ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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RADIATIVE CORRECTIONS TO THE BHABHA SCATTERING

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Large-angle Bhabha scattering cross section and related processes are considered. Standard procedure of calculation of radiative corrections (RC) with emission of one and two real photons as well as lepton pairs is considered in detail. Special attention is paid to the case of radiative large-angle Bhabha scattering, where we show the validity of factorization theorem for one-loop level RC. This is the basic assumption of the Drell–Yan form of the cross sections when hard particles are detected at large angles. The cases of collinear and semicollinear kinematics for the final particles are considered explicitly. At the end of the review we consider some contributions to the large-angle Bhabha elastic scattering cross section at two-loop level. Among them, the emission of two hard photons in collinear and semicollinear kinematics regions and the contribution of a set of gauge-invariant FD of two-loop level were calculated.

Изучены радиационные поправки к различным постановкам баба-рассеяния на большие углы, в частности с дополнительным излучением одного и двух реальных фотонов. Показано, что учет радиационных поправок согласуется с факторизационной теоремой и сечение процесса представимо в виде сечения процесса Дрелла–Яна. Рассмотрены случаи коллинеарного и полуколлинеарного излучения фотона, получены радиационные поправки и явный вид k-фактора, включающего нелидирующие члены. В отдельном пункте рассмотрены основные вклады в двухпетлевые поправки к баба-рассеянию, излучение дополнительных одного и двух коллинеарных жестких и мягких фотонов.

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To the blessed memory of Teachers Alexandr Ilich Akhiezer, Sergej Semenovich Sannikov, and Vladimir Naumovich Gribov

INTRODUCTION

Nowadays we could see the renewal of interest to Bhabha scattering. For example, at BEPC facility (Bejing, China), the e^+e^- collider with extremely high luminosity (10^{33} cm$^{-1} \cdot$ s^{-1}) is build, which gives us the possibility of obtaining with much more accuracy J/Ψ decays and maybe some new physics such as 4-quark resonances, heavy quarkonia.

For such a precise measurements we have to know with accuracy at the level of 0.1% the beam luminosity and background processes such as Bhabha scattering at small angles, with additional soft and hard photon or lepton pair emission. All these processes are considered at this review. At the end of each Section we give the explicit expression of the processes in the frames of Drell–Yan picture.

At the beginning we consider the well-known lowest order RC to the large-angle Bhabha scattering and put the final answer in terms of DGLAP evolution equation kernels.

Then we put the expressions for the large-angle Bhabha scattering accompanied with the soft, hard photon and soft real lepton pair. The relevant virtual one-loop corrections are also considered. The final answer is given in the form of explicit leading logarithm contribution and the so-called K factor, which includes nonleading RC corrections. Due to its cumbersome form we put some numerical estimations of it. In the next Section we consider also the hard real lepton pair emission, and we provide the check that the final answer is in accordance with the renorm-group predictions.

The remainig part is devoted to the radiative Bhabha scattering. In Subsec. 1.4 we start with the consideration of one-loop RC (box-type, vertex-type, vacuum polarization, etc.). Again, we convince that our final result is consistent with the Drell–Yan picture of this process. The nonleading terms are presented in the numerical form. In Subsecs. 1.5, 1.6 we consider the emission of one and two real hard photons in collinear and semicollinear kinematics.

At the end of the review we put some parts of two-loop calculation for Bhabha scattering. All two-loop contributions are divided to the gauge-invariant classes, and addition soft photon emission is also included.

In Sec. 2 we give the tables for one-loop Feynman integrals of scalar, vector, and tensor types, with two, three, four, and five denominators. All formulae are presented with the accuracy up to the terms of the order of the ratio of the electron to the muon masses squared, and the kinematic invariants are assumed to be large compared to the electron mass squared.

Throughout our paper we use the next designations:

DIS — deep inelastic scattering

FD — Feynman diagram

LLA — leading logarithmic approximation

NLO — next-to-leading order

QCD — quantum chromodynamics

QED — quantum electrodynamics

RC — radiative corrections.

1. LARGE-ANGLE BHABHA SCATTERING

The cross section of Bhabha scattering (corrected by the vacuum polarization factor), which enters into the Drell–Yan form of corrected cross section, has a bit more complicated form, as far as the scattering and annihilation amplitudes and their interference are to be taken into account. We remind here the form of the Lorentz-invariant matrix element module squared in the Born approximation:

$$R_0(s,t,u) = \frac{1}{16(4\pi\alpha)^4} \sum_{\text{spins}} \left|\mathcal{M}(e^-(p_-) + e^+(p_+) \to e^-(p'_-) + e^+(p'_+))\right|^2 = \\ = \frac{s^2+u^2}{2t^2} + \frac{u^2+t^2}{2s^2} + \frac{u^2}{st}, \quad (1.1)$$

$$s = (p_- + p_+)^2, \quad t = (p_- - p'_-)^2, \quad u = (p_- - p'_+)^2, \\ s + t + u = \mathcal{O}(m_e^2), \quad (1.2)$$

here and below we neglect the terms of order $m_e^2/s \sim m_e^2/(-t) \sim m_e^2/(-u)$ compared with ones of order of unity.

The first term in the right-hand side of Eq. (1.1) describes the scattering-type Feynman diagram square. The second one corresponds to the square of the annihilation-type diagram. And the third one deals with the interference of the two diagrams. A more compact representation of R_0 is also useful, $R_0 = (1 + s/t + t/s)^2$. In the center of mass of initial particles (further implied) we have

$$s = 4\epsilon^2, \quad t = -2\epsilon^2(1-c), \quad u = -2\epsilon^2(1+c),$$

where $c = \cos\theta$, and the scattering angle θ is the angle between the initial and the scattered electron momenta, ε is the energy of initial electron.

The differential cross section in the Born approximation has the form

$$\frac{d\sigma_0^{\text{Born}}}{d\Omega_-} = \frac{\alpha^2}{4s}\left(\frac{3+c^2}{1-c}\right)^2. \quad (1.3)$$

Consider the case when the initial electron and positron lost certain energy fractions. Supposing the scattering angle remains the same and using the conservation laws, we will obtain the value of R_0 in terms of the relevant invariants.

The corresponding kinematics is defined as follows:

$$e^-(z_1p_-) + e^+(z_2p_+) \longrightarrow e^-(\tilde{p}_-) + e^+(\tilde{p}_+),$$

$$\tilde{s} = sz_1z_2, \quad \tilde{t} = -\frac{1}{2}sz_1Y_1(1-c), \quad \tilde{u} = -\frac{1}{2}sz_2Y_1(1+c),$$

$$Y_1 = \frac{\tilde{p}_-^0}{\varepsilon} = \frac{2z_1z_2}{a}, \quad a = z_1 + z_2 - (z_1 - z_2)c.$$

For definiteness we put here the *shifted* Born cross section of two types:

$$\frac{d\sigma_0((1-x)p_-,p_+)}{dc} = \frac{2\pi\alpha^2}{\varepsilon^2}\left\{\frac{3-3x+x^2+2x(2-x)c+c^2(1-x+x^2)}{(1-x)(1-c)(2-x+xc)^2}\right\}^2,$$

$$\frac{d\sigma_0(p_-,(1-x)p_+)}{dc} = \frac{2\pi\alpha^2}{\varepsilon^2}\left\{\frac{3-3x+x^2-2x(2-x)c+c^2(1-x+x^2)}{(1-x)(1-c)(2-x-xc)^2}\right\}^2. \quad (1.4)$$

The *shifted* Born cross section corrected by vacuum polarization insertions into virtual photon propagators reads

$$d\tilde{\sigma}_0(z_1,z_2) = \frac{4\alpha^2}{sa^2}\left\{\frac{1}{|1-\Pi(\tilde{t})|^2}\,\frac{a^2+z_2^2(1+c)^2}{2z_1^2(1-c)^2} + \right.$$
$$+\frac{1}{|1-\Pi(\tilde{s})|^2}\,\frac{z_1^2(1-c)^2+z_2^2(1+c)^2}{2a^2} -$$
$$\left. -\operatorname{Re}\frac{1}{(1-\Pi(\tilde{t}))(1-\Pi(\tilde{s}))^*}\,\frac{z_2^2(1+c)^2}{az_1(1-c)}\right\}d\Omega_-. \quad (1.5)$$

Quantities $\Pi(s)$ and $\Pi(t)$ are the vacuum polarization operators in the s and t channels.

1.1. Lowest Order Radiative Corrections and the Leading Logarithmic Approximation. Rewriting the known results [14] (see as well [16]) for the cross section in the Born approximation with one-loop virtual corrections to it and with the other ones arising due to soft photon emission, we obtain

$$\frac{d\sigma_{B+S+V}}{d\Omega_-} = \frac{d\tilde{\sigma}_0(1,1)}{d\Omega_-}\left\{1+\frac{2\alpha}{\pi}(L-1)\left[2\ln\frac{\Delta\varepsilon}{\varepsilon}+\frac{3}{2}\right] - \right.$$
$$\left. -\frac{8\alpha}{\pi}\ln\left(\cot\frac{\theta}{2}\right)\ln\frac{\Delta\varepsilon}{\varepsilon}+\frac{\alpha}{\pi}K_{SV}\right\}, \quad L=\ln\frac{s}{m_e^2}, \quad (1.6)$$

where

$$K_{SV} = -1-2\mathrm{Li}_2\left(\sin^2\frac{\theta}{2}\right)+2\mathrm{Li}_2\left(\cos^2\frac{\theta}{2}\right)+\frac{1}{(3+c^2)^2}\left[\frac{\pi^2}{3}(2c^4-3c^3-15c)+\right.$$
$$+2(2c^4-3c^3+9c^2+3c+21)\ln^2\left(\sin\frac{\theta}{2}\right)-4(c^4+c^2-2c)\ln^2\left(\cos\frac{\theta}{2}\right)-$$
$$-4(c^3+4c^2+5c+6)\ln^2\left(\tan\frac{\theta}{2}\right)+2(c^3-3c^2+7c-5)\ln\left(\cos\frac{\theta}{2}\right)+$$
$$\left.+2(3c^3+9c^2+5c+31)\ln\left(\sin\frac{\theta}{2}\right)\right] \quad (1.7)$$

is the part of the K factor coming from soft and virtual photon corrections, $d\tilde{\sigma}_0(1,1)$ is defined in (1.5). Quantity $\Delta\varepsilon$ in Eq. (1.6) is the maximal energy of emitted soft photons.

Consider now the process of hard photon (with the energy $\omega = k_0 > \Delta\varepsilon$ in the center-of-mass system) emission

$$e^+(p_+) + e^-(p_-) \to e^+(p'_+) + e^-(p'_-) + \gamma(k).$$

We start with the differential cross section in the form suggested by F. A. Berends et al. [4, 5] which is valid for scattering angles being large compared with m_e/ε (the case of extremely small scattering angles was treated in [32]):

$$d\sigma_{\rm hard} = \frac{\alpha^3}{2\pi^2 s}\, R_{e\bar{e}\gamma}\, d\Gamma, \quad d\Gamma = \frac{d^3p'_+ d^3p'_- d^3k}{\varepsilon'_+\varepsilon'_- k^0}\delta^{(4)}(p_+ + p_- - p'_+ - p'_- - k),$$

$$R_{e\bar{e}\gamma} = \frac{WT}{4} - \frac{m_e^2}{(\chi'_+)^2}\left(\frac{s}{t} + \frac{t}{s} + 1\right)^2 - \frac{m_e^2}{(\chi'_-)^2}\left(\frac{s}{t_1} + \frac{t_1}{s} + 1\right)^2 - \\ - \frac{m_e^2}{\chi_+^2}\left(\frac{s_1}{t} + \frac{t}{s_1} + 1\right)^2 - \frac{m_e^2}{\chi_-^2}\left(\frac{s_1}{t_1} + \frac{t_1}{s_1} + 1\right)^2, \quad (1.8)$$

where

$$W = \frac{s}{\chi_+\chi_-} + \frac{s_1}{\chi'_+\chi'_-} - \frac{t_1}{\chi'_+\chi_+} - \frac{t}{\chi'_-\chi_-} + \frac{u}{\chi'_+\chi_-} + \frac{u_1}{\chi'_-\chi_+},$$
$$T = \frac{ss_1(s^2+s_1^2) + tt_1(t^2+t_1^2) + uu_1(u^2+u_1^2)}{ss_1tt_1},$$

and the invariants are defined as

$$s = 2p_-p_+, \quad s_1 = 2p'_-p'_+, \quad t = -2p_-p'_-, \quad t_1 = -2p_+p'_+,$$
$$u = -2p_-p'_+, \quad u_1 = -2p_+p'_-, \quad \chi_\pm = kp_\pm, \quad \chi'_\pm = kp'_\pm.$$

It is convenient to extract the contribution of the collinear kinematics. We do that for the following reasons. First, it is natural to separate the region with a very sharp behavior of the cross section and to consider it carefully. Second, we keep in mind the idea of the leading logarithm factorization, which is valid in all orders of the perturbation theory. We will evaluate the collinear kinematical regions in two different ways. The first one (the quasi-real electron approximation) is suitable for generalization in order to account higher order leading corrections by means of the structure function method. In this way we will obtain below the leading logarithmic contributions and the compensating terms, which will provide the cancellation of auxiliary parameters. The second one (the direct calculation) is more rigorous, it can be used as a check of the first one. We discuss it in detail in Subsubsec. 1.1.1.

To obtain explicit formulae for compensators it is needed to consider four kinematical regions corresponding to hard photon emission inside narrow cones, surrounding the initial and final charged particle momenta. The vertices of the cones are taken in the interaction point. We introduce a small auxiliary parameter θ_0, it should obey the restriction

$$\frac{m_e}{\sqrt{s}} \ll \theta_0 \ll 1. \tag{1.9}$$

So, we define a collinear kinematical region, as a part of the whole phase space, in which the hard photon is emitted within the cone of θ_0 polar angle with respect to the direction of motion of one of the charged particles.

Using the method of quasi-real electrons [6, 7], the matrix element $\mathcal{M}$ (squared and summed up over polarization states) of the process of hard photon emission can be expressed through a *shifted* matrix element of the process without photon emission:

$$\sum |\mathcal{M}(p_1, k, p_1', \mathcal{X})|^2 = \\ = 4\pi\alpha \left[\frac{1 + (1-x)^2}{x(1-x)} \frac{1}{kp_1} - \frac{m^2}{(kp_1)^2} \right] \sum |\mathcal{M}_0(p_1 - k, p_1', \mathcal{X})|^2,$$

$$\sum |\mathcal{M}(p_1, p_1', k, \mathcal{X})|^2 = \\ 4\pi\alpha \left[\frac{y^2 + Y^2}{\omega Y} \frac{\varepsilon}{kp_1'} - \frac{m^2}{(kp_1')^2} \right] \sum |\mathcal{M}_0(p_1, p_1' + k, \mathcal{X})|^2, \tag{1.10}$$

$$x = \frac{\omega}{\varepsilon}, \quad p_1^0 = \varepsilon, \quad y = \frac{{p_1'}^0}{\varepsilon}, \quad Y = x + y,$$

where $\mathcal{X}$ denotes the momenta of nonradiating incoming and outgoing particles in the given process. The integration over the phase volume of the emitted photon inside the narrow cone, surrounding its parent charged particle momentum, gives the following factors:

$$\frac{4\alpha}{16\pi^2} \int \frac{d^3k}{\omega} \left[\frac{1 + (1-x)^2}{x(1-x)} \frac{1}{kp_1} - \frac{m^2}{(kp_1)^2} \right] = \\ = \frac{\alpha}{2\pi} \frac{dz_1}{z_1} \left[P_\Theta^{(1)}(z_1) \left(L - 1 + \ln\frac{\theta_0^2}{4} \right) + 1 - z_1 \right], \quad z_1 = 1 - x,$$

$$\frac{4\alpha}{16\pi^2}\int\frac{d^3k}{\omega}\left[\frac{y^2+Y^2}{xY}\,\frac{1}{kp_1'}-\frac{m^2}{(kp_1')^2}\right]=$$
$$=\frac{\alpha}{2\pi}\,\frac{dz_3}{z_3}\left[P_\Theta^{(1)}(z_3)\left(L-1+\ln\frac{\theta_0^2}{4}+2\ln z_3\right)+1-z_3\right], \quad (1.11)$$

$$z_3=1-\frac{\omega}{{p_1'}^0+\omega}=1-\frac{x}{Y}.$$

Note that the terms proportional to $(L-1)$ contain the Θ-part of the kernel $P^{(1)}=P_\Delta^{(1)}+P_\Theta^{(1)}$ of Altarelli–Parisi–Lipatov evolution equations:

$$P_\Theta^{(1)}(z)=\frac{1+z^2}{1-z}\Theta(1-z-\Delta).$$

Collecting the contributions of the four collinear regions, we obtain

$$\frac{d\sigma_{\rm coll}}{d\Omega_-}=\frac{\alpha}{\pi}\int\limits_\Delta^1\frac{dx}{x}\left\{\left[\left(1-x+\frac{x^2}{2}\right)\left(L-1+\ln\frac{\theta_0^2}{4}+2\ln(1-x)\right)+\frac{x^2}{2}\right]\times\right.$$
$$\times\,2\,\frac{d\tilde\sigma_0(1,1)}{d\Omega_-}+\left[\left(1-x+\frac{x^2}{2}\right)\left(L-1+\ln\frac{\theta_0^2}{4}\right)+\frac{x^2}{2}\right]\times$$
$$\left.\times\left[\frac{d\tilde\sigma_0(1-x,1)}{d\Omega_-}+\frac{d\tilde\sigma_0(1,1-x)}{d\Omega_-}\right]\right\}, \quad (1.12)$$

where the shifted Born cross section is defined in Eq. (1.5).

