \frac{1}{l_u} l_{(\nabla_u a)_{\perp z}} \right\}. \quad (535)$$

The explicit form of  $\bar{C}$  could now be found as

$$\bar{C} = \frac{1}{l_{k_{\perp}}} g(\text{rel}k_{\perp}, n_{\perp}) = \pm \frac{1}{l_u} \left\{ \frac{dl}{l_{\xi_{\perp}}} [\alpha l_{v_z} + \beta l_{v_{\eta z}}] - \frac{dl}{l_u} l_{(a_{\perp})_z} \right\} \mp \frac{1}{2} \frac{dl^2}{l_u^2} \left\{ \frac{1}{l_{\xi_{\perp}}} [\alpha l_{a_z} + \beta l_{a_{\eta z}}] - \frac{1}{l_u} l_{(\nabla_u a)_{\perp z}} \right\}. \quad (536)$$

If we introduce the abbreviations

$$\bar{l}_{v_z} = \frac{dl}{l_{\xi_{\perp}}} l_{v_z}, \quad \bar{l}_{v_{\eta z}} = \frac{dl}{l_{\xi_{\perp}}} l_{v_{\eta z}}, \quad \bar{l}_{(a_{\perp})_z} = \frac{dl}{l_{\xi_{\perp}}} l_{(a_{\perp})_z}, \quad (537)$$

$$\bar{l}_{a_z} = \frac{dl}{l_{\xi_{\perp}}} l_{a_z}, \quad \bar{l}_{a_{\eta z}} = \frac{dl}{l_{\xi_{\perp}}} l_{a_{\eta z}}, \quad \bar{l}_{(\nabla_u a)_{\perp z}} = \frac{dl}{l_{\xi_{\perp}}} l_{(\nabla_u a)_{\perp z}}, \quad (538)$$

then  $\bar{C}$  could be represented in the form

$$\bar{C} = \pm \frac{1}{l_u} \left[ (\alpha \bar{l}_{v_z} + \beta \bar{l}_{v_{\eta z}}) - \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})_z} \right] \mp \frac{1}{2} \frac{dl}{l_u^2} \left[ (\alpha \bar{l}_{a_z} + \beta \bar{l}_{a_{\eta z}}) - \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp z}} \right]. \quad (539)$$

In analogous way, we can find the explicit form of  $\bar{S}$ .

7. By the use of the relations

$$g(\text{rel}k_{\perp}, m_{\perp}) = \frac{l_{k_{\perp}}}{l_u} \left\{ \mp \frac{dl}{l_u} [g((a_{\perp})_z, m_{\perp}) + g((a_{\perp})_c, m_{\perp})] \pm \frac{dl}{l_{\xi_{\perp}}} [\alpha g(\text{rel}v, m_{\perp}) + \beta g(\text{rel}v_{\eta}, m_{\perp})] \right\} \pm \frac{1}{2} \frac{dl^2}{l_u^2} l_{k_{\perp}} \left\{ \frac{1}{l_u} [g((\nabla_u a)_{\perp z}, m_{\perp}) + g((\nabla_u a)_{\perp c}, m_{\perp})] - \frac{1}{l_{\xi_{\perp}}} [\alpha g(\text{rel}a, m_{\perp}) + \beta g(\text{rel}a_{\eta}, m_{\perp})] \right\}, \quad (540)$$

$$g(\text{rel}k_{\perp}, m_{\perp}) = \frac{l_{k_{\perp}}}{l_u} \left\{ \mp \frac{dl}{l_u} g((a_{\perp})_c, m_{\perp}) \pm \frac{dl}{l_{\xi_{\perp}}} [\alpha g(\text{rel}v, m_{\perp}) + \beta g(\text{rel}v_{\eta}, m_{\perp})] \right\} \pm \frac{1}{2} \frac{dl^2}{l_u^2} l_{k_{\perp}} \left\{ \frac{1}{l_u} g((\nabla_u a)_{\perp c}, m_{\perp}) - \frac{1}{l_{\xi_{\perp}}} [\alpha g(\text{rel}a, m_{\perp}) + \beta g(\text{rel}a_{\eta}, m_{\perp})] \right\}, \quad (541)$$

$$(a_{\perp})_c = \mp l_{(a_{\perp})_c} m_{\perp}, \quad (542)$$

$$g((a_{\perp})_c, m_{\perp}) = \mp l_{(a_{\perp})_c} g(m_{\perp}, m_{\perp}) = l_{(a_{\perp})_c}, \quad (543)$$

$$v_c = \mp l_{v_c} m_{\perp}, \quad (544)$$

$$g(\text{rel}v, m_{\perp}) = g(v_z + v_c, m_{\perp}) = g(v_c, m_{\perp}) = \mp l_{v_c} g(m_{\perp}, m_{\perp}) = l_{v_c}, \quad (545)$$

$$v_{\eta c} = \mp l_{v_{\eta c}} m_{\perp}, \quad (546)$$

$$g(\text{rel}v_{\eta}, m_{\perp}) = g(v_{\eta z} + v_{\eta c}, m_{\perp}) = g(v_{\eta c}, m_{\perp}) = \mp l_{v_{\eta c}} g(m_{\perp}, m_{\perp}) = l_{v_{\eta c}}, \quad (547)$$

$$(\nabla_u a)_{\perp c} = \mp l_{(\nabla_u a)_{\perp c}} m_{\perp}, \quad (548)$$