Adding the contributions of virtual and soft photon emission, we restore the complete kernel. Generalizing the procedure for the case of photon emission by all charged particles, we come to the representation of the cross section in the leading logarithmic approximation. The final expression for the cross section therefore has the form

$$\frac{d\sigma^{e^+e^-\to e^+e^-(\gamma)}}{d\Omega_-}=\int\limits_{\bar z_1}^1 dz_1\int\limits_{\bar z_2}^1 dz_2\,\mathcal{D}(z_1)\mathcal{D}(z_2)\frac{d\tilde\sigma_0(z_1,z_2)}{d\Omega_-}\left(1+\frac{\alpha}{\pi}K_{SV}\right)\Theta\times$$
$$\times\int\limits_{y_{\rm th}}^{Y_1}\frac{dy_1}{Y_1}\int\limits_{y_{\rm th}}^{Y_2}\frac{dy_2}{Y_2}\mathcal{D}\left(\frac{y_1}{Y_1}\right)\mathcal{D}\left(\frac{y_2}{Y_2}\right)+$$
$$+\frac{\alpha}{\pi}\int\limits_\Delta^1\frac{dx}{x}\left\{\left[\left(1-x+\frac{x^2}{2}\right)\ln\frac{\theta_0^2(1-x)^2}{4}+\frac{x^2}{2}\right]2\frac{d\sigma_0^{\rm Born}}{d\Omega_-}+\right.$$

$$+\left[\left(1-x+\frac{x^2}{2}\right)\ln\frac{\theta_0^2}{4}+\frac{x^2}{2}\right]\left[\frac{4\alpha^2}{s(1-x)^2[2-x(1-c)]^4}\times\right.$$

$$\times\left(\frac{3-3x+x^2+2x(2-x)c+c^2(1-x+x^2)}{1-c}\right)^2+$$

$$\left.\left.+\frac{4\alpha^2}{s[2-x(1+c)]^4}\left(\frac{3-3x+x^2-2x(2-x)c+c^2(1-x+x^2)}{1-c}\right)^2\right]\right\}\Theta-$$

$$-\frac{\alpha^2}{4s}\left(\frac{3+c^2}{1-c}\right)^2\frac{8\alpha}{\pi}\ln\left(\cot\frac{\theta}{2}\right)\ln\frac{\Delta\varepsilon}{\varepsilon}+\frac{\alpha^3}{2\pi^2 s}\int\limits_{\substack{k^0>\Delta\varepsilon\\ \pi-\theta_0>\theta>\theta_0}}\frac{WT}{4}\,\Theta\frac{d\Gamma}{d\Omega_-}, \quad (1.13)$$

$$Y_1=\frac{2z_1z_2}{z_1+z_2-c(z_1-z_2)}, \quad Y_2=\frac{z_1^2+z_2^2-(z_1^2-z_2^2)c}{z_1+z_2-c(z_1-z_2)},$$

$$\bar{z}_1=\frac{y_{\rm th}(1+c)}{2-y_{\rm th}(1-c)}, \qquad \bar{z}_2=\frac{z_1y_{\rm th}(1-c)}{2z_1-y_{\rm th}(1+c)}.$$

Symbol Θ denotes above possible experimental cuts on the final state particle phase space. The last term describes hard photon emission process, provided that the photon energy fraction x is larger than $\Delta=\Delta\varepsilon/\varepsilon$, and its emission angle with respect to any charged particle direction is larger than some small quantity θ_0. The sum of the last three terms in Eq. (1.13) does not depend on the auxiliary parameters Δ and θ_0, if they are sufficiently small. We omitted the effects due to vacuum polarization in the last three terms which describe real hard photon emission; because the theoretical uncertainty, coming from this approximation, has the order $\delta(d\sigma)/d\sigma\sim(\alpha/\pi)^2L\lesssim 10^{-4}$. Nevertheless, if the center-of-mass energy is close to some resonance mass (say to m_φ), the effect due to vacuum polarization may become visible. According to the master formula (1.13) a Monte Carlo event generator [33] was created. The generator is applied to data simulation and analysis at electron–positron colliders such as VEPP-2M and VEPP-2000.

1.1.1. Calculation of Real Collinear Photon Radiation. Here we present the direct evaluation of the collinear region contribution to the Bhabha scattering cross section. Let us write the contribution of the collinear kinematics in the form:

$$(d\sigma)_{\rm coll}=d\sigma_{\mathbf{k}\|\mathbf{p}_-}+d\sigma_{\mathbf{k}\|\mathbf{p}_+}+d\sigma_{\mathbf{k}\|\mathbf{p}'_-}+d\sigma_{\mathbf{k}\|\mathbf{p}'_+}\equiv$$
$$\equiv d\sigma_a+d\sigma_b+d\sigma_c+d\sigma_d. \quad (1.14)$$

For the case of photon emission along the initial electron we have (see Eqs. (1.4), (1.8)):

$$W_a = \frac{2}{\omega^2}\,\frac{1}{1-\beta c_2}, \quad T_a = \frac{1+(1-x)^2}{1-x}R(s_1,t^a,u^a), \quad d\Gamma_a = \frac{d^3k}{\omega}\,\frac{y_1^a}{a_a}\,d\Omega_-,$$
$$c_2 = \cos(\widehat{\mathbf{k}\mathbf{p}_-}),\ \beta = \sqrt{1-\frac{m_e^2}{\varepsilon^2}},\ \omega = k^0 = x\varepsilon,\ s_1 = s(1-x),\ t_1^a = t^a(1-x), \tag{1.15}$$
$$u_1^a = u^a(1-x),\ t^a = -s\frac{(1-x)^2(1-c)}{a_a},\ u^a = -s\frac{(1-x)(1+c)}{a_a},\ s = 4\varepsilon^2,$$
$$a_a = 2 - x(1-c), \quad y_1^a = \frac{2(1-x)}{a_a}, \quad c = \cos(\widehat{\mathbf{p}_-\mathbf{p}'_-}).$$

Performing the angular integration over photon angles inside the narrow cone, surrounding the direction of the initial electron beam, we get

$$\int W_a d\Gamma_a = 4\pi\frac{d\omega}{\omega}d\Omega_-\times \\ \times \int\limits_{1-\theta_0^2/2}^{1}\frac{dc_2}{1-\beta c_2} = 4\pi\frac{dx}{x}d\Omega_-\frac{y_1^a}{a_a^2}\left(L+\ln\frac{\theta_0^2}{4}+\mathcal{O}(\theta_0^2)\right). \tag{1.16}$$

We neglect the terms proportional to θ_0^2. Collecting all the factors and reminding the contribution of the terms proportional to m_e^2 (see Eq. (1.8)), we obtain the contribution of the first collinear region:

$$\frac{d\sigma_a}{d\Omega_-} = \frac{4\alpha^2}{s}\,\frac{\alpha}{\pi}\int\limits_{\Delta}^{1}\frac{dx}{x}\left[\left(1-x+\frac{x^2}{2}\right)\left(L-1+\ln\frac{\theta^2}{4}\right)+\frac{x^2}{2}\right]\frac{1}{a_a^2}\times \\ \times\frac{[a_a^2+(1-c)^2(1-x)^2-a_a(1-c)(1-x)]^2}{a_a^2(1-x)^2(1-c)^2}. \tag{1.17}$$

For the case of photon emission inside the narrow cone, surrounding $\mathbf{p}_+$, in a similar way one gets

$$\frac{d\sigma_b}{d\Omega_-} = \frac{4\alpha^2}{s}\,\frac{\alpha}{\pi}\int\limits_{\Delta}^{1}\frac{dx}{x}\left[\left(1-x+\frac{x^2}{2}\right)\left(L-1+\ln\frac{\theta^2}{4}\right)+\frac{x^2}{2}\right]\frac{1}{a_b^2}\times \\ \times\frac{[a_b^2+(1-c)^2(1-x)^2-a_b(1-c)(1-x)]^2}{a_b^2(1-c)^2}, \quad a_b = 2 - x(1+c). \tag{1.18}$$

For the cases $\mathbf{k} \parallel \mathbf{p}'_-$ and $\mathbf{k} \parallel \mathbf{p}'_+$, quantity R (if suppose $\Pi = 0$) is simple:

$$R_c = R_d = \frac{1}{4}\left(\frac{3+c^2}{1-c}\right)^2, \quad T_c = T_d = \frac{1+(1-x)^2}{1-x} R_c,$$

$$d\Gamma_{c,d} = \varepsilon^2 dx\, d\Omega_-\, d\varphi_1\, dc_1\, \frac{x y^{c,d}}{2-x+xc_1}, \quad c_1 = \cos\widehat{\mathbf{k}\mathbf{p}'_-},$$

$$y^c = \left.\frac{p'^0_-}{\varepsilon}\right|_{\mathbf{k}\parallel\mathbf{p}'_-} \approx \left.\frac{2(1-x)}{2-x+xc_1}\right|_{c_1\to 1} = 1-x,$$

$$y^d = \left.\frac{p'^0_-}{\varepsilon}\right|_{\mathbf{k}\parallel\mathbf{p}'_+} \approx \frac{2(1-x)}{2-x+xc_1} + c_1\frac{m_e^2}{4\varepsilon^2}\,\frac{x}{1-x},$$

$$W_c(kp'_-) = W_d(kp'_+) = \frac{2(1-x)}{x}.$$

Note that $kp'_+ = 2\varepsilon^2(1-y)$. In the evaluation of the rest multipliers for these cases one has to be careful:

$$\int W_c\, d\Gamma_c\Big|_{1-\theta_0^2/2\leqslant c_1\leqslant 1} = \int W_d\, d\Gamma_d\Big|_{-1+\theta_0^2(1-x)^2/2\geqslant c_1\geqslant -1} =$$

$$= 2\pi\frac{dx}{x}d\Omega_-\frac{1-x}{2}\left[L + \ln\frac{\theta_0^2}{4} + 2\ln(1-x)\right].$$

Note that the collinear region d is defined by the condition $1-\theta_0^2/2 \leqslant \cos\widehat{\mathbf{k}\mathbf{p}'_+} \leqslant 1$, which leads to the bounds on c_1 shown above. So, the contributions of these two collinear regions are

$$\frac{d\sigma_c + d\sigma_d}{d\Omega_-} = 2\,\frac{\alpha^2}{4s}\left(\frac{3+c^2}{1-c}\right)^2\,\frac{\alpha}{\pi}\int\limits_{\Delta}^{1}\frac{dx}{x}\left[\left(1-x+\frac{x^2}{2}\right)\times\right.$$

$$\left.\times\left(L - 1 + \ln\frac{\theta_0^2}{4} + 2\ln(1-x)\right) + \frac{x^2}{2}\right]. \quad (1.19)$$

Note that there is an asymmetry between the contributions due to the emission along the directions of the (initial or final) electron and the ones due to production of collinear photons along the positron momenta. The symmetry was broken when we decided to write a differential cross section with respect to the electron scattering angles ($d\Omega_- = d\cos(\widehat{\mathbf{p}_-\mathbf{p}'_-})\, d\varphi$). After an integration over a symmetrical angular acceptance, the contributions would become equal. Compensating terms are to be extracted from Eqs. (1.17)–(1.19) by omitting the terms proportional to $(L-1)$.

1.2. Virtual and Soft Real Pair Production in Large-Angle Bhabha Scattering. This Subsection is devoted to the calculations of the QED $\mathcal{O}(\alpha^2)$ RC to the LABS process accompanied by the production of a virtual or a soft real e^+e^- pair. We work within the logarithmic accuracy and drop all the terms of the order α^2 which are not reinforced by the *large logarithm* $L = \ln(4\varepsilon^2/m_e^2)$ (ε is the beam energy in the center-of-mass reference frame). We consider here only the contribution of e^+e^- pairs. The contributions of muon and other pairs are less than the latter (they contain only linear in L terms), they will be considered separately.

Recently analytic calculations for complete two-loop virtual pair corrections to Bhabha scattering have been completed including the contributions of electron–positron loop [38] and those of heavier leptons and hadrons [34–37].

The general expression for the cross section with the corrections under consideration can be presented in the form

$$d\sigma = d\sigma_0 \left\{ 1 + \left(\frac{\alpha}{\pi}\right)^2 \left[\sum_{i=1}^{7} \delta_i + \delta^{\gamma}_{\text{soft}} + \delta^{\gamma}_{\text{hard}} + \delta^{e^+e^-}_{\text{soft}} \right] \right\}, \tag{1.20}$$

where $d\sigma_0$ is the Born cross section, δ_i arises from virtual corrections, $\delta^{\gamma}_{\text{soft}}$ — from soft photon emission, $\delta^{\gamma}_{\text{hard}}$ — from hard photon emission, and $\delta^{e^+e^-}_{\text{soft}}$ — from soft pair production.

1.2.1. Virtual Corrections. The Feynman diagrams describing the $\mathcal{O}(\alpha^2)$ order RC to LABS process

$$e^-(p_1) + e^+(p_2) \to e^-(q_1) + e^+(q_2), \tag{1.21}$$

which contain a vacuum polarization bubble, can be split into seven classes. In Fig. 1, one can see some representatives of the diagrams from different classes (any multiplication of diagrams has to be considered as a multiplication of a diagram by a conjugated one).

The first five contributions $\delta_{1,5}$ can be written down using the known expressions for vacuum polarization operators and vertex functions (only the Dirac form factor is relevant: the contribution of the Pauli one is proportional to m_e^2/ε^2). In the scattering channel one has for the vacuum polarization operators Π_{1t} (for a one-loop bubble) and Π_{2t} (for a two-loop bubble):

$$\begin{gathered}\Pi_{1t} = \frac{1}{3}l_t - \frac{5}{9}, \quad \Pi_{2t} = \frac{1}{4}l_t + \mathcal{O}(1), \quad l_t = \ln\left(\frac{-t}{m_e^2}\right), \\ t = -2p_1q_1, \quad s = 2p_1p_2, \quad u = -2p_1q_2, \quad -t \sim s \gg m_e^2,\end{gathered} \tag{1.22}$$

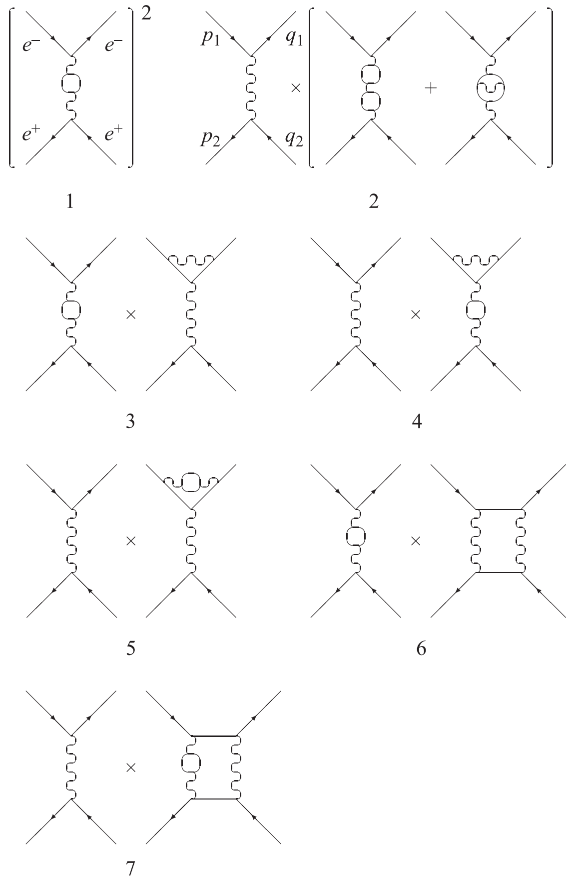

Fig. 1. Representatives of Feynman diagrams for virtual pair production

and for the vertex functions F_{1t} (for a one-loop vertex correction) and F_{2t} (for a two-loop vertex correction which includes a vacuum polarization insertion (see Fig. 5.(4, 5)):

$$F_{1t} = (l_t - 1)(l_\lambda + 1) - \frac{1}{4}l_t - \frac{1}{4}l_t^2 + \frac{\pi^2}{12}, \quad l_\lambda = \ln\left(\frac{\lambda}{m_e}\right),$$
$$F_{2t} = -\frac{1}{36}l_t^3 + \frac{19}{72}l_t^2 - \left(\frac{\pi^2}{36} + \frac{265}{216}\right)l_t + \mathcal{O}(1), \tag{1.23}$$

λ is an auxiliary parameter — the fictitious photon mass. Similar expressions can be written in the annihilation channel (for $\Pi_{1s,2s}$ and $F_{1s,2s}$) using the substitution

$$l_t \to \tilde{l}_s = l_s - i\pi, \quad l_s = \ln\left(\frac{s}{m_e^2}\right). \tag{1.24}$$

After simple algebraic transformations within the logarithmic accuracy we obtain

$$\sum_{i=1}^{5} \delta_i = \left[2\left(\frac{s}{t} + \frac{t}{s} + 1\right)^2\right]^{-1} \left\{\frac{s^2+u^2}{t^2}\Phi_t + \frac{t^2+u^2}{s^2}\Phi_s + \frac{2u^2}{st}\Phi_{st}\right\}, \tag{1.25}$$

$$\Phi_t = 3\Pi_{1t}^2 + 2\Pi_{2t} + 8\Pi_{1t}F_{1t} + 4F_{2t} = \\ = \frac{8}{3}l_t\left(l_t - \frac{8}{3}\right)l_\lambda - \frac{7}{9}l_t^3 + \frac{9}{2}l_t^2 + l_t\left(\frac{\pi^2}{9} - \frac{311}{27}\right),$$

$$\Phi_s = 3|\Pi_{1s}|^2 + 2\mathrm{Re}\,\Pi_{2s} + 8\,\mathrm{Re}\,\Pi_{1s}^* F_{1s} + 4\,\mathrm{Re}\,F_{2s} = \\ = \frac{8}{3}l_s\left(l_s - \frac{8}{3}\right)l_\lambda - \frac{7}{9}l_s^3 + \frac{9}{2}l_s^2 + l_s\left(-\frac{2\pi^2}{9} - \frac{311}{27}\right),$$

$$\Phi_{st} = \mathrm{Re}\left\{\Pi_{1t}^2 + \Pi_{1s}^2 + \Pi_{1t}\Pi_{1s} + 2(\Pi_{1t} + \Pi_{1s})(F_{1t} + F_{1s}) + \Pi_{2t} + \Pi_{2s} + \right. \\ \left. + 2(F_{2t} + F_{2s})\right\} = \left[\frac{2}{3}(l_s + l_t)^2 - \frac{32}{9}(l_s + l_t)\right]l_\lambda - \frac{2}{9}(l_t^3 + l_s^3) - \\ - \frac{1}{6}l_t l_s(l_t + l_s) + \frac{61}{36}(l_t^2 + l_s^2) + \frac{10}{9}l_t l_s + \left(\frac{17}{36}\pi^2 - \frac{311}{54}\right)(l_t + l_s).$$

Consider now the virtual corrections of the sixth class: the ones due to the interference of the Born amplitude corrected by a vacuum polarization insertion with the box amplitude. One has to consider the scalar, vector, and tensor loop integrals over the box virtual momentum k. As an example we present here the integrals for scattering channel box diagram with uncrossed photon lines (see Fig. 1(6)):

$$b,\ J_\sigma,\ J_{\rho\sigma} = \int \frac{d^4k}{i\pi^2}\ \frac{1,\ k_\sigma,\ k_\rho k_\sigma}{(k^2 - \lambda^2 + i0)((q-k)^2 - \lambda^2 + i0)} \times \\ \times \frac{1}{((p_1+k)^2 - m_e^2 + i0)((-p_2+k)^2 - m_e^2 + i0)}, \tag{1.26}$$

$$J_\sigma = b_1\Delta_\sigma + b_2 q_\sigma,$$
$$J_{\rho\sigma} = b_3\Delta_\rho\Delta_\sigma + b_4 P_\rho P_\sigma + b_5(\Delta_\rho q_\sigma + \Delta_\sigma q_\rho) + b_6 q_\rho q_\sigma + b_7 g_{\rho\sigma},$$
$$q = q_1 - p_1 = p_2 - q_2, \quad \Delta = \frac{1}{2}(p_2 - p_1), \quad P = \frac{1}{2}(p_1 + p_2).$$

The explicit expressions for the coefficients $b - b_7$ are given in Subsec. 2.1.

After some algebraic work with traces one gets

$$\delta_6 = \left[2\left(\frac{s}{t} + \frac{t}{s} + 1\right)^2\right]^{-1} \mathrm{Re}\,\big(1 + P(s,t)\big)\times \\ \times\left\{\frac{\Pi_{1t}}{t}\big(1 - P(s,u)\big)f_1(s,t) - \frac{\Pi_{1s}^*}{s}\left(f_2(s,t) + f_3(u,t)\right)\right\}, \quad (1.27)$$

where

$$f_1(s,t) = -s(s^2+u^2)[b-b_1] + 2s^2t[-b_2+b_5+b_6]+ \\ + \frac{s}{4}(2s^2+u^2)[-b_3+b_4] + 2(4s^2+u^2)b_7,$$

$$f_2(s,t) = u^2\left(s\left[b - b_1 + \frac{1}{4}b_3 - \frac{1}{4}b_4\right] - b_7\right),$$
$$f_3(u,t) = u^2\left(u\left[-\tilde b + \tilde b_1 - \frac{1}{2}\tilde b_3 - \frac{1}{2}\tilde b_4\right] + 2t[-\tilde b_2 + \tilde b_5 + \tilde b_6] + 8\tilde b_7\right),$$
$$\tilde b, \tilde b_i = P(s,u)b, b_i \quad \Pi_{1s}^* = \frac{1}{3}(l_s + i\pi) - \frac{5}{9}.$$

The interchange operators $P(s,t)$ and $P(s,u)$ act in the following way:

$$P(s,t)f(s,t,u;\tilde l_s,l_t,l_u) = f(t,s,u;l_t,\tilde l_s,l_u), \\ P(s,u)f(s,t,u;\tilde l_s,l_t,l_u) = f(u,t,s;l_u,l_t,\tilde l_s). \quad (1.28)$$

Consider now the corrections of the seventh class. In the calculations we use the substitution suggested by J. Schwinger for the photon propagator (with 4-momentum k) corrected by a one-loop vacuum polarization insertion (see Subsec. 2.2 for implementation of this method):

$$\frac{1}{k^2 - \lambda^2 + i0} \to \frac{\alpha}{\pi}\int\limits_0^1 \frac{dv\varphi_i(v)}{1-v^2}\,\frac{1}{k^2 - M_i^2}, \quad M_i^2 = \frac{4m_i^2}{1-v^2}. \quad (1.29)$$

For contribution of leptons we have

$$\varphi_l(v) = \frac{1}{3}(2 - (1-v^2)(2-v^2)), \quad m_i = m_l, \quad l = e, \mu, \tau. \quad (1.30)$$

The interference of the eight box diagrams with the Born ones gives the following contribution to the summed over spin states matrix element square:

$$\sum_{\text{spin}} |\mathcal{M}|^2 = \alpha^4 2^8 \int\limits_0^1 \frac{dv\varphi(v)}{1-v^2} \left(1+P(s,t)\right)\times \times \int \frac{d^4k}{i\pi^2} \frac{1}{(k^2-\lambda^2+i0)((k+q)^2-M^2)} \times \times \left\{ \left(\frac{S_1}{t} - \frac{A_1}{4s}\right)\frac{1}{a_1 a_2} + \left(\frac{S_2}{t} - \frac{A_2}{4s}\right)\frac{1}{a_1 a_3} \right\}, \quad (1.31)$$

$$a_1 = (k+q_1)^2 - m_e^2 + i0,$$
$$a_2 = (k-q_2)^2 - m_e^2 + i0,$$
$$a_3 = (k+p_2)^2 - m_e^2 + i0,$$

where $S_{1,2}$ and $A_{1,2}$ are relevant traces of γ matrixes.

Some scalar, vector, and tensor integrals calculated within the logarithmic accuracy are necessary. We use the notations

$$I(aba_1a_2),\, I_\rho,\, I_{\rho\sigma} = \int\limits_0^1 \frac{dv\varphi(v)}{1-v^2} \int \frac{d^4k}{i\pi^2} \frac{1,\, k_\rho,\, k_\rho k_\sigma}{aba_1a_2}, \quad (1.32)$$

$$I(aba_2) = \int\limits_0^1 \frac{dv\varphi(v)}{1-v^2} \int \frac{d^4k}{i\pi^2} \frac{1}{aba_2}, \quad I(ba_1a_2) = \int\limits_0^1 \frac{dv\varphi(v)}{1-v^2} \int \frac{d^4k}{i\pi^2} \frac{1}{ba_1a_2},$$

$$a = k^2 - \lambda^2 + i0, \quad b = (k+q)^2 - M^2,$$

$$I_\rho = \alpha(p_{2\rho} - p_{1\rho}) + \beta(p_{1\rho} - q_{1\rho}), \quad \alpha = \frac{1}{2u}\left[-I - 2I(aba_2) + I(ba_1a_2)\right],$$

$$\beta = \frac{1}{2tu}\left[(t-u)I + sI(ba_1a_2) + 2tI(aba_2)\right],$$

$$I = \frac{1}{3s}\mathrm{Re}\left\{ -\frac{1}{6}\tilde{l}_s^3 + \frac{5}{6}\tilde{l}_s^2 + \tilde{l}_s\left[l_t^2 - \frac{10}{3}l_t + \frac{28}{9} + \frac{8}{9}\pi^2\right] + \mathcal{O}(1)\right\},$$

$$I_{\rho\sigma} = f_0 g_{\rho\sigma} + f_1(q_{1\rho}q_{1\sigma} + q_{2\rho}q_{2\sigma}) + f_3 q_\rho q_\sigma + f_4(q_{1\rho}q_{2\sigma} + q_{2\rho}q_{1\sigma}) + + f_5(q_\rho(q_{2\sigma} - q_{1\sigma}) + q_\sigma(q_{2\rho} - q_{1\rho})).$$

In Subsec. 2.1 we give the list of scalar integrals. It appears that only two tensor coefficients, namely f_0 and f_4, are relevant. They contain only the first power of the large logarithm. Infrared parameter λ is contained only in $I(aba_1a_2)$:

$$I(aba_1a_2) = \frac{1}{3st}\,\mathrm{Re}\times$$
$$\times\left\{-\frac{1}{6}\tilde{l}_s^3 + \frac{1}{2}\tilde{l}_s^2 l_t + \tilde{l}_s l_t^2 - \frac{10}{3}\tilde{l}_s l_t - \frac{28}{9} l_s - \frac{\pi^2}{6} l_t - 2\tilde{l}_s\left(l_t - \frac{5}{3}\right) l_\lambda\right\}. \quad (1.33)$$

1.2.2. The Soft Photon Emission. To eliminate the dependence on the fictitious photon mass λ we have to consider also the cross section of the additional emission of a soft photon with the energy

$$\omega < d\varepsilon, \quad d \ll 1, \quad (1.34)$$

corrected by vacuum polarization insertion into the virtual photon propagator. The correction can be obtained using the standard technique [2, 3], where we use the same reference frame to calculate the soft and hard photon emission contribution:

$$\delta^\gamma = \left[2\left(\frac{s}{t} + \frac{t}{s} + 1\right)^2\right]^{-1} \mathrm{Re}\left[\frac{s^2+u^2}{t^2}\Pi_{1t} + \frac{t^2+u^2}{s^2}\Pi_{1s} + \right.$$
$$\left. + \frac{u^2}{st}(\Pi_{1s} + \Pi_{1t})\right]\left\{4(\ln d - l_\lambda)(l_s + l_t - l_u - 1) + l_s^2 + l_t^2 - \right.$$
$$\left. - l_u^2 - \frac{2}{3}\pi^2 - 2\mathrm{Li}_2\left(\frac{1-c}{2}\right) + 2\mathrm{Li}_2\left(\frac{1+c}{2}\right)\right\}, \quad (1.35)$$
$$c = \cos\theta, \quad \theta = \widehat{\mathbf{p}_1, \mathbf{q}_1}.$$

1.2.3. The Semicollinear Kinematics of Hard Photon Emission. To cancel the auxiliary parameter d we have to consider also the case of a hard photon (with the energy $\omega > d\varepsilon$) emission. Our method of calculation here consists in a splitting of the total kinematical region of the emitted photon into two ones: the collinear one, when the photon is emitted within a small cone with respect to one charged particle, and the semicollinear one, when the photon moves outside of any such a cone. Then we show explicitly that the small auxiliary parameter θ_0, describing that cones, cancels in the sum of the contributions of two regions. The procedure allows us to extract explicitly the radiative corrections to the process under considerations of the orders $\mathcal{O}(\alpha^2 L^2)$ and $\mathcal{O}(\alpha^2 L)$.

Consider at first the case, when the photon moves in respect to the directions of the charged particles (as of the initial ones as well as of the final ones) with the angles satisfying the following conditions:

$$\widehat{\mathbf{kp}}_{1,2} > \theta_0, \quad \widehat{\mathbf{kq}}_{1,2} > \theta_0. \quad (1.36)$$

Here the matrix element of the process is not singular and the contribution of this region in the $\mathcal{O}(\alpha)$ order does not contain the large logarithm. In the next order in α we can just write down the contribution in the next-to-leading approximation multiplying the well-known differential cross section of a single hard photon emission by the factor $2\alpha L/(3\pi)$, coming from the vacuum polarization insertion into the virtual photon propagator:

$$d\sigma^{\gamma}_{\text{semi-coll}} = \frac{\alpha^3}{4s\pi^2}\frac{\alpha}{3\pi}\,L\,\frac{d^3\mathbf{q}_1\,d^3\mathbf{q}_2\,d^3\mathbf{k}}{q_1^0 q_2^0 k^0}\,WB\delta^{(4)}(p_1+p_2-q_1-q_2-k),$$
$$W = \frac{s}{p_1k\,p_2k} + \frac{s_1}{q_1k\,q_2k} - \frac{t}{p_1k\,q_1k} - \frac{t_1}{p_2k\,q_2k} + \frac{u}{p_1k\,q_2k} + \frac{u_1}{p_2k\,q_1k},$$
$$B = \frac{ss_1(s^2+s_1^2) + tt_1(t^2+t_1^2) + uu_1(u^2+u_1^2)}{ss_1tt_1}, \tag{1.37}$$
$$s = 2p_1p_2, \quad t = -2p_1q_1, \quad u = -2p_1q_2, \quad p_i^2 = q_i^2 = 0,$$
$$s_1 = 2q_1q_2, \quad t_1 = -2p_2q_2, \quad u_1 = -2p_2q_1.$$

The contribution should be integrated over the phase volume of the final particles, which is defined by experimental conditions, restrictions (1.36) should be fulfilled.

1.2.4. The Collinear Kinematics of Hard Photon Emission. The contribution of the collinear kinematics of a photon emission is divided naturally into four ones (in the correspondence with the cases of the photon motion in the four directions of the charged particles): 1) $\mathbf{k} \parallel \mathbf{p}_1$, 2) $\mathbf{k} \parallel \mathbf{p}_2$, 3) $\mathbf{k} \parallel \mathbf{q}_1$, 4) $\mathbf{k} \parallel \mathbf{q}_2$. So, we write the differential cross section in the form

$$\frac{d\sigma^{\gamma}_{\text{coll}}}{dy_1\,dc_-} = \frac{d\sigma_1 + d\sigma_2 + d\sigma_3 + d\sigma_4}{dy_1\,dc_-}, \tag{1.38}$$

where $y_1 = q_1^0/\varepsilon$ is the energy fraction of the scattered electron; $c_- = \cos\theta_-$, $\theta_- = \widehat{\mathbf{p}_1\mathbf{q}_1}$ is the electron scattering angle in the CM reference frame of the initial particles; subscripts in σ_i denote correspondent kinematical regions.

In the first region we get

$$\frac{d\sigma_1}{dy_1\,dc_-} = \frac{\alpha^3 y_1}{sx(1-x)(2-y_1(1-c_-))}\big[(1+(1-x)^2)L_0 - 2(1-x)\big]\times$$
$$\times\Bigg[P_{t_1}^2\frac{4+(2-y_1(1-c_-))^2}{(y_1(1-c_-))^2} + |P_{s_1}|^2\frac{1}{4}\big[(y_1(1-c_-))^2 +$$
$$+ (2-y_1(1-c_-))^2\big] - \mathrm{Re}\,(P_{t_1}P_{s_1})\frac{(2-y_1(1-c_-))^2}{y_1(1-c_-)}\Bigg], \tag{1.39}$$
$$P_{t_1} = \left(1 - \frac{\alpha}{3\pi}\big(l_{t_1} - \frac{5}{3}\big)\right)^{-1}, \qquad P_{s_1} = \left(1 - \frac{\alpha}{3\pi}\big(l_{s_1} - \frac{5}{3} - i\pi\big)\right)^{-1},$$
$$l_{t_1} = \ln\!\left(\frac{2y_1(1-c_-)(1-x)\varepsilon^2}{m_e^2}\right), \quad l_{s_1} = \ln\!\left(\frac{4\varepsilon^2(1-x)}{m_e^2}\right),$$
$$x = \frac{2(1-y_1)}{2-y_1(1-c_-)}, \quad L_0 = \ln\left(\frac{\varepsilon\theta_0}{m_e}\right)^2 \gg 1,$$

x is the energy fraction of the emitted photon. The energy fraction and the scattering angle of the positron in this kinematical region are

$$y_2 = \frac{q_2^0}{\varepsilon} = \frac{1+(1-y_1)^2+y_1(2-y_1)c_-}{2-y_1(1-c_-)}, \qquad y_2c_+ = -x - y_1c_-,$$
$$c_+ = \cos(\widehat{\mathbf{p}_1\mathbf{q}_2}), \quad y_2\sqrt{1-c_+^2} = y_1\sqrt{1-c_-^2}. \tag{1.40}$$

One can check that in the sum of the above contribution with the one of the semicollinear region, when the photon is emitted close to the cone of the angle θ_0 around the initial electron beam direction, the terms of the order $\sim L\ln\theta_0^2$ will disappear.

The contribution of the second collinear region has the form

$$\begin{aligned}\frac{d\sigma_2}{dy_1\,dc_-} &= \frac{\alpha^3 y_1}{sx(1-x)(2-y_1(1+c_-))}\big[(1+(1-x)^2)L_0 - 2(1-x)\big]\times\\ &\times\bigg[P_t^2\frac{4+(2-y_1(1-c_-))^2}{(y_1(1-c_-))^2} + |P_{s_1}|^2\frac{1}{4}\big[(y_1(1-c_-))^2+\\ &+(2-y_1(1-c_-))^2\big] - \mathrm{Re}\,(P_tP_{s_1})\frac{(2-y_1(1-c_-))^2}{y_1(1-c_-)}\bigg],\\ P_t &= \left(1-\frac{\alpha}{3\pi}\big(l_t-\frac{5}{3}\big)\right)^{-1}, \quad l_t = \ln\left(\frac{2y_1(1-c_-)\varepsilon^2}{m_e^2}\right).\end{aligned} \tag{1.41}$$

We put here also the expressions for the photon and positron energy fractions and for the positron scattering angle:

$$1-x = \frac{y_1(1-c_-)}{2-y_1(1+c_-)}, \quad y_2 = \frac{1+(1-y_1)^2-y_1(2-y_1)c_-}{2-y_1(1+c_-)},$$
$$y_2c_+ = x - y_1c_-. \tag{1.42}$$

The contribution of the third collinear region and the parameters of the scattered positron are:

$$\begin{aligned}\frac{d\sigma_3}{dy_1\,dc_-} &= \frac{\alpha^3y_1^2}{2sx(1-x)}\left[\frac{1+(1-x)^2}{1-x}\big(L_0+2\ln(1-x)\big)-2\right]\times\\ &\times\left[P_t^2\frac{4+(1+c_-)^2}{(1-c_-)^2} + |P_s|^2\frac{(1-c_-)^2+(1+c_-)^2}{4}-\right.\\ &\left.-\mathrm{Re}\,(P_tP_s)\frac{(1+c_-)^2}{1-c_-}\right], \quad P_s = \left(1-\frac{\alpha}{3\pi}\left(l_s-\frac{5}{3}-i\pi\right)\right)^{-1},\\ l_s &= \ln\left(\frac{4\varepsilon^2}{m_e^2}\right), \quad y_2 = 1, \quad 1-x = y_1, \quad c_+ = -c_-.\end{aligned} \tag{1.43}$$

Finally, in the fourth collinear region the energy fraction of the scattered electron is unity and the final particles move back to back as well as in the third region. The correspondent contribution reads

$$\frac{d\sigma_4}{dy_1\,dc_-} = \int \frac{\alpha^3\delta(1-y_1)}{2sx(1-x)} y_2^2\,dy_2 \left[\frac{1+(1-x)^2}{1-x}\big(L_0 + 2\,\ln(1-x)\big) - 2\right]\times$$
$$\times \left[P_t^2 \frac{4+(1+c_-)^2}{(1-c_-)^2} + |P_s|^2 \frac{(1-c_-)^2+(1+c_-)^2}{4} - \right.$$
$$\left. - \operatorname{Re}(P_t P_s^*)\frac{(1+c_-)^2}{1-c_-}\right], \quad 1-x=y_2, \qquad c_+ = -c_-. \quad (1.44)$$

In all cases one can be convinced in the cancellation of $L\ln\theta_0^2$ in the sum with the relevant terms of the semicollinear contributions.

So, the total contribution to the LABS process differential cross section due to a hard photon emission with the vacuum polarization correction of the virtual photon propagator reads

$$\delta^{\gamma}_{\text{hard}} = \left(\frac{d\sigma_0}{dy_1\,dc_-}\right)^{-1}\left[\frac{d\sigma^{\gamma}_{\text{semi-coll}}}{dy_1\,dc_-} + \frac{d\sigma^{\gamma}_{\text{coll}}}{dy_1\,dc_-}\right]. \quad (1.45)$$

The auxiliary parameter d (see Eq. (1.34)) cancels in the above sum.

1.2.5. Soft Pair Production. Here we consider the process

$$e^-(p_1) + e^+(p_2) \to e^-(q_1) + e^+(q_2) + e^-(p_-) + e^+(p_+), \quad (1.46)$$

where $e^-(p_-)\ +\ e^+(p_+)$ is the created soft pair. It gives to the cross section an important contribution, which contains terms cubic in the large logarithm. The maximum energy of the soft pair is taken as $D\varepsilon$, it is assumed to be large compared with the electron mass:

$$2m_e < D\varepsilon \ll \varepsilon. \quad (1.47)$$

The contributions containing L^3 will cancel with the terms due to virtual corrections, and the dependence on the auxiliary parameter D will disappear in the sum with the contribution of the hard (with the energy of pair components larger than $D\varepsilon$) pair emission.

Recently the contribution of the soft pair production was calculated in two limiting cases: for the process of e^+e^- annihilation into hadrons [11] and for the case of small-angle Bhabha scattering [1, 9]. Here we carry out the calculations for arbitrary scattering angles.