$$g((\nabla_u a)_{\perp c}, m_{\perp}) = \mp l_{(\nabla_u a)_{\perp c}} g(m_{\perp}, m_{\perp}) = l_{(\nabla_u a)_{\perp c}}, \quad (549)$$

$$\text{rel}a = a_z + a_c, \quad a_z = \mp l_{a_z} n_{\perp}, \quad a_c = \mp l_{a_c} m_{\perp}, \quad (550)$$

$$g(\text{rel}a, m_{\perp}) = g(a_z + a_c, m_{\perp}) = g(a_c, m_{\perp}) = \mp l_{a_c} g(m_{\perp}, m_{\perp}) = l_{a_c}, \quad (551)$$

$$\text{rel}a_{\eta} = a_{\eta z} + a_{\eta c}, \quad a_{\eta z} = \mp l_{a_{\eta z}} n_{\perp}, \quad a_{\eta c} = \mp l_{a_{\eta c}} m_{\perp}, \quad (552)$$

$$g(\text{rel}a_{\eta}, m_{\perp}) = g(a_{\eta c}, m_{\perp}) = \mp l_{a_{\eta c}} g(m_{\perp}, m_{\perp}) = l_{a_{\eta c}}, \quad (553)$$

we can find the explicit forms of  $g(\text{rel}v_{\eta}, m_{\perp})$  related to the centrifugal (centripetal) and Coriolis velocities and accelerations:

$$g(\text{rel}k_{\perp}, m_{\perp}) = \frac{l_{k_{\perp}}}{l_u} \left\{ \mp \frac{dl}{l_u} l_{(a_{\perp})_c} \pm \frac{dl}{l_{\xi_{\perp}}} [\alpha l_{v_c} + \beta l_{v_{\eta c}}] \right\} \pm \frac{1}{2} \frac{dl^2}{l_u^2} l_{k_{\perp}} \left\{ \frac{1}{l_u} l_{(\nabla_u a)_{\perp c}} - \frac{1}{l_{\xi_{\perp}}} [\alpha l_{a_c} + \beta l_{a_{\eta c}}] \right\}, \quad (554)$$

$$g(\text{rel}k_{\perp}, m_{\perp}) = \pm \frac{l_{k_{\perp}}}{l_u} \left[ \frac{dl}{l_{\xi_{\perp}}} (\alpha l_{v_c} + \beta l_{v_{\eta c}}) - \frac{dl}{l_u} l_{(a_{\perp})_c} \right] \mp \frac{1}{2} \frac{dl^2}{l_u^2} l_{k_{\perp}} \left[ \frac{1}{l_{\xi_{\perp}}} (\alpha l_{a_c} + \beta l_{a_{\eta c}}) - \frac{1}{l_u} l_{(\nabla_u a)_{\perp c}} \right]. \quad (555)$$

The explicit form of  $\bar{S}$  could be found as

$$\begin{aligned} \bar{S} := \frac{1}{l_{k_{\perp}}} g_{(\text{rel}k_{\perp}, m_{\perp})} &= \pm \frac{1}{l_u} \left[ \frac{dl}{l_{\xi_{\perp}}} (\alpha l_{v_c} + \beta l_{v_{\eta c}}) - \frac{dl}{l_u} l_{(a_{\perp})_c} \right] \mp \\ &\mp \frac{1}{2} \frac{dl^2}{l_u^2} \left[ \frac{1}{l_{\xi_{\perp}}} (\alpha l_{a_c} + \beta l_{a_{\eta c}}) - \frac{1}{l_u} l_{(\nabla_u a)_{\perp c}} \right]. \end{aligned} \quad (556)$$

If we introduce the abbreviations

$$\bar{l}_{v_c} = \frac{dl}{l_{\xi_{\perp}}} l_{v_c}, \quad \bar{l}_{v_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} l_{v_{\eta c}}, \quad \bar{l}_{(a_{\perp})_c} = \frac{dl}{l_{\xi_{\perp}}} l_{(a_{\perp})_c}, \quad (557)$$

$$\bar{l}_{a_c} = \frac{dl}{l_{\xi_{\perp}}} l_{a_c}, \quad \bar{l}_{a_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} l_{a_{\eta c}}, \quad \bar{l}_{(\nabla_u a)_{\perp c}} = \frac{dl}{l_{\xi_{\perp}}} l_{(\nabla_u a)_{\perp c}}, \quad (558)$$

$$\bar{l}_{\diamond} \lesseqgtr 0, \quad (559)$$

then  $\bar{S}$  could be represented in the form

$$\begin{aligned} \bar{S} &= \pm \frac{1}{l_u} \left[ (\alpha \bar{l}_{v_c} + \beta \bar{l}_{v_{\eta c}}) - \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(a_{\perp})_c} \right] \mp \\ &\mp \frac{1}{2} \frac{dl}{l_u^2} \left[ (\alpha \bar{l}_{a_c} + \beta \bar{l}_{a_{\eta c}}) - \frac{l_{\xi_{\perp}}}{l_u} \bar{l}_{(\nabla_u a)_{\perp c}} \right]. \end{aligned} \quad (560)$$

The explicit forms of  $\bar{C}$  and  $\bar{S}$  determine the relations describing the aberration, the Doppler effect, and the Hubble effect in spaces with affine connections and metrics.

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