Due to the smallness of the energy of the pair components, the matrix element M of a hard process with the charged particles with momenta p_1, q_1, accompanied by soft pair emission, can be expressed through the matrix element of the hard subprocess M_0 without pair production in the way

$$M = M_0 \frac{4\pi\alpha}{k^2}\,\bar{v}(p_+)\gamma_\mu u(p_-)\,J_\mu, \quad k = p_+ + p_-. \tag{1.48}$$

The classic accompanied radiation current approximation can be applied to put J_μ in the form:

$$J_\mu = -\frac{p_{1\mu}}{p_1 k + \frac{1}{2}k^2} + \frac{q_{1\mu}}{q_1 k - \frac{1}{2}k^2} + \frac{p_{2\mu}}{p_2 k - \frac{1}{2}k^2} - \frac{q_{2\mu}}{q_2 k + \frac{1}{2}k^2}. \tag{1.49}$$

Performing the covariant integration of the summed over spin states modulus of the matrix element over the pair components momenta, we obtain

$$\sum_{\text{spin}} |\bar{v}(p_+)\gamma_\mu u(p_-)|^2 = 4\left(p_+^\mu p_-^\nu + p_+^\nu p_-^\mu - \frac{k^2}{2}\,g^{\mu\nu}\right),$$

$$\int \frac{d^3\mathbf{p}_+\,d^3\mathbf{p}_-}{p_+^0 p_-^0}\delta^4(p_+ + p_- - k)\left(p_+^\mu p_-^\nu + p_+^\nu p_-^\mu - \frac{k^2}{2}g_{\mu\nu}\right) =$$

$$= \left(-\frac{2\pi}{3}(k^2 + 2m_e^2)\sqrt{1 - \frac{4m_e^2}{k^2}}\right)\left(g_{\mu\nu} - \frac{1}{k^2}k_\mu k_\nu\right). \tag{1.50}$$

At first, we parameterize the phase volume of the pair momentum as

$$d^4k = dk_0(\mathbf{k})^2 d\,|\mathbf{k}|\,d\Omega_k = dk_0\,dk^2\,\sqrt{k_0^2 - k^2}\pi\,dc_k. \tag{1.51}$$

Neglecting the invariant mass of the pair $\sqrt{k^2}$ compared with the energies of its components and omitting the terms of order $m_e^2/(p_1k)^2$ (this simplification does not violate the logarithmical accuracy that we keep here) we perform the angular integration:

$$I = \int \frac{d\Omega_k}{2\pi}\,\frac{2p_1q_1}{2p_1k\,2q_1k} = \int_0^1 dx \int_{-1}^1 \frac{dc_k\,2p_1q_1}{4\varepsilon^2(k_0 - |\mathbf{k}||\mathbf{n}|c_k)^2} = \frac{2p_1q_1}{2\varepsilon^2}\int_0^1 \frac{dx}{k_0^2 - (\mathbf{k})^2(\mathbf{n})^2},$$

$$\mathbf{n} = x\frac{\mathbf{p}_1}{\varepsilon} + (1-x)\frac{\mathbf{q}_1}{\varepsilon}, \quad (\mathbf{n})^2 = 1 - 4x(1-x)z^2, \quad z = \sqrt{\frac{1-c}{2}}, \tag{1.52}$$

$$c = \cos\theta, \quad \theta = \widehat{\mathbf{p}_1\mathbf{q}_1}.$$

Integrating over auxiliary variable x we obtain

$$I = \frac{2J}{k_0^2(1-y)},$$

$$J = \left(1 + \frac{y}{(1-y)z^2}\right)^{-1/2} \times \\ \times \left[\ln z - \frac{1}{2}\ln y + \ln 2 + \ln\left(\frac{\sqrt{1-y} + \sqrt{1+y(z^{-2}-1)}}{2}\right)\right]. \quad (1.53)$$

The result can be expressed as a ratio of the cross sections of the processes of electron scattering in an external field with soft pair production to the Born one:

$$\frac{d\sigma^{\rm sp}}{d\sigma_0} = \frac{\alpha^2}{3\pi^2}\int\limits_1^N \frac{dt}{t}\int\limits_{1/t}^1 \frac{dy}{y^2}(y + \frac{1}{2t})\sqrt{1 - \frac{1}{yt}}(1 + y(z^{-2}-1))^{-1/2} \times \\ \times \left[2\ln z + 2\ln 2 - \ln y + 2\ln\left(\frac{\sqrt{1-y} + \sqrt{1+y(z^{-2}-1)}}{2}\right)\right], \quad (1.54)$$

$$N = \left(\frac{D\varepsilon}{2m}\right)^2 \gg 1, \quad t = \frac{k_0^2}{4m^2}, \quad y = \frac{k^2}{k_0^2}.$$

At first integrating over t we omit terms of the order N^{-1}. Then we introduce variable $x = 1/(Ny)$ and split the integration using parameter η ($1 \gg \eta \gg N^{-1}$). Within the logarithmic accuracy we obtain

$$\frac{d\sigma^{\rm sp}}{d\sigma_0} = \frac{\alpha^2}{3\pi^2}\left\{\int\limits_{1/N}^{\eta} \frac{dx}{\sqrt{x(x + N^{-1}(z^{-2}-1))}}\left(-\frac{5}{3} + 2\ln 2 - \ln x\right) \times \right. \\ \times \left[2\ln z + \ln N + 2\ln 2 + \ln x + \right. \\ \left. + 2\ln\left(\frac{\sqrt{1-(Nx)^{-1}} + \sqrt{1+(Nx)^{-1}(z^{-2}-1)}}{2}\right)\right] + \\ \left. + \ln N \int\limits_{\eta}^{1} \frac{dx}{x}\left[\left(-\frac{5}{3} - \frac{1}{3}x\right)\sqrt{1-x} + \ln\frac{(1+\sqrt{1-x})^2}{x}\right]\right\}. \quad (1.55)$$

The final expression reads

$$\frac{d\sigma^{\rm sp}}{d\sigma_0} = \frac{\alpha^2}{6\pi^2}\Bigg\{\frac{1}{3}L^3 + L^2\left(2\ln D - \frac{5}{3}\right) + L\left[4\ln^2 D - \frac{20}{3}\ln D + \frac{56}{9} - \frac{2}{3}\pi^2 + 2{\rm Li}_2\left(\frac{1+c}{2}\right)\right] + \mathcal{O}(1)\Bigg\}, \quad (1.56)$$

where

$$L = \ln\frac{2\varepsilon^2(1-c)}{m_e^2} \gg 1, \quad z = \sqrt{\frac{1-c}{2}}. \quad (1.57)$$

Using general expression (1.56) we reproduce the results [11] obtained earlier in the annihilation channel ($c = -1,\ z = 1$):

$$\frac{d\sigma^{\rm sp}}{d\sigma_0}\Bigg|_{z=1} = \frac{\alpha^2}{6\pi^2}\Bigg\{\frac{1}{3}(\rho + 2\ln D)^3 - \frac{5}{3}(\rho + 2\ln D)^2 + 4(\rho + 2\ln D)\left(\frac{14}{9} - \frac{\pi^2}{6}\right)\Bigg\}, \quad (1.58)$$

$$\rho = \ln\left(\frac{4\varepsilon^2}{m_e^2}\right),$$

and in the small-angle scattering channel ($c \to 1$) [8]:

$$\frac{d\sigma^{\rm sp}}{d\sigma_0}\Bigg|_{z\ll 1} = \frac{\alpha^2}{6\pi^2}\Bigg\{\frac{1}{3}(L + 2\ln D)^3 - \frac{5}{3}(L + 2\ln D)^2 + 4(L + 2\ln D)\left(\frac{14}{9} - \frac{\pi^2}{12}\right)\Bigg\}, \quad (1.59)$$

$$L = \ln\left(\frac{\varepsilon^2\theta^2}{m_e^2}\right), \quad z \approx \frac{\theta}{2} \to 0, \quad L \gg 1.$$

To obtain the total contribution of the soft pair production we have to multiply by a factor of 2 (to account the pair emission from the positron line) and to add the t- and u-channel contributions, which can be obtained by simple substitutions. In this way one gets

$$\begin{aligned}\delta^{e^+e^-} = \frac{1}{3}\Bigg\{&\frac{1}{3}l_s^3 + l_s^2\left(2\ln D - \frac{5}{3}\right) + l_s\left(4\ln^2 D - \frac{20}{3}\ln D + A_s\right) + \\ &+ \frac{1}{3}l_t^3 + l_t^2\left(2\ln D - \frac{5}{3}\right) + l_t\left(4\ln^2 D - \frac{20}{3}\ln D + A_t\right) - \\ &- \frac{1}{3}l_u^3 - l_u^2\left(2\ln D - \frac{5}{3}\right) - l_u\left(4\ln^2 D - \frac{20}{3}\ln D + A_u\right)\Bigg\}, \quad (1.60)\end{aligned}$$

where

$$A_s = \frac{56}{9} - \frac{2}{3}\pi^2,$$
$$A_t = \frac{56}{9} - \frac{2}{3}\pi^2 + 2\mathrm{Li}_2\left(\frac{1+c}{2}\right), \qquad (1.61)$$
$$A_u = \frac{56}{9} - \frac{2}{3}\pi^2 + 2\mathrm{Li}_2\left(\frac{1-c}{2}\right).$$

1.2.6. Total Sum of the Contributions and Numerical Estimation. The total sum of the considered corrections does not contain parameter λ and cubic in large logarithm terms. It reads

$$d\sigma = d\sigma_0 \left\{1 + \left(\frac{\alpha}{\pi}\right)^2 (\delta + \delta^{\gamma}_{\mathrm{hard}})\right\},$$

$$\delta = l_s\left[\frac{8}{3}l_s \ln d - \frac{64}{9}\ln d + \frac{8}{3}\ln d \ln\left(\frac{s}{-u}\right) + \frac{4}{3}\ln^2 D + \right.$$
$$+ \ln D\left(\frac{4}{3}\ln\left(\frac{t}{u}\right) + \frac{2}{3}l_s - \frac{20}{9}\right) + \frac{17}{6}l_s -$$
$$- \frac{4}{3}\mathrm{Li}_2\left(\frac{1-c}{2}\right) + \frac{4}{3}\mathrm{Li}_2\left(\frac{1+c}{2}\right) - \frac{311}{27} + \frac{1}{3}(A_s + A_t - A_u) -$$
$$\left. - \frac{4}{3}\ln d \ln\left(\frac{s}{-t}\right)\left(1 + \frac{s}{t} + \frac{t}{s}\right)^{-2}\left(4\frac{s^2}{t^2} + 7\frac{s}{t} + 5\frac{t}{s} + 9\right) + H(c)\right], \qquad (1.62)$$

where $H(c)$ is a function of the scattering angle (for analytical exression see Subsec. 2.1), the table of values in several points in Table 1 shows us that $H(c)$ is not small. That convinces us in the importance of the nonleading terms. Parameters d and D will cancel in the sum with the contributions due to the emission of a hard photon $\delta^{\gamma}_{\mathrm{hard}}$ and a hard pair [29].

Table 1. **$H(c)$ as a function of c**

c	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
$H(c)$	−13.9	−12.7	−12.2	−12.2	−12.5	−13.0	−13.7	−14.7	−17.1

In this Subsection we calculated the correction within the logarithmic accuracy due to all mechanisms of virtual and soft pair production to the large-angle Bhabha differential cross section. The correction is one of the important contributions, which are to be taken in an analysis of experimental data.

1.3. Hard Pair Production in Large-Angle Bhabha Scattering. In this Subsection we calculate the cross section of hard pair production in large-angle Bhabha scattering in the leading and next-to-leading logarithmic approximations.

Eight regions of the collinear kinematics, when the final particles imitate a process of the $2 \to 2$ type, three semicollinear regions, when the final particles imitate a process of the $2 \to 3$ type, are considered. Analytical formulae for differential cross sections are presented in [15].

Large-angle Bhabha scattering (LABS) process $e^+e^- \to e^+e^-$ is used for mobile luminosity measurements at electron–positron colliders of intermediate energies ($\sqrt{s} \sim 1{-}3$ GeV). The experimental accuracy is estimated to be better than 0.5% [18, 19]. Adequate calculations of the cross section in the framework of the Standard electroweak theory are in general rather poor. We perform systematic analytical calculations of RC to the process at the $\mathcal{O}(\alpha^2)$ level. Due to the complexity of the problem we separate it into several parts. Here we consider the process of the $2 \to 4$ type:

$$e^-(p_1) + e^+(p_2) \to e^-(q_1) + e^+(q_2) + e^-(p_-) + e^+(p_+). \tag{1.63}$$

We assume for definiteness that two final particles $e^-(q_1)$ and $e^+(q_2)$ hit the detectors, allowing the following angular aperture and energy thresholds:

$$\begin{gathered} \Psi_0 < \theta_1, \quad \theta_2 < \pi - \Psi_0, \quad \theta_{1,2} = \widehat{\mathbf{q}_{1,2}\mathbf{p}_1}, \\ y_{\rm th} < y_{1,2} < 1, \quad y_{1,2} = \frac{q^0_{1,2}}{\varepsilon}, \end{gathered} \tag{1.64}$$

where the *dead* angle Ψ_0 depends on the detector ($\Psi_0 \sim 20^\circ$ for DAFNE and $\Psi_0 \sim 35^\circ$ for CMD-1 [18]), $y_{\rm th} \gtrsim 0.1$, ε is the beam energy in the center-of-mass (CM) reference frame of the initial particles.

In paper [15] similar problems were considered for the case of small-angle Bhabha scattering (SABS). We have there at least three simplifications: i) the generalized eikonal form of the amplitude allowed to omit all scattering-type Feynman diagrams with more than one exchanged photons in the t channel; ii) at the $\mathcal{O}(\alpha^2)$ level it was possible to omit all annihilation-type Feynman diagrams and contributions connected with heavy Z, W, and H bosons; iii) the interference of the *emission from the positron line* with *the emission from the electron line* was suppressed for real photon or pair production. Calculations for the LABS case are considerably more complicated. Only the possibility to omit heavy-boson contributions in the $\mathcal{O}(\alpha^2)$ order remains here.

1.3.1. Definitions of Kinematical Regions. There are 36 tree-type Feynman (Fig. 2) diagrams which describe e^+e^- pair production in the LABS process. A lot of attention was paid to this process in the literature [10, 20], where different cross sections were obtained in terms of chiral amplitudes. It was found that in the

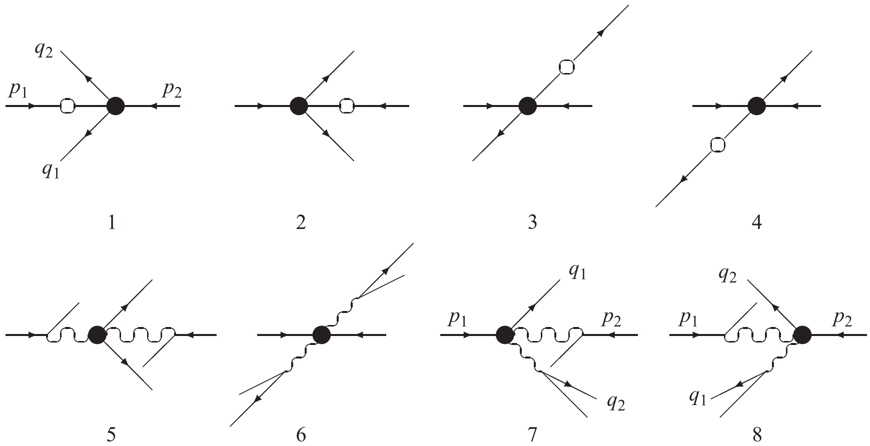

Fig. 2. Kinematical diagrams for collinear pair production

general kinematics the cross section has a rather complicated form. Fortunately in the general case, when the angles between each of two final particles are not small, the correspondent RC contribution to the Born cross section will have the value $(\alpha/\pi)^2 \sim 10^{-5}$):

$$d\sigma^{e\bar{e}\to 2e2\bar{e}} \sim \frac{\alpha^2}{\pi^2} d\sigma_0^{e\bar{e}\to e\bar{e}}. \tag{1.65}$$

It can be safely omitted working within an accuracy of 0.1%. In RC contributions due to pair production some enhancements appear in the cases when one or two final particles move within a small angle $\theta_i \sim m_e/\varepsilon$ to the direction of one of the tagged (initial or final registered) particles. In these cases one will have logarithmical enhanced contributions of the orders $(\alpha L/\pi)^2$ and $(\alpha/\pi)^2 L$, where $L = \ln s/m_e^2$ is the *large logarithm*, $s = 4\varepsilon^2$ ($L \sim 15$ for $\sqrt{s} \sim 1$ GeV). The aim is to extract contributions of that sort because of their importance at the 0.1% accuracy level.

Our method of calculation is to separate the contributions of the collinear and semicollinear kinematical regions. In the collinear kinematics two of the final particles (which are not registered) go within the narrow cone about the direction of one of the initial particles or about the direction of one of the registered final particles:

$$\theta_i < \theta_0, \quad \frac{m_e^2}{\varepsilon^2} \ll \theta_0 \ll 1, \tag{1.66}$$

where θ_i, $i = 1, 2$ are the polar angles of the two particles with respect to the chosen direction. As the semicollinear case we define the kinematics when only one of the nonregistered final particles moves within this cone and the second one does not (with respect to all tagged directions). The contribution of the collinear kinematics has the form

$$a\left(\frac{\alpha}{\pi}(L+\ln\theta_0^2)\right)^2 + b\left(\frac{\alpha}{\pi}\right)^2(L+\ln\theta_0^2), \tag{1.67}$$

while the semicollinear one reads

$$\left(\frac{\alpha}{\pi}\right)^2 f(\theta_0)L, \quad f(\theta_0) = -2a\ln\theta_0^2 + C, \tag{1.68}$$

where C is finite for $\theta_0 \to 0$. The sum of the contributions does not depend on the auxiliary parameter θ_0 within the logarithmic accuracy (we omit the terms $(\alpha/\pi)^2\ln^2\theta_0^2$ and $(\alpha/\pi)^2\ln\theta_0^2$). The cancellation of the dependence provides a test of our calculations.

Consider now the structure of the collinear region contribution to the cross section. It could be presented as a sum of the cross sections of hard subprocesses multiplied by the so-called collinear factors. In the case of the emission of one or two hard collinear photons, the hard subprocess is just the Bhabha scattering. This is the manifestation of the known factorization theorem in the simplest form [15]. In the case of pair production, besides Bhabha scattering there three other types of hard subprocesses appear: Compton scattering, two-quantum annihilation of the initial particles, and the subprocess of the creation of the final registered particles by two photons moving close to the directions of the initial beams. Note that this rather complicated form of the factorization theorem appears for Bhabha scattering first in the process under consideration.

The contributions of semicollinear regions could as well be expressed in terms of hard subprocesses of the $2 \to 3$ type [4]: a single-photon emission in e^+e^- scattering, and the process of pair creation in a photon–electron (–positron) scattering. In Figs. 2 and 3 we show the kinematical schemes for the collinear and semicollinear regions (empty circles denote the production of a collinear undetected pair; the full ones, hard subprocesses).

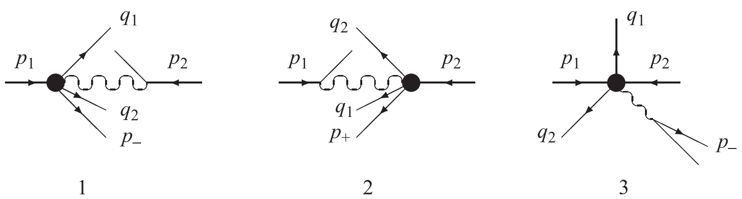

Fig. 3. Kinematical diagrams for semicollinear pair production

Our method, we believe, saves a lot of computation work. Really, instead of 8-fold integration of very complicated expressions with sharp singularities it provides 2(3)-fold integrals of smooth functions within the same accuracy.

1.3.2. Collinear Regions. Consider first the set of the collinear kinematics. We will see that there are eight different cases. As we underlined above, the experimental criterion of an event consists in the kinematics of the final particles with at least one electron and one positron moving at large angles to the beam direction in the opposite hemispheres. In the case of the emission of a particle with momentum k, moving along the direction of its parent particle with momentum p, a small quantity $2pk$ appears in the denominator of the matrix element. Evidently, at least, two such small denominators are necessary to obtain a nonzero contribution integrating over the small phase volume of the two emitted particles in the collinear kinematics ($d\Gamma_2 \sim \theta_0^4$). Our criterion of the Feynman-diagram selection from the total 36 ones (or from 18 gauge-invariant pairs of diagrams) is to choose such gauge-invariant sets which have one diagram with two small denominators.

We verified explicitly the validity of the criterion for the first kinematics considering the full set of 36 diagrams. Note that collinear regions (5–8) (see Fig. 2) are specific for the pair-production process and arise due to the presence of identical particles in the final state.

Calculation of the collinear factors for regions (1–4) (see Fig. 2) was described in detail in papers [21, 22], so here we present only the main points of the derivations. We start with the general form of the cross section in region 1:

$$d\sigma^{(1)}_{\mathrm{coll}} = \frac{\alpha^4}{8\pi^4 s} \sum_{\mathrm{spin}} \left| M^{(1)} \right|^2 \frac{d^3q_1\, d^3q_2}{4q_1^0 q_2^0} \frac{d^3p_-\, d^3p_+}{4p_-^0 p_+^0} \times \\ \times \delta^{(4)}(yp_1 + p_2 - q_1 - q_2), \quad (1.69)$$

$$y = 1 - x_- - x_+, \quad x_\pm = \frac{p_\pm^0}{\varepsilon},$$

where

$$\sum_{\mathrm{spin}} |M^{(1)}|^2 = \frac{4}{y} \frac{I^{(1)}}{m_e^4}\, 16 \left(\frac{s_1}{t_1} + \frac{t_1}{s_1} + 1 \right)^2,$$
$$s_1 = ys = 4y\varepsilon^2, \quad t_1 = yt = -2yy_1\varepsilon^2(1 - c_-), \quad (1.70)$$
$$c_- = \cos \widehat{\mathbf{q}_1 \mathbf{p}_1}, \quad y_{1,2} = \frac{q_{1,2}^0}{\varepsilon},$$

and quantity $I^{(1)}$ is a rather complicated function of $z_\pm = \varepsilon^2 \theta_\pm^2 / m_e^2$ and $x_\pm$, it is given explicitly in [19, 21].

Transforming the phase volume of the created pair into the form

$$\int d\Phi = \int \frac{d^3p_-\, d^3p_+}{4p_-^0 p_+^0} = \frac{\pi^2}{4} m_e^4 \int\limits_0^{2\pi} \frac{d\varphi}{2\pi} \int\limits_0^{z_0} dz_+ \int\limits_0^{z_0} dz_- \int\limits_0^{1-y} dx_+ \int\limits_0^{1-y-x_-} x_- x_+ dx_-, \tag{1.71}$$

$$z_0 = \left(\frac{\varepsilon\theta_0}{m_e}\right)^2 \gg 1$$

and performing all integrations over variables of pair components except its total energy fraction $(1-y)$, one obtains

$$d\sigma_{\text{coll}}^{(1)} = \frac{2\alpha^4}{\pi^2 s} \frac{dy}{y} F^{(1)}(y) \left(\frac{s_1}{t_1} + \frac{t_1}{s_1} + 1\right)^2 \frac{d^3q_1\, d^3q_2}{4q_1^0 q_2^0} \times \\ \times \delta^{(4)}(yp_1 + p_2 - q_1 - q_2). \tag{1.72}$$

The next step is to rewrite the contribution in terms of the scattered-electron observable variables c_- and y_1.

The conservation law gives

$$1 + y = y_1 + y_2, \quad -1 + y = y_1 c_- + y_2 c_+, \\ y_1 \sin\theta_- = y_2 \sin\theta_+, \quad c_+ = \cos\widehat{\mathbf{q}_2\mathbf{p}_1} = \cos\theta_+. \tag{1.73}$$

The final result for the contribution of the first collinear kinematics region reads

$$\frac{d\sigma^{(1)}}{dy_1\, dc_-} = \frac{\alpha^4}{s\pi} \frac{F^{(1)}(y, z_0)}{y} \frac{y_1}{[2 - y_1(1-c_-)]} \left(1 - y_1 \frac{1-c_-}{2} - \frac{2}{y_1(1-c_-)}\right)^2, \tag{1.74}$$

$$y = \frac{y_1(1+c_-)}{2 - y_1(1-c_-)}.$$

The quantity $F^{(1)}(y, z_0)$ could be found in papers [19,21] and it has the following form:

$$F^{(1)}(y, z_0) = L\left(\frac{1}{2}R(y)L + f(y)\right), \quad L = \ln z_0,$$

$$R(y) = \frac{2}{3}\frac{1+y^2}{1-y} + \frac{1-y}{3y}(4 + 7y + 4y^2) + 2(1+y)\ln y,$$

$$f(y) = \frac{1}{9}\left(-107 + 136y - 6y^2 - \frac{12}{y} - \frac{20}{1-y}\right) + \frac{2}{3}\left(-4y^2 - 5y + 1 + \right.$$
$$\left. + \frac{4}{y(1-y)}\right) \ln(1-y) + \frac{1}{3}\left(8y^2 + 5y - 7 - \frac{13}{1-y}\right) \ln y -$$
$$- \frac{2}{1-y} \ln^2 y + 4(1+y) \ln y \ln(1-y) + \frac{2(1-3y^2)}{1-y} \mathrm{Li}_2(1-y). \quad (1.75)$$

Remind the way in which this differential cross section enters into the experimentally observable one:

$$\Delta\sigma^{(1)}_{\mathrm{exp}} = \int\limits_{-c_0}^{c_0} dc_- \int\limits_{y_{\mathrm{th}}}^{1} dy_1 \Theta(c_0^2 - c_+^2)\Theta(y_2 - y_{\mathrm{th}})\Theta(1-y_2)\frac{d\sigma^{(1)}_{\mathrm{coll}}}{dy_1\, dc_-}, \quad (1.76)$$

where

$$y_2 = \frac{1 + (1-y_1)^2 + y_1(2-y_1)c_-}{2 - y_1(1-c_-)},$$
$$c_+ = \frac{-1 + y - y_1 c_-}{y_2}, \quad c_0 = \cos\Psi_0. \quad (1.77)$$

Let us consider as a check that our formula for $d\sigma_c^{(1)}$ agrees with the corresponding contribution to the case of small-angle Bhabha scattering cross section. Really, the correspondence would take place if we took the small-angle limit:

$$c_- = 1 - \frac{\theta_-^2}{2}, \quad \theta_+ = y\theta_-, \quad z = \frac{\theta_+^2}{\theta_1^2}, \quad Q_1^2 = \varepsilon^2\theta_1^2. \quad (1.78)$$

In this way we obtain

$$d\sigma^{(1)} = \frac{\alpha^2}{4\pi^2}\frac{4\pi\alpha^2}{Q_1^2}F^{(1)}(y, z_0)\, dy \frac{dz}{z^2}. \quad (1.79)$$

This formula agrees with Eq. (39) from [15], where two directions were taken into account (we have to note that the expression for $f(y)$ in this paper contains some misprints, they are corrected above).

The third collinear region gives the same contribution:

$$\Delta\sigma^{(3)}_{\mathrm{exp}} = \Delta\sigma^{(1)}_{\mathrm{exp}}. \quad (1.80)$$

Also, the contributions of the collinear regions 2 and 4 are equal:

$$\Delta\sigma^{(2)}_{\rm exp} = \Delta\sigma^{(4)}_{\rm exp}, \tag{1.81}$$

$$\Delta\sigma^{(2)}_{\rm exp} = \int\limits_{-c_0}^{c_0} dc_- \int\limits_{y_{\rm th}}^{1} dy_1 \frac{d\sigma^{(2)}_{\rm coll}}{dy_1\, dc_-}, \quad y_2 = 1, \quad c_+ = -c_-, \quad y_1 = y,$$

$$\frac{d\sigma^{(2)}}{dy_1\, dc_-} = \frac{\alpha^4}{2s\pi} F^{(2)}(y, z_0)\left(1 - \frac{1-c_-}{2} - \frac{2}{1-c_-}\right)^2,$$

$$F^{(2)}(y, z_0) = -yF^{(1)}\left(\frac{1}{y}, z_0 y^2\right) = L\left(\frac{1}{2}R(y)L + 2R(y)\ln y + f_1(y)\right),$$

$$f_1(y) = \frac{1}{9}\left(-116 + 127y + 12y^2 + \frac{6}{y} - \frac{20}{1-y}\right) + \frac{2}{3}\left(-4y^2 - 5y + 1 + \right.$$
$$\left. + \frac{4}{y(1-y)}\right)\ln(1-y) + \frac{1}{3}\left(8y^2 - 10y - 10 + \frac{5}{1-y}\right)\ln y - (1+y)\ln^2 y +$$
$$+ 4(1+y)\ln y \ln(1-y) + \frac{2(3-y^2)}{1-y}\,{\rm Li}_2(1-y).$$

Again one can check the correspondence of this result with the case of SABS (see Eq. (39) in [15]).

We underline that neglecting terms of order α^2/π^2 permits us within the accuracy of 0.1% to express the contribution to $\sigma_{\rm exp}$ in terms of two-fold integrals of smooth functions.

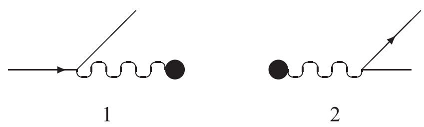

Fig. 4. Diagrams for collinear factors in a space-like kinematics (1) and in a time-like one (2)

Consider now the collinear region (see Fig. 2(5)) in which two of the final particles move close to the directions of the initial beams and the registered pair is created by two almost real photons moving also very close to the initial particle directions. The method of the collinear-factor calculations in this case can be considered as an essential generalization of the Weizsaecker–Williams approximation [6, 7]. Let us consider the block of the kinematical diagram, Fig. 2(5), that describes the emission of an undetected fermion and an almost real photon (both close to the initial direction). The photon enters then into a hard block (see Fig. 4(1)). The corresponding matrix element reads

$$M = \frac{1}{q^2} J_\nu g^{\mu\nu} I_\mu, \quad J_\nu = \bar{u}(p_1')\gamma_\nu u(p_1), \tag{1.82}$$

where I_μ is the current corresponding to the hard block. Let us expand, following V. Sudakov [23], the 4-momentum of the emitted fermion:

$$p_1' = \alpha\tilde{p}_2 + \beta\tilde{p}_1 + p_{1\perp}', \quad p_{1\perp}'p_1 = p_{1\perp}'p_2 = 0,$$
$$\tilde{p}_{1,2} = p_{1,2} - p_{2,1}\frac{m^2}{s}, \quad s = 2p_1p_2 \gg m^2. \tag{1.83}$$

The 4-momentum $\tilde{p}_{1,2}$ is almost light-like. The parameter β here is the quantity of an order of unity. It has the meaning of energy fraction of the scattered electron; $1-\beta$ is the energy fraction of our almost real photon; $p_{1\perp}'$ is the two-dimensional vector describing the components of the scattered electron momentum transverse with respect to the initial direction (and further we denote transverse momentum components by symbol «$\perp$»). Parameter $\alpha = ((\mathbf{p}_1)^2 + m^2)/(s\beta)$ is small: $\alpha \ll 1$. It could be found from the mass shell condition for the scattered electron: ${p_1'}^2 = m^2$. In that way we obtain also the useful equation

$$q^2 = -\frac{(\mathbf{p}_1')^2 + m^2(1-\beta)^2}{\beta} < 0. \tag{1.84}$$

Representing identically the metric tensor, entering into the photon Green function, in the form

$$g^{\mu\nu} = g_\perp^{\mu\nu} + \frac{2}{s}(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu), \tag{1.85}$$

we note it could be effectively written as

$$g^{\mu\nu} \approx g_\perp^{\mu\nu} + \frac{2}{s}\,p_1^\mu p_2^\nu, \tag{1.86}$$

since the contribution of the omitted term is suppressed by an additional factor of order q^2/s. Taking that into account, one obtains

$$M = \frac{1}{q^2}\left\{(JI)_\perp + \frac{2}{s}(Jp_2)\left(-\frac{Ip_{1\perp}'}{1-\beta}\right)\right\}, \tag{1.87}$$

where the current conservation condition

$$Iq = I(\alpha_1 p_2 + (1-\beta)p_1 + q_\perp) \approx I((1-\beta)p_1 + q_\perp) = 0 \tag{1.88}$$

was used. Now we sum up over fermion spin states:

$$\sum_{\text{spin}} |(JI)_\perp|^2 = \mathrm{Tr}\,(\hat{p}_1' + m)I_\perp(\hat{p}_1 + m)I_\perp = -2q^2 I_\perp^2 > 0,$$
$$\sum_{\text{spin}} |Jp_2|^2 = 2s^2\beta, \tag{1.89}$$
$$\sum_{\text{spin}} (Jp_2)(JI)_\perp^* = 2s(q_\perp I_\perp), \quad q_\perp = -p_{1\perp}'.$$

And we obtain

$$\sum_{\rm spin} |M|^2 = \frac{1}{(q^2)^2}\left[-2q^2 I_\perp^2 + \frac{8}{(1-\beta)^2}(p'_{1\perp} I_\perp)^2\right], \qquad (1.90)$$

where q^2 are to be taken from Eq. (1.84). The phase volume of the scattered electron can be presented in the form

$$\int \frac{d^3 p'_1}{2\varepsilon'_1} = \int \frac{d\beta}{2\beta} \int \frac{d\varphi}{2\pi}\, 2\pi \int\limits_0^{(\varepsilon\beta\theta_0)^2} \frac{d(\mathbf{p}'_1)^2}{2}. \qquad (1.91)$$

Then we carry out a simple integration and obtain

$$\int \sum_{\rm spin} |M|^2 \frac{d^3 p'_1}{2\varepsilon'_1} = \pi (I_\perp)^2 Q(\beta, z_0)\, d\beta, \qquad (1.92)$$

where the collinear factor $Q(\beta, z_0)$ for a space-like virtual photon reads

$$Q(\beta, z_0) = \frac{1+\beta^2}{1-\beta}\left[L + 2\ln\frac{\beta}{1-\beta}\right] - \frac{2\beta}{(1-\beta)^2}. \qquad (1.93)$$

Now we are ready to calculate the cross section in the collinear region (see Fig. 2 (5)), where we have two collinear factors $Q(\beta, z_0)$. We need also the matrix element squared of the hard block describing hard e^+e^- pair creation by two photons:

$$\gamma((1-\beta_1)p_1) + \gamma((1-\beta_2)p_2) \to e_+(q_2) + e_-(q_1). \qquad (1.94)$$

Taking the phase volume in terms of the detected electron as follows:

$$d\beta_2 \frac{d^3 q_1\, d^3 q_2}{2q_1^0 2q_2^0} \delta^{(4)}(q_1 + q_2 - p_1(1-\beta_1) - p_2(1-\beta_2)) = \\ = \frac{(\pi/2) y_1\, dy_1\, dc_-}{2\beta_1 - y_1(1+c_-)}, \qquad (1.95)$$

we obtain for the cross section

$$\frac{d\sigma^{(5)}_{\rm coll}}{dy_1\, dc_-} = \frac{\alpha^4}{\pi s} \int\limits_0^1 \frac{d\beta_1 y_2 (1 - c_- c_+)}{\beta_1^2 \beta_2^2 (2\beta_1 - y_1(1+c_-)) y_1 (1 - c_-^2)} \times \\ \times \left\{(1 + (1-\beta_1)^2)\left(L + 2\ln\frac{1-\beta_1}{\beta_1}\right) - 2(1-\beta_1)\right\} \times \\ \times \left\{(1 + (1-\beta_2)^2)\left(L + 2\ln\frac{1-\beta_2}{\beta_2}\right) - 2(1-\beta_2)\right\}, \qquad (1.96)$$

where

$$2\beta_1 - y_1(1+c_-) > 0, \quad \beta_2 = \frac{y_1\beta_1(1-c_-)}{2\beta_1 - y_1(1+c_-)},$$
$$y_2 = \frac{2\beta_1^2 + y_1(y_1 - 2\beta_1)(1+c_-)}{2\beta_1 - y_1(1+c_-)}, \quad c_+ = \frac{1}{y_2}(\beta_1 - \beta_2 - y_1 c_-). \tag{1.97}$$

For the hard block we used the following expression:

$$\sum_{\text{spin}} |M^{\gamma\gamma\to e^+e^-}|^2 \sim \frac{t_1}{u_1} + \frac{u_1}{t_1} = \frac{y_1(1-c_-)}{y_2(1-c_+)} + \frac{y_2(1-c_+)}{y_1(1-c_-)} = \\ = \frac{2y_2(1-c_+c_-)}{y_1(1-c_-^2)}. \tag{1.98}$$

And for the contribution to the experimental cross section we get

$$\Delta\sigma^{(5)} = \int\limits_{y_{\text{th}}}^{1} dy_1 \int\limits_{-c_0}^{c_0} dc_- \frac{d\sigma^{(5)}_{\text{coll}}}{dy_1\, dc_-}\, \Theta(y_2 - y_{\text{th}})\, \Theta(1-y_2)\, \Theta(c_0^2 - c_+^2). \tag{1.99}$$

A similar situation takes place for the collinear kinematics (see Fig. 2(6)), when the initial electron and positron annihilate into two almost real photons, that convert then into two electron–positron pairs.

The matrix element describing the emission of a time-like almost real photon with its subsequent conversion into a pair (see Fig. 3(2)) has the form

$$M = \frac{g^{\mu\nu}}{k^2} I_\mu J_\nu, \quad J_\nu = \bar{v}(p_-)\gamma_\nu u(q_+). \tag{1.100}$$

We use again the Sudakov representation for the momenta of the pair components and the photon:

$$g^{\mu\nu} \approx g_\perp^{\mu\nu} + \frac{2}{s_1} q^\nu q_+^\mu, \quad q^2 = 0, \quad 2qq_+ = s_1,$$
$$p_- = \alpha_1 q + \beta_1 \tilde{q}_+ + (p_-)_\perp, \quad \tilde{q}_+ = q_+ - q\frac{m^2}{s_1}, \tag{1.101}$$
$$k = q_+ + p_- = \alpha_2 q + \beta_2 \tilde{q}_+ + k_\perp, \quad \beta_1 = \beta_2 - 1 > 0.$$

The current conservation condition here reads

$$kI \approx (\beta_1 \tilde{q}_+ + p_-^\perp) I = 0. \tag{1.102}$$

Using the above definitions we get the matrix element squared summed over spin states in the following form:

$$\sum_{\text{spin}} |M|^2 = 2\frac{(I^\perp)^2}{(k^2)^2}\,\frac{[(1+(\beta_2-1)^2)(k^\perp)^2 + m^2\beta_2^4]}{\beta_2^2(\beta_2-1)},$$
$$k^2 = \frac{(k^\perp)^2 + m^2\beta_2^2}{\beta_2 - 1} > 0. \qquad (1.103)$$

Integrating over the transverse momentum components $(p_-)_\perp$ of the electron from the created pair, we obtain

$$\int \frac{d^2p_-^\perp}{2p_-^0}\sum_{\text{spin}}|M|^2 = \frac{\pi(I_\perp)^2 d\beta_2}{\beta_2^2}\times$$
$$\times\left\{(1+(\beta_2-1)^2)\left(L + 2\ln\left(y_2\left(1-\frac{1}{\beta_2}\right)\right)\right) + 2(\beta_2-1)\right\}. \qquad (1.104)$$

Note that due to the character of the hard $e^+e^- \to \gamma\gamma$ block we have $k_1^0 = k_2^0 = \varepsilon$, and the relation between the detected positron energy fraction $y_2 = q_+^0/\varepsilon$ and parameter β_2 reads:

$$\beta_2 = 1/y_2. \qquad (1.105)$$

The cross section for the collinear region 6 takes the form

$$\frac{d\sigma_{\text{coll}}^{(6)}}{dy_1\, dc_-\, dy_2} =$$
$$= \frac{\alpha^4}{4\pi s}\,\frac{1+c_-^2}{1-c_-}\big\{(y_1^2 + (1-y_1)^2)\big(L + 2\ln\big(y_1(1-y_1)\big)\big) + 2y_1(1-y_1)\big\}\times$$
$$\times\big\{(y_2^2 + (1-y_2)^2)\big(L + 2\ln\big(y_2(1-y_2)\big)\big) + 2y_2(1-y_2)\big\}. \qquad (1.106)$$

The corresponding contribution to the experimentally observable cross section has the following form:

$$\Delta\sigma^{(6)} = N\int\limits_{y_{\text{th}}}^{1} dy_1 \int\limits_{y_{\text{th}}}^{1} dy_2 \int\limits_{-c_0}^{c_0} dc_-\,\frac{d\sigma_{\text{coll}}^{(6)}}{dy_1\, dc_-\, dy_2}, \quad c_+ = -c_-. \qquad (1.107)$$

Quantity N depends on the concrete experimental set-up. Namely, $N = 1/2$ when one requires registration of two leptons with opposite charges going back to back. In a charge-blind set-up one would have $N = 1$.

Consider now two remaining collinear regions (Fig. 2 (7, 8)). They contain, as a hard block, the Compton scattering amplitude. Combining the expressions for the collinear factors for time-like and space-like photons one obtains

$$\Delta\sigma^{(7)} = \Delta\sigma^{(8)} = \int\limits_{y_{\rm th}}^{1} dy_1 \int\limits_{-c_0}^{c_0} dc_- \, \frac{\sigma^{(8)}}{dy_1 dc_-} \Theta(1-y_2)\Theta(y_2-y_{\rm th})\Theta(c_0^2-c_+^2),$$

$$\frac{\sigma^{(8)}}{dy_1\, dc_-} = \frac{\alpha^4}{2\pi s} \int\limits_{\beta_{1\,\min}}^{1} \frac{d\beta_1}{\beta_1^2\beta_2^2(1+\beta_1+c_-(1-\beta_1))} \times$$

$$\times \left(\frac{y_2(1-c_+)}{2} + \frac{2}{y_2(1-c_+)}\right)\left\{(1+(1-\beta_1)^2)\left(L+2\ln\frac{1-\beta_1}{\beta_1}\right)-2(1-\beta_1)\right\}\times$$

$$\times \left\{(1+(\beta_2-1)^2)\left(L+2\ln\left(y_2\frac{(\beta_2-1)}{\beta_2}\right)\right)+2(\beta_2-1)\right\}, \quad (1.108)$$

$$\beta_2 = \frac{2\beta_1}{y_1(1+\beta_1+(1-\beta_1)c_-)}, \quad y_2 = \frac{1+\beta_1^2+c_-(1-\beta_1^2)}{1+\beta_1+c_-(1-\beta_1)},$$

$$c_+ = \frac{1}{y_2}[\beta_1 - 1 - \beta_2 y_1 c_-], \quad \beta_{1\,\min} = \frac{y_1(1+c_-)}{2-y_1(1-c_-)}.$$

1.3.3. Semicollinear Regions. The differential cross section of the pair production process in large-angle Bhabha scattering (see Fig. 2) has the following form:

$$\Delta\sigma_{\rm s-coll} = 2\frac{\alpha}{2\pi}L \int\limits_{0}^{1-\beta_0} \frac{d\beta(1+\beta^2)}{1-\beta}\Sigma_1(\Omega_1,\Omega_2,\Omega_+,\theta_0)\times$$

$$\times\, d\sigma(\gamma(p_1(1-\beta)) + e^+(p_2) \to e^+(p_+) + e^+(q_2) + e^-(q_1)) +$$

$$+ \frac{\alpha}{2\pi}L \int\limits_{1}^{1/y} \frac{d\beta}{\beta^2}(1+(\beta-1)^2)\Sigma_2(\Omega_1,\Omega_2,\theta_0)\times$$

$$\times\, d\sigma(e^-(p_1) + e^+(p_2) \to e^-(q_1) + e^+(q_2) + \gamma(\beta p_+)), \quad (1.109)$$

where we used the collinear factors considered above within the logarithmic accuracy, $y = p_+^0/\varepsilon$, and the hard subprocess cross sections [15, 20] are

$$d\sigma^{\gamma(q)+e^+(p_2)\to e^+(p_+)+e^+(q_2)+e^-(q_1)} =$$

$$= \frac{(4\pi\alpha)^3}{16(2\pi)^5 2(qp_2)} \frac{1}{(p_2p_+)(p_2q_2)(q_1q_2)(p_+q_1)}\big((p_2q_2)(q_1p_+)((p_2q_2)^2+(q_1p_+)^2)+$$

$$+ (p_2p_+)(q_1q_2)((q_1q_2)^2 + (p_2p_+)^2) + (p_2q_1)(q_2p_+)((p_2q_1)^2 + (q_2p_+)^2)\big)\times$$
$$\times\left(\frac{2p_2p_+}{(p_2q)(p_+q)} + \frac{2p_2q_2}{(p_2q)(q_2q)} + \frac{2q_1p_+}{(q_1q)(p_+q)} + \frac{2q_1q_2}{(q_1q)(q_2q)} - \frac{2p_2q_1}{(p_2q)(q_1q)} - \right.$$
$$\left. - \frac{2p_+q_2}{(p_+q)(q_2q)}\right)\delta^{(4)}(q + p_2 - p_+ - q_1 - q_2)\frac{d^3q_1\, d^3q_2\, d^3p_+}{q_1^0q_2^0p_+^0}, \quad (1.110)$$

$$d\sigma^{e^-(p_1)+e^+(p_2)\to e^-(q_1)+e^+(q_2)+\gamma(q)} =$$
$$= \frac{(4\pi\alpha)^3}{16(2\pi)^5 2(p_1p_2)}\frac{1}{(p_1p_2)(q_1q_2)(p_1q_1)(p_2q_2)}\big((p_1p_2)(q_1q_2)((p_1p_2)^2 + (q_1q_2)^2) +$$
$$+ (p_1q_1)(p_2q_2)((p_1q_1)^2 + (p_2q_2)^2) + (p_1q_2)(p_2q_1)((p_1q_2)^2 + (p_2q_1)^2)\big)\times$$
$$\times\left(\frac{2p_1p_2}{(p_1q)(p_2q)} + \frac{2q_1q_2}{(q_1q)(q_2q)} + \frac{2p_1q_1}{(p_1q)(q_1q)} + \frac{2p_2q_2}{(p_2q)(q_2q)} - \frac{2p_1q_2}{(p_1q)(q_2q)} - \right.$$
$$\left. - \frac{2p_2q_1}{(p_2q)(q_1q)}\right)\delta^{(4)}(p_1 + p_2 - q - q_1 - q_2)\frac{d^3q_1\, d^3q_2\, d^3q}{q_1^0q_2^0q^0}. \quad (1.111)$$

The quantity $1 - \beta_0$ in Eq. (1.109) is the minimum energy fraction of the virtual photon in the pair creation process $\gamma^*\bar{e} \to e\bar{e}\bar{e}$ provided that fermions with momenta q_1 and q_2 are to be detected. Multipliers Σ_1 and Σ_2 provide the emission angles of every final state of a fermion with respect to the beam directions; also with respect to each other in order the angles to be larger than θ_0. Note that because of the integration over the phase space of the final particles, the identity of two positrons is taken into account automatically. The numerical integration of $\Delta\sigma_{\text{s-coll}}$ (1.109) and different contributions to $\Delta\sigma_{\text{coll}}$ (see Eqs. (1.76), (1.81), (1.99), (1.107), (1.108)) will show that the total sum does not depend on the auxiliary parameter θ_0.

1.3.4. Renormalization Group Approach. In the leading logarithmic approximation, i.e., for the terms of order $(\alpha L)^2$, the parton picture of the cross section is valid: that could be just seen from the above expressions for different collinear kinematics.

Radiative corrections to the considered process, i.e., terms of order $(\alpha L)^3$ could be obtained using the renormalization group methods. But their contribution is beyond the required accuracy.

1.4. Large-Angle Radiative Bhabha Scattering. Let us consider the calculation of RC to a single hard-photon emission process [25]. We consider the kinematics essentially of type $2 \to 3$, in which all possible scalar products of 4-momenta of external particles are large compared to the electron mass squared.

Considering virtual corrections, we identify several gauge-invariant sets of FD. Loop corrections associated with emission and absorption of virtual photons by the same fermion line are called as Glass-type (G) corrections. The case in which a loop involves exchange of two virtual photons between different fermion lines is called Box-type (B) FD. The third class includes the vertex function and vacuum polarization contributions ($\Gamma\Pi$-type). We see explicitly that all terms that contain the square of large logarithms $\ln(s/m^2)$, as well as those that contain the infrared singularity parameter (fictitious photon mass λ), are cancelled out in the total sum, where the emission of an additional soft photon is also considered.

We note here that the part of the general result associated with scattering-type diagrams (see Fig. 5(1, 5)) was used to describe radiative DIS with RC taken into account in [16]. A similar set of FD can be used to describe the annihilation channel [25].

The problem of virtual RC calculations at the one-loop level is cumbersome for the process

$$e^+(p_2) + e^-(p_1) \longrightarrow e^+(p_2') + e^-(p_1') + \gamma(k_1). \qquad (1.112)$$

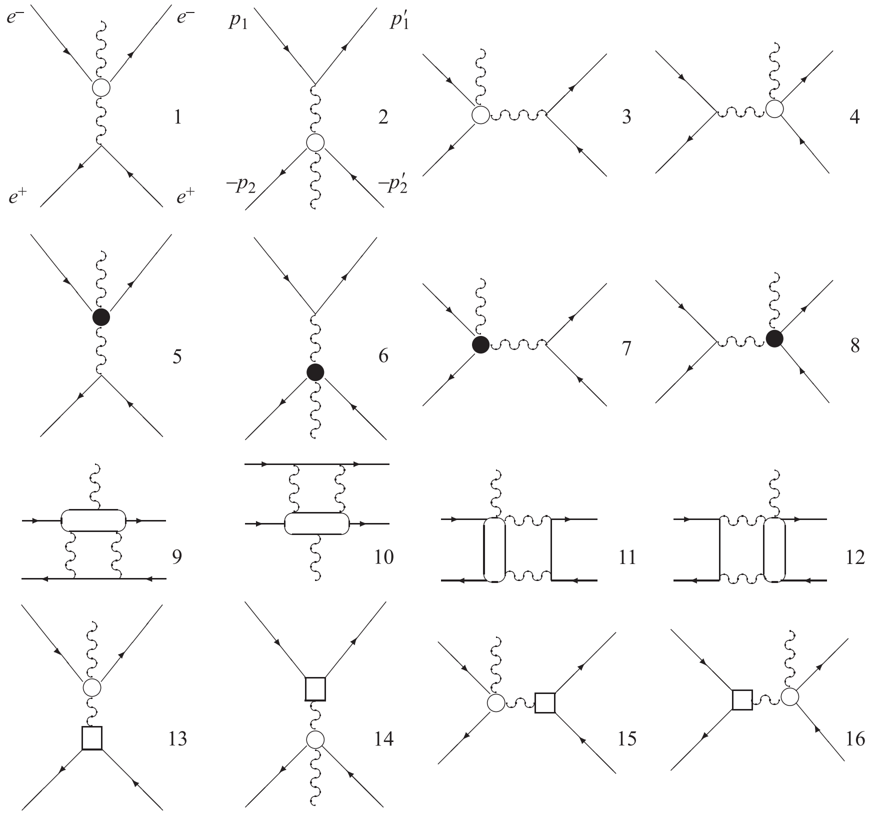

Fig. 5. G- and B-type Feynman diagrams for radiative Bhabha scattering. For designations see Fig. 6

Specifically, if at the Born level we need to consider eight FD, then at the one-loop level we have as many as 72. Furthermore, performing loop momentum integration, we introduce scalar, vector, and tensor integrals up to the third rank with 2, 3, 4, and 5 denominators (a set of relevant integrals is given in Subsec. 2.3). A high degree of topological symmetry of FD for a cross section can be exploited to calculate the matrix element squared. Using them, we can restrict ourselves to the consideration of interferences of the Born-level amplitudes (Fig. 5(1–4)) with those that contain one-loop integrals (Fig. 5(5–16)). Our calculation is simplified since we omit the electron mass m in evaluating the corresponding traces due to the kinematic region under consideration:

$$\begin{gathered} s \sim s_1 \sim -t_1 \sim -t \sim -u \sim -u_1 \sim \chi_{1,2} \sim \chi'_{1,2} \gg m^2, \\ s = 2p_1p_2, \quad t = -2p_2p'_2, \quad u = -2p_1p'_2, \quad s_1 = 2p'_1p'_2, \\ t_1 = -2p_1p'_1, \quad u_1 = -2p_2p'_1, \quad \chi_{1,2} = 2k_1p_{1,2}, \quad \chi'_{1,2} = 2k_1p'_{1,2}, \\ s + s_1 + t + t_1 + u + u_1 = 0, \quad s + t + u = \chi'_1, \\ s_1 + t + u_1 = -\chi_1, \quad t + \chi_1 = t_1 + \chi'_1. \end{gathered} \tag{1.113}$$

We found that some kind of local factorization took place both for the G- and B-type FD: the leading logarithmic contribution to the matrix element squared, summed over spin states, arising from interference of one of the four FD at the Born level (Fig. 5(1–4)) with some one-loop corrected FD (Fig. 5(5–16)), turns out to be proportional to the interference of the corresponding amplitudes at the Born level. The latter has the form

$$\begin{aligned} E_0 &= (4\pi\alpha)^{-3} \sum |M_1|^2 = -\frac{16}{t^2}\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}'_1 O_{11'} \hat{p}_1 \tilde{O}_{11'})\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_2\gamma_\sigma \hat{p}'_2 \gamma_\rho) = \\ &= -\frac{16}{t\chi_1\chi'_1}(u^2 + u_1^2 + s^2 + s_1^2), \\ O_0 &= (4\pi\alpha)^{-3} \sum M_1 M_2^* = \frac{8}{tt_1}\left(\frac{s}{\chi_1\chi_2} + \frac{s_1}{\chi'_1\chi'_2} + \frac{u}{\chi_1\chi'_2} + \frac{u_1}{\chi_2\chi'_1}\right)\times \\ &\times (u^2 + u_1^2 + s^2 + s_1^2), \end{aligned} \tag{1.114}$$

$$\begin{aligned} I_0 &= (4\pi\alpha)^{-3} \sum M_1(M_3^* + M_4^*) = -(1+\hat{Z})\frac{4}{ts_1}\left\{-\frac{4u_1\chi'_2}{\chi_1} + \right. \\ &+ \frac{4u(s_1+t_1)(s+t)}{\chi_2\chi'_1} - \frac{2}{\chi_1\chi_2}[2suu_1 + (u+u_1)(uu_1 + ss_1 - tt_1)] + \\ &\left. + \frac{2}{\chi_1\chi'_1}[2t_1uu_1 + (u+u_1)(uu_1 + tt_1 - ss_1)]\right\}, \\ O_{11'} &= \gamma_\rho \frac{\hat{p}'_1 + \hat{k}_1}{\chi'_1}\gamma_\mu - \gamma_\mu \frac{\hat{p}_1 - \hat{k}_1}{\chi_1}\gamma_\rho, \quad \tilde{O}_{11'} = O_{11'}(\rho \leftrightarrow \mu), \end{aligned} \tag{1.115}$$

where the $\hat{Z}$ operator acts as follows:

$$\hat{Z} = \begin{vmatrix} p_1 \longleftrightarrow p_1' & s \longleftrightarrow s_1 \\ p_2 \longleftrightarrow p_2' & u \longleftrightarrow u_1 \\ k_1 \rightarrow -k_1 & t, t_1 \rightarrow t, t_1 \end{vmatrix}.$$

It can be shown that the total matrix element squared, summed over spin states, can be obtained using symmetry properties realized by means of the permutation operations:

$$\sum |M|^2 = (4\pi\alpha)^3 F,$$

$$F = (1 + \hat{P} + \hat{Q} + \hat{R})\Phi = 16\,\frac{ss_1(s^2+s_1^2) + tt_1(t^2+t_1^2) + uu_1(u^2+u_1^2)}{ss_1tt_1} \times$$
$$\times \left(\frac{s}{\chi_1\chi_2} + \frac{s_1}{\chi_1'\chi_2'} - \frac{t}{\chi_2\chi_2'} - \frac{t_1}{\chi_1\chi_1'} + \frac{u}{\chi_1\chi_2'} + \frac{u_1}{\chi_2\chi_1'}\right),$$

$$\Phi = E_0 + O_0 - I_0. \quad (1.116)$$

The explicit form of the $\hat{P}, \hat{Q}, \hat{R}$ operators is

$$\hat{P} = \begin{vmatrix} p_1 \longleftrightarrow -p_2' & s \longleftrightarrow s_1 \\ p_2 \longleftrightarrow -p_1' & t \longleftrightarrow t_1 \\ k_1 \rightarrow k_1 & u, u_1 \rightarrow u, u_1 \end{vmatrix},$$
$$\hat{Q} = \begin{vmatrix} p_2 \longleftrightarrow -p_1' & s \longleftrightarrow t_1 \\ p_2' \rightarrow p_2' & s_1 \longleftrightarrow t \\ p_1, k_1 \rightarrow p_1, k_1 & u, u_1 \rightarrow u, u_1 \end{vmatrix}, \quad (1.117)$$
$$\hat{R} = \begin{vmatrix} p_1 \longleftrightarrow -p_2' & s \longleftrightarrow t \\ p_1' \rightarrow p_1' & s_1 \longleftrightarrow t_1 \\ p_2, k_1 \rightarrow p_2, k_1 & u, u_1 \rightarrow u, u_1 \end{vmatrix}.$$

The differential cross section at the Born level in the case of large-angle kinematics (1.113) was found in [4, 5]:

$$d\sigma_0(p_1, p_2) = \frac{\alpha^3}{32s\pi^2} F \frac{d^3p_1'\, d^3p_2'\, d^3k_1}{\varepsilon_1'\varepsilon_2'\omega_1} \delta^{(4)}(p_1 + p_2 - p_1' - p_2' - k_1), \quad (1.118)$$

where ε_1, ε_2, and ω_1 are the energies of the outgoing fermions and photon, respectively. The collinear kinematic regions (real photon emitted in the direction of one of the charged particles) corresponding to the case in which one of the invariants χ_i, χ_i' is of order m^2 yields the main contribution to the total cross section. These require separate investigation, and will be considered elsewhere.

Here we consider the cross section in the kinematic region (1.113), in principle, with the power-law accuracy, i.e., neglecting terms that are

$$\mathcal{O}\left(\frac{\alpha}{\pi}\frac{m^2}{s}L_s^2\right), \tag{1.119}$$

as compared to $\mathcal{O}(1)$ terms calculated in this Section. Note that the terms in (1.119) are less than 10^{-4} for typical moderately high energy colliders (DAΦNE, VEPP-2M, BEPS). Unfortunately, the nonleading terms are too complicated to be presented analytically, so we have estimated them numerically in Table 2.

1.4.1. Contribution of G-Type Diagrams. The set of FD (Fig. 5 (5–8)) we call *glasses* here (G-type diagrams). Using crossing symmetry, we can construct the whole G-type contribution from the gauge-invariant set of FD in Fig. 5(5). Moreover, only the set of FD depicted in Fig. 6, *d* can be considered in practical calculations, due to an additional mirror symmetry in the diagrams of Fig. 6, *d*, *e*. We therefore start by checking the gauge invariance of the Compton tensor described by the FD of Fig. 6, *d*, *e* for all fermions and one of the photons:

$$\bar{u}(p_1')R^{\sigma\mu}_{1,1'}u(p_1). \tag{1.120}$$

This was done indirectly in [8], where the Compton tensor for a heavy photon was written in terms of explicitly gauge-invariant tensor structures. We use the expression

$$R^{\sigma\mu}_{1,1'} = R^{\chi_1} + R^{\chi_1'}, \tag{1.121}$$

$$R^{\chi_1} = A_2\gamma_\sigma\hat{k}_1\gamma_\mu + \int\frac{d^4k}{i\pi^2}\left\{\frac{\gamma_\lambda(\hat{p}_1'-\hat{k})\gamma_\sigma(\hat{p}_1-\hat{k}_1-\hat{k})\gamma_\lambda(\hat{p}_1-\hat{k}_1)\gamma_\mu}{-\chi_1(0)(2)(q)}+\right.$$
$$\left.+\frac{\gamma_\lambda(\hat{p}_1'-\hat{k})\gamma_\sigma(\hat{p}_1-\hat{k}_1-\hat{k})\gamma_\mu(\hat{p}_1-\hat{k})\gamma_\lambda}{(0)(1)(2)(q)}\right\}, \tag{1.122}$$

where

$$(0) = k^2-\lambda^2, \quad (2) = (p_1'-k)^2-m^2, \quad (1) = (p_1-k)^2-m^2,$$
$$(q) = (p_1-k_1-k)^2-m^2, \quad A_2 = \frac{2}{\chi_1}\left(L_{\chi_1}-\frac{1}{2}\right), \quad L_{\chi_1} = \ln\frac{\chi_1}{m^2}. \tag{1.123}$$

The quantity R^{χ_1} corresponds to the FD depicted in Fig. 6, *d*, while $R^{\chi_1'}$ corresponds to the FD in Fig. 6, *e*. The first term on the right-hand side of Eq. (1.122) corresponds to the first two FD of Fig. 6, *d*. The gauge invariance condition $R^{\sigma\mu}_{1,1'}k_\mu = 0$ is clearly satisfied. The gauge invariance condition regarding the heavy photon Lorentz index provides some check of the loop momentum

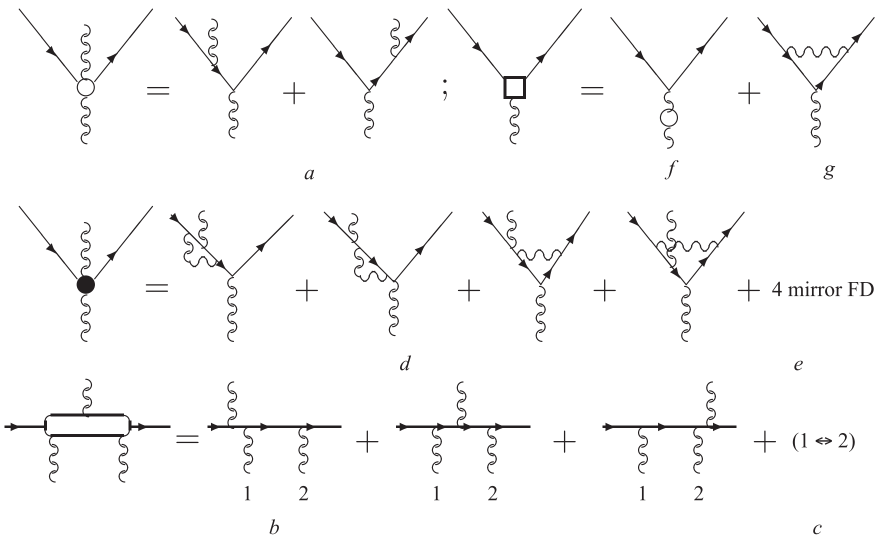

Fig. 6. Content of the notation for Fig. 5

integrals, which can be found in Subsubsec. 2.3.1:

$$\bar{u}(p_1')R^{\sigma\mu}_{1,1'}u(p_1)q_\sigma e_\mu(k_1) = Ak_1^\mu e_\mu(k_1),\quad A = -2\frac{L_{\chi_1}-2}{\chi_1} - 6\frac{L_{\chi_1'}-1}{\chi_1'}. \tag{1.124}$$

The gauge-invariance thus satisfied due to the Lorentz condition for the on-shell photon, $e(k_1)k_1 = 0$. As stated above, the use of crossing symmetries of amplitudes permits us to consider only R^{χ_1}. For interference of amplitudes at the Born level (see Fig. 5 (1–4) and Fig. 5 (5–8)), we obtain in terms of the replacement operators

$$(\Delta|M|^2)_G = = 2^5\alpha^4\pi^2(1+\hat{P}+\hat{Q}+\hat{R})(1+\hat{Z})[E_{15}^{\chi_1} + O_{25}^{\chi_1} - I_{35}^{\chi_1} - I_{45}^{\chi_1}], \tag{1.125}$$

with

$$E_{15}^{\chi_1} = \frac{16}{t^2}\,\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_1' R^{\chi_1}\hat{p}_1 O_{11'})\cdot\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_2\gamma_\rho\hat{p}_2'\gamma_\sigma),$$

$$O_{25}^{\chi_1} = \frac{16}{tt_1}\,\frac{1}{4}\,\mathrm{Tr}(\hat{p}_1' R^{\chi_1}\hat{p}_1\gamma_\rho)\cdot\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_2\gamma_\sigma\hat{p}_2' O_{22'}),$$

$$I_{35}^{\chi_1} = \frac{4}{ts_1}\,\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_1' R^{\chi_1}\hat{p}_1 O_{12}\hat{p}_2\gamma_\sigma\hat{p}_2'\gamma_\rho),$$

$$I_{45}^{\chi_1} = \frac{4}{ts}\,\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_1' R^{\chi_1}\hat{p}_1\gamma_\rho\hat{p}_2\gamma_\sigma\hat{p}_2' O_{1'2'}),$$

$$
\begin{aligned}
O_{11'} &= \gamma_\rho \frac{\hat{p}_1' + \hat{k}_1}{\chi_1'} \gamma_\mu - \gamma_\mu \frac{\hat{p}_1 - \hat{k}_1}{\chi_1} \gamma_\rho, \\
O_{22'} &= \gamma_\mu \frac{-\hat{p}_2' - \hat{k}_1}{\chi_2'} \gamma_\rho - \gamma_\rho \frac{-\hat{p}_2 + \hat{k}_1}{\chi_2} \gamma_\mu, \\
O_{12} &= -\gamma_\mu \frac{\hat{p}_1 - \hat{k}_1}{\chi_1} \gamma_\rho - \gamma_\rho \frac{-\hat{p}_2 + \hat{k}_1}{\chi_2} \gamma_\mu, \\
O_{1'2'} &= \gamma_\rho \frac{\hat{p}_1' + \hat{k}_1}{\chi_1'} \gamma_\mu + \gamma_\mu \frac{-\hat{p}_2' - \hat{k}_1}{\chi_2'} \gamma_\rho.
\end{aligned}
\tag{1.126}
$$

In the logarithmic approximation, the G-type amplitude contribution to the cross section has the form

$$
\begin{aligned}
d\sigma_G &= \frac{d\sigma_0}{F} \frac{\alpha}{\pi} (1 + \hat{P} + \hat{Q} + \hat{R}) \Phi \left[-\frac{1}{2} L_{t_1}^2 + \frac{3}{2} L_{t_1} + 2 L_{t_1} \ln \frac{\lambda}{m} \right], \\
L_{t_1} &= \ln \frac{-t_1}{m^2}.
\end{aligned}
\tag{1.127}
$$

1.4.2. Vacuum Polarization and Vertex Insertion Contributions. Let us examine a set of $\Gamma\Pi$-type FD. The contribution of the Dirac form factor of fermions and vacuum polarization can be parameterized as $(1 + \Gamma_t)/(1 - \Pi_t)$, while the contribution of the Pauli form factor is proportional to the fermion mass, and is omitted here. We obtain

$$
d\sigma_{\Gamma\Pi} = \frac{d\sigma_0}{F} \frac{\alpha}{\pi} 2 (1 + \hat{P} + \hat{Q} + \hat{R}) (\Gamma_t + \Pi_t) \Phi,
\tag{1.128}
$$

where

$$
\begin{aligned}
\Gamma_t &= \frac{\alpha}{\pi} \left\{ \left(\ln \frac{m}{\lambda} - 1 \right) (1 - L_t) - \frac{1}{4} L_t - \frac{1}{4} L_t^2 + \frac{1}{2} \zeta_2 \right\}, \\
\Pi_t &= \frac{\alpha}{\pi} \left(\frac{1}{3} L_t - \frac{5}{9} \right), \quad L_t = \ln \frac{-t}{m^2}.
\end{aligned}
\tag{1.129}
$$

In realistic calculations, the vacuum polarization due to hadrons and muons can be taken into account in a very simple fashion, just by adding it to Π_t.

1.4.3. Contribution of the B-Type Set of Feynman Diagrams. The contribution of FD with virtual two-photon exchange, shown in Fig. 5(9–12) are called *boxes* here (B-type diagrams). Again, using the crossing symmetry of FD, we can use only the FD of Fig. 5(9) in calculations.

A procedure resembling the one used in the Subsubsec. 1.4.1, applied to the B-type set of FD, enables us to use only one-loop diagrams in the scattering channel with uncrossed exchanged photon legs:

$$(\Delta|M|^2)_B = \\ = 2^5\alpha^4\pi^2\,\mathrm{Re}\,(1+\hat{P}+\hat{Q}+\hat{R})[(1-\hat{P}_{22'})I_{19}^{\chi_1}+(1+\hat{P}_{22'})I_{29}^{\chi_1}-I], \quad (1.130)$$

where

$$\hat{P}_{22'} = \begin{vmatrix} p_2 \longleftrightarrow -p_2' & s \longleftrightarrow u \\ p_1 \longleftrightarrow p_1 & s_1 \longleftrightarrow u_1 \\ p_1', k_1 \to p_1', k_1 & t, t_1 \to t, t_1 \end{vmatrix} \quad (1.131)$$

and

$$I_{19}^{\chi_1} = \int \frac{d^4k}{i\pi^2}\frac{1}{(0)(q)((p_2+k)^2-m^2)}\frac{16}{t}\,\frac{1}{4}\mathrm{Tr}(\hat{p}_1' B^{\chi_1}\hat{p}_1 O_{11'})\times \\ \times\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_2\gamma_\sigma(-\hat{p}_2-\hat{k})\gamma_\lambda\hat{p}_2'\gamma_\rho),$$

$$I_{29}^{\chi_1} = \int \frac{d^4k}{i\pi^2}\frac{1}{(0)(q)((p_2+k)^2-m^2)}\frac{16}{t_1}\,\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_1' B^{\chi_1}\hat{p}_1\gamma_\rho)\times \\ \times\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_2\gamma_\sigma(-\hat{p}_2-\hat{k})\gamma_\lambda\hat{p}_2' O_{22'}), \quad (1.132)$$

$$I = \int \frac{d^4k}{i\pi^2}\frac{1}{(0)(q)}\left\{\frac{4}{s_1}\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_2'\gamma_\rho\hat{p}_1' B^{\chi_1}\hat{p}_1 O_{12}\hat{p}_2(\hat{A}+\hat{B})+ \right. \\ \left. +\frac{4}{s}\frac{1}{4}\,\mathrm{Tr}\,(\hat{p}_2' O_{1'2'}\hat{p}_1 B^{\chi_1}\hat{p}_1\gamma_\rho\hat{p}_2(\hat{A}+\hat{B}))\right\},$$

$$\hat{A} = \frac{\gamma_\sigma(-\hat{p}_2-\hat{k})\gamma_\lambda}{(p_2+k)^2-m^2}, \quad \hat{B} = \frac{\gamma_\lambda(-\hat{p}_2'+\hat{k})\gamma_\sigma}{(-p_2'+k)^2-m^2}.$$

Here

$$B^{\chi_1} = \frac{\gamma_\lambda(\hat{p}_1-\hat{k}_1-\hat{k})\gamma_\sigma(\hat{p}_1-\hat{k}_1)\gamma_\mu}{-\chi_1(d)} + \frac{\gamma_\lambda(\hat{p}_1-\hat{k}_1-\hat{k})\gamma_\mu(\hat{p}_1-\hat{k})\gamma_\sigma}{(d)(1)} + \\ + \frac{\gamma_\mu(\hat{p}_1'+\hat{k}_1)\gamma_\lambda(\hat{p}_1-\hat{k})\gamma_\sigma}{\chi_1'(1)}, \quad (1.133)$$

$$(q) = (p_2-p_2'+k)^2-\lambda^2, \\ (d) = (p_1-k_1-k)^2-m^2, \\ (1) = (p_1-k)^2-m^2, \quad (0) = k^2-\lambda^2.$$

Box contribution does not contain double logarithm ($\sim L_s^2$) and infrared logarithm ($\sim \ln(\lambda/m)L$) terms. The correction coming from the B-type FD is:

$$d\sigma_B = d\sigma_0\frac{\alpha}{\pi}L_s\Delta_B, \;\; \Delta_B = \frac{1}{F}\left[(\Phi+\Phi_P)\ln\frac{uu_1}{ss_1}+(\Phi_Q+\Phi_R)\ln\frac{uu_1}{tt_1}\right]. \quad (1.134)$$

The total virtual correction to the cross section has the form:

$$
\begin{aligned}
d\sigma^{\rm virt} &= d\sigma_G + d\sigma_{\Gamma\Pi} + d\sigma_B = \\
&= \frac{\alpha}{\pi}\left[-L_s^2 + L_s\left(\frac{11}{3} + 4\ln\frac{\lambda}{m} + \Delta_G + \Delta_{\Gamma\Pi} + \Delta_B\right) + \mathcal{O}(1)\right], \\
&\Delta_G + \Delta_{\Gamma\Pi} = \frac{1}{F}\left((\Phi + \Phi_P)\ln\frac{s^2}{tt_1} + (\Phi_R + \Phi_Q)\ln\frac{s}{s_1}\right),
\end{aligned}
\qquad (1.135)
$$

where $\Phi_P = \hat{P}\Phi$, $\Phi_Q = \hat{Q}\Phi$, and $\Phi_R = \hat{R}\Phi$.

1.4.4. Contribution from Additional Soft Photon Emission. Consider now radiative Bhabha scattering accompanied by emission of an additional soft photon in the center-of-mass reference frame. By *soft* we mean that its energy does not exceed some small quantity $\Delta\varepsilon$, compared to the energy ε of the initial beams. The corresponding cross section has the form

$$
\begin{aligned}
& d\sigma^{\rm soft} = d\sigma_0\delta^{\rm soft}, \\
& \delta^{\rm soft} = -\frac{4\pi\alpha}{16\pi^3}\int\frac{d^3k_2}{\omega_2}\left(-\frac{p_1}{p_1k_2} + \frac{p_1'}{p_1'k_2} + \frac{p_2}{p_2k_2} - \frac{p_2'}{p_2'k_2}\right)^2\Bigg|_{\omega_2<\Delta\varepsilon}.
\end{aligned}
\qquad (1.136)
$$

The soft photon energy does not exceed $\Delta\varepsilon \ll \varepsilon_1 = \varepsilon_2 \equiv \varepsilon \sim \varepsilon_1' \sim \varepsilon_2'$. In order to calculate the right-hand side of Eq. (1.136), we use [25]:

$$
\begin{aligned}
-\frac{4\pi\alpha}{16\pi^3}\int\frac{d^3k}{\omega}\frac{(q_i)^2}{(q_ik)^2}\Bigg|_{\omega<\Delta\varepsilon} &= -\frac{\alpha}{\pi}\ln\left(\frac{\Delta\varepsilon m}{\lambda\varepsilon_i}\right), \quad \omega = \sqrt{k^2+\lambda^2}, \\
\frac{4\pi\alpha}{16\pi^3}\int\frac{d^3k}{\omega}\frac{2q_1q_2}{(kq_1)(kq_2)}\Bigg|_{\omega<\Delta\varepsilon} &= \frac{\alpha}{\pi}\left[L_q\ln\left(\frac{m^2(\Delta\varepsilon)^2}{\lambda^2\varepsilon_1\varepsilon_2}\right) + \frac{1}{2}L_q^2 - \right. \\
&\left. - \frac{1}{2}\ln^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right) - \frac{\pi^2}{3} + {\rm Li}_2\left(\cos^2\frac{\theta}{2}\right)\right].
\end{aligned}
\qquad (1.137)
$$

Here we used the notation

$$
\begin{aligned}
L_q = \ln\frac{-q^2}{m^2}, \quad q_1^2 = q_2^2 = m^2, \quad -q^2 = -(q_1-q_2)^2 \gg m^2, \\
q_{1,2} = (\varepsilon_{1,2}, \mathbf{q}_{1,2}), \quad \theta = \widehat{\mathbf{q}_1\mathbf{q}_2},
\end{aligned}
\qquad (1.138)
$$

where ε_1, ε_2, and θ are the energies and angle between the three-momenta $\mathbf{q}_1$, $\mathbf{q}_2$, respectively, and λ is the fictitious photon mass (all defined in the center-of-mass system).

Factorized out the large logarithms, we obtain

$$\delta^{\text{soft}} = \frac{\alpha}{\pi}\Bigg\{(L_s - 1)\left[4\ln\frac{m}{\lambda} + 2\ln\frac{\Delta\varepsilon}{\varepsilon} + \ln\frac{\Delta\varepsilon}{\varepsilon_1'} + \ln\frac{\Delta\varepsilon}{\varepsilon_2'}\right] + \\ + L_s^2 + L_s\ln\frac{tt_1s_1}{uu_1s} + \mathcal{O}(1)\Bigg\}. \quad (1.139)$$

This can be written in another form, using experimentally measurable quantities, the relative energies of the scattered leptons and the scattering angles:

$$y_i = \frac{\varepsilon_i'}{\varepsilon}, \quad c_i = \cos\theta_i', \quad \theta_i' = \widehat{\mathbf{p}_1, \mathbf{p}_i'},$$
$$-\frac{t}{s} = y_2\frac{1+c_2}{2}, \quad -\frac{u}{s} = y_2\frac{1-c_2}{2}, \quad -\frac{t_1}{s} = y_1\frac{1-c_1}{2}, \quad (1.140)$$
$$\frac{s_1}{s} = y_1 + y_2 - 1, \quad -\frac{u_1}{s} = y_1\frac{1+c_1}{2}, \quad \frac{1}{2}(1 - c_{1'2'}) = \frac{y_1+y_2-1}{y_1y_2}.$$

1.4.5. Renormalization Group Approach. The double logarithmic terms of type L_s^2 and those proportional to $L_s\ln(\lambda/m)$ cancel in the overall sum with the corresponding terms from the soft photon contribution (1.139). Omitting vacuum polarization, we obtain in the logarithmic approximation

$$d\sigma^{\text{soft+virt}} = d\sigma_0\frac{\alpha}{\pi}\left[L_s\left(\ln\frac{(\Delta\varepsilon)^4}{\varepsilon^2\varepsilon_1'\varepsilon_2'} + 3\right) + \Delta(y_1, y_2, c_1, c_2)\right]. \quad (1.141)$$

The function $\Delta(y_1, y_2, c_1, c_2)$ is quite complicated. We give the numerical values in Table 2 (omitting vacuum polarization) for a certain set of points from physical regions:

$$y_1 + y_2 > 1, \quad D > 0, \quad 0 < y_i < 1, \quad -1 < c_{1,2} < 1. \quad (1.142)$$

Table 2. **Numerical estimates of Δ versus y_1, y_2, c_1, c_2 (see (1.141))**

y_1	y_2	c_1	c_2	Δ
0.36	0.89	−0.70	−0.10	−8.89
0.59	0.66	0.29	−0.06	2.00
0.67	0.67	0.50	0.30	−1.47
0.68	0.65	0.60	−0.50	7.80

After performing loop integration and shifting of logarithm argument ($L_i = L_s + L_{is}$), one can see that the terms containing infrared singularities and double logarithmic terms $\sim L_s^2$, are associated with a factor equal to the corresponding Born contribution. This is true of all types of contributions.

The phase volume

$$d\Gamma = \frac{d^3p_1' d^3p_2' d^3k_1}{\varepsilon_1'\varepsilon_2'\omega_1}\delta^{(4)}(p_1 + p_2 - p_1' - p_2' - k_1)$$

can be transformed in various ways. The phase volume then takes the form in the variables (1.140):

$$\begin{gathered} d\Gamma = \frac{\pi s\, dy_1\, dy_2\, dc_1\, dc_2}{2\sqrt{D(y_1, y_2, c_1, c_2)}}\Theta(y_1 + y_2 - 1)\Theta(D(y_1, y_2, c_1, c_2)), \\ D(y_1, y_2, c_1, c_2) = \rho^2 - c_1^2 - c_2^2 - 2c_{1'2'}c_1c_2, \\ \rho^2 = 2(1 - c_{1'2'})\frac{(1 - y_1)(1 - y_2)}{y_1y_2}. \end{gathered} \tag{1.143}$$

The allowed region of integration is a triangle in the y_1, y_2 plane and the interior of the ellipse $D > 0$ in the c_1, c_2 plane.

We now discuss the relation of our result to the renormalization group approach. The dependence on $\Delta\varepsilon/\varepsilon$ in (1.141) disappears when one takes into account hard two-photon emission. The leading contribution arises from the kinematics when the second hard photon (with the energy ω_2) is emitted close to the direction of motion of one of the incoming or outgoing particles:

$$\begin{gathered} d\sigma^{\text{hard}} = \frac{\alpha}{2\pi}L_s\left[\frac{1 + z^2}{1 - z}\Big(d\sigma_0(zp_1, p_2, p_1', p_2') + d\sigma_0(p_1, zp_2, p_1', p_2')\Big)dz + \right. \\ \left. + \frac{1 + z_1^2}{1 - z_1}\, d\sigma_0\left(p_1, p_2, \frac{p_1'}{z_1}, p_2'\right) dz_1 + \frac{1 + z_2^2}{1 - z_2}\, d\sigma_0\left(p_1, p_2, p_1', \frac{p_2'}{z_2}\right) dz_2\right], \\ z = 1 - x_2, \quad z_i = \frac{y_i}{y_i + x_2}, \quad x_2 = \frac{\omega_2}{\varepsilon}. \end{gathered} \tag{1.144}$$

The fractional energy of the additional photon varies within the limits $\Delta\varepsilon/\varepsilon < x_2 = \omega_2/\varepsilon < 1$. This formula agrees with the Drell–Yan form of radiative Bhabha scattering (with switched-off vacuum polarization)

$$\begin{aligned} d\sigma(p_1, p_2, p_1', p_2') = \int dx_1\, dx_2\, dz_1\, dz_2\, D(x_1, \beta)D(x_2, \beta)D(z_1, \beta)D(z_2, \beta)\times \\ \times\, d\sigma_0\left(x_1p_1, x_2p_2, \frac{p_1'}{z_1}, \frac{p_2'}{z_2}\right), \end{aligned} \tag{1.145}$$

where the nonsinglet structure functions $D(x, \beta)$, $\beta = \alpha/(2\pi)(L_s - 1)$ are explicitly given in [17].

1.5. Radiative Large-Angle Bhabha Scattering in Collinear Kinematics. In this Subsection we consider the process of large-angle high-energy electron–positron scattering with emission of one hard photon almost collinear to one of the charged particles momenta. We derive the differential cross section with radiative corrections due to emission of virtual and soft real photons with a power accuracy. At the end of Subsection we consider the emission of two hard photons and total expressions for radiative correction in LLA [25].

1.5.1. Born Expressions in Collinear Kinematics. Physical Gauge. Let us begin with revising the radiative Bhabha scattering process

$$e^-(p_1) + e^+(p_2) \to e^-(p_1') + e^+(p_2') + \gamma(k_1) \tag{1.146}$$

at the tree level. We define the collinear kinematical domains as those in which the hard photon is emitted close (within a narrow cone with opening angle $\theta_0 \ll 1$) to the incident ($\theta_{1(2)} = \widehat{\mathbf{p}_{1(2)}\mathbf{k}_1} < \theta_0$) or the outgoing electron (positron) ($\theta'_{1(2)} = \widehat{\mathbf{p}'_{1(2)}\mathbf{k}_1} < \theta_0$) direction of motion. Because of the symmetry between electron and positron, we may restrict ourselves to a consideration of only two collinear regions, which correspond to the emission of the photon along the electron momenta. The two remaining contributions to the differential cross section of the process (1.146) can be obtained by the substitution $\mathcal{Q}$:

$$d\sigma_{\text{coll}} = \left[1 + \mathcal{Q}\begin{pmatrix} p_1 \leftrightarrow p_2 \\ p_1' \leftrightarrow p_2' \end{pmatrix}\right]\left\{d\sigma^\gamma(\mathbf{k}_1 \parallel \mathbf{p}_1) + d\sigma^\gamma(\mathbf{k}_1 \parallel \mathbf{p}_1')\right\}. \tag{1.147}$$

In the collinear kinematical domain in which $\mathbf{k}_1 \parallel \mathbf{p}_1$, the above formula takes the form

$$d\sigma_0^\gamma(\mathbf{k}_1 \parallel \mathbf{p}_1) = \frac{\alpha^3}{\pi^2 s}\frac{d^3\mathbf{k}_1}{\omega_1}\frac{1}{\chi_1}\Upsilon F\frac{d^3\mathbf{p}_1' d^3\mathbf{p}_2'}{\varepsilon_1'\varepsilon_2'}\delta^{(4)}((1-x)p_1 + p_2 - p_1' - p_2') =$$
$$= dW_{p_1} d\sigma_0((1-x)p_1, p_2), \tag{1.148}$$
$$\Upsilon = \frac{1+(1-x)^2}{x(1-x)} - \frac{2m^2}{\chi_1}, \quad F = \left(\frac{s_1}{t} + \frac{t}{s_1} + 1\right)^2,$$

where

$$s_1 = s(1-x), \quad y_1 = \frac{\varepsilon_1'}{\varepsilon} = 2\frac{1-x}{a}, \quad y_2 = \frac{\varepsilon_2'}{\varepsilon} = \frac{2-2x+x^2+cx(2-x)}{a},$$
$$a = 2 - x + cx, \quad \omega_1 = \varepsilon x, \quad s = 4\varepsilon^2, \quad \chi_1 = \frac{s}{2}x(1 - c_1\beta), \quad \beta = \sqrt{1 - \frac{m^2}{\varepsilon^2}}, \tag{1.149}$$
$$t = t_1(1-x) = -s\frac{(1-x)^2(1-c)}{a}, \quad c = \cos(\widehat{\mathbf{p}_1\mathbf{p}_1'}), \quad c_1 = \cos(\widehat{\mathbf{p}_1\mathbf{k}_1}),$$
$$dW_{p_1} = \frac{\alpha}{2\pi^2}\frac{1-x}{\chi_1}\Upsilon\frac{d^3\mathbf{k}_1}{\omega_1}.$$

Here y_i are the energy fractions of the scattered leptons and $d\sigma_0(p_1(1-x), p_2)$ is the cross section of the elastic Bhabha scattering process.

Throughout this Subsection we use the following relations among invariants:

$$s_1 + t + u_1 = 4m^2 - \chi_1 \approx 0, \qquad s + t_1 + u = 4m^2 + \chi_1 \approx 0.$$

In the case $\mathbf{k}_1 \parallel \mathbf{p}_1'$ we have

$$d\sigma_0^\gamma(\mathbf{k}_1 \parallel \mathbf{p}_1') = \frac{\alpha}{2\pi^2}\frac{1}{\chi_1'}\tilde{\Upsilon}\frac{d^3\mathbf{k}_1}{\omega_1}(1-x)\,d\sigma_0(p_1, p_2),$$
$$\tilde{\Upsilon} = \frac{1+(1-x)^2}{x} - \frac{2m^2}{\chi_1'}. \tag{1.150}$$

These expressions could also be inferred by using the method of quasi-real electrons [6, 7, 30] and starting from the nonradiative Bhabha cross section.

After integration over a hard collinear ($\mathbf{k}_1 \parallel \mathbf{p}_1$) photon angular phase space, the cross section of radiative Bhabha scattering in the Born approximation is found to be

$$\left.\frac{d\sigma_0^\gamma}{dx\,dc}\right|_{\mathbf{k}_1\parallel\mathbf{p}_1} = \frac{4\alpha^3}{s}\left[\frac{1+(1-x)^2}{x}L_0 - 2\frac{1-x}{x}\right]\times$$
$$\times\left(\frac{3-3x+x^2+2cx(2-x)+c^2(1-x(1-x))}{(1-x)(1-c)a^2}\right)^2(1+\mathcal{O}(\theta_0^2)),$$

where $L_0 = \ln{(\varepsilon\theta_0/m)}^2$. And in the case $\mathbf{k}_1 \parallel \mathbf{p}_1'$ it reads

$$\left.\frac{d\sigma_0^\gamma}{dx\,dc}\right|_{\mathbf{k}_1\parallel\mathbf{p}_1'} = \frac{\alpha^3}{4s}\left[\frac{1+(1-x)^2}{x}L_0' - 2\frac{1-x}{x}\right]\left(\frac{3+c^2}{1-c}\right)^2(1+\mathcal{O}(\theta_0^2)),$$
$$L_0' = \ln\left(\frac{\varepsilon_1'\theta_0}{m}\right)^2, \quad \varepsilon_1' = \varepsilon(1-x). \tag{1.151}$$

The simplest way to reproduce these results is to use the physical gauge for the real photon which in the beam cms sets the photon polarization vector to be a space-like 3-vector $\mathbf{e}_\lambda$ having density matrix

$$\sum_\lambda e_\mu^\lambda e_\nu^{\lambda *} = \begin{cases} 0, & \text{if } \mu \text{ or } \nu = 0, \\ \delta_{\mu\nu} - n_\mu n_\nu, & \mu = \nu = 1, 2, 3, \end{cases} \qquad \mathbf{n} = \frac{\mathbf{k}_1}{\omega_1}$$

with the properties

$$\sum_\lambda |e_\lambda|^2 = -2, \qquad \sum_\lambda |p_1 e_\lambda|^2 = \varepsilon^2(1-c_1^2),$$
$$\sum_\lambda |p_1' e_\lambda|^2 = \frac{t_1 u_1}{s}, \quad \sum_\lambda (p_1 e_\lambda)(p_1' e_\lambda)^* \Big|_{\theta\to 0} \sim 0. \tag{1.152}$$

These properties enable us to omit mass terms in the calculations of traces and, besides, to restrict ourselves to the consideration of *singular* terms (see Eq. (1.153)) only, both at the Born and one-loop level. As shown in [26], this gauge is proved useful for a description of jet production in QCD; it is also very well suited to our case because it allows one to simplify a lot of the calculations with respect, for instance, to the Feynman gauge. What is more, it possesses another very attractive feature related with the structure of the correction to be mentioned below (see (2.68)).

With these tools at our disposal, let us turn now to the main point. The contributions, which survive the limit $\theta_0 \to 0$, arise from the terms containing

$$\frac{(p_1 e)^2}{\chi_1^2}, \quad \frac{e^2}{\chi_1}, \quad \frac{(p_1' e)^2}{\chi_1}. \tag{1.153}$$

Other omitted terms (in particular those which do not contain a factor χ_1^{-1}) can be safely neglected since they give a contribution of the order of θ_0^2 which determines the accuracy of our calculations

$$1 + \mathcal{O}\left(\frac{\alpha}{\pi}\theta_0^2 L_s\right), \quad \frac{m}{\varepsilon} \ll \theta_0 \ll 1, \quad L_s = \frac{s}{m^2}. \tag{1.154}$$

In the realistic case this corresponds to an accuracy of the order of per mill.

1.5.2. Crossing Relations. In this and the next Section we shall consider the case $\mathbf{k}_1 \parallel \mathbf{p}_1$. In the case of photon emission along p_1' one can get the desired expression by using the *left-to-right* permutation

$$|M|^2_{\mathbf{k}_1\parallel\mathbf{p}_1'} = \mathcal{Q}\begin{pmatrix} p_1 \leftrightarrow -p_1' \\ p_2 \leftrightarrow -p_2' \end{pmatrix} |M|^2_{\mathbf{k}_1\parallel\mathbf{p}_1}. \tag{1.155}$$

From now on, we deal with scattering-type amplitudes (FD) with the emission of hard photon by initial electron. This is possible due to the properties of the physical gauge. The contribution of annihilation-type amplitudes may be derived by applying the momenta replacement operation as follows:

$$\Delta|M|^2_{\text{annihilation}} = \\ = \{\mathcal{Q}(p_1' \leftrightarrow -p_2)\}\, \Delta|M|^2_{\text{scattering}} \equiv \{Q_1\}\, \Delta|M|^2_{\text{scattering}}. \tag{1.156}$$

When considering FD with two photons in the scattering channel (box FD) one may examine only those with uncrossed photons because a contribution of the others may be obtained by the permutation $p_2 \leftrightarrow -p_2'$. Thus the general answer becomes

$$|M|^2_{\mathbf{k}_1\|\mathbf{p}_1} = \mathrm{Re}\left\{(1+Q_1)[G+L] + \frac{1}{s_1 t}(1+Q_1)(1+Q_2)[s_1 t(B+P)]\right\}, \tag{1.157}$$

with the permutation operators acting as

$$Q_1 F(s_1, t_1, s, t) = F(t, s, t_1, s_1), \quad Q_2 F(s, u, s_1, u_1) = F(u, s, u_1, s_1).$$

1.5.3. Virtual and Soft Photon Emission in $\mathbf{k}_1 \parallel \mathbf{p}_1$ *Kinematics.* One-loop QED RC (which are described by seventy two Feynman diagrams) can be classified out into the two gauge-invariant subsets (see Fig. 7):

- single-photon exchange between electron and positron lines (G, L-type);
- double-photon exchange between electron and positron lines (B, P-type).

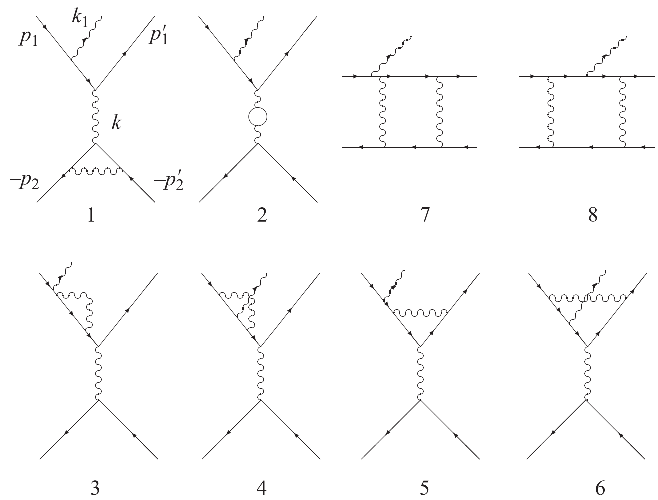

Fig. 7. Some representatives of FD for radiative Bhabha scattering up to the second order: *1*) is the vertex insertion; *2*) is the vacuum polarization insertion; graphs denoted by *3*), *4*) are of the L-type, *5*) is of G_1-type, *6*) is of G_2-type, *7*) is of B-type and *8*) is of P-type

For L-type FD (see Fig. 7(3, 4)) the initial spinor $u(p_1)$ is replaced by $(\alpha/(2\pi))A_2\hat{k}_1\hat{e}u(p_1)$, with

$$A_2 = \frac{1}{\chi_1}\left\{-\frac{\rho}{2(\rho-1)} + \frac{2\rho^2 - 3\rho + 2}{2(\rho-1)^2}L_\rho + \frac{1}{\rho}\left[-\mathrm{Li}_2(1-\rho) + \frac{\pi^2}{6}\right]\right\},$$

$$L_\rho = \ln\rho, \qquad \rho = \frac{\chi_1}{m^2}.$$

The relevant contribution to the matrix element squared and summed over spin states reads

$$\Delta|M|^2_L = 2^9\pi^2\alpha^4 \frac{A_2}{\chi_1}\frac{s_1^3-u_1^3}{s_1t^2}\left[Y-\frac{2(2-x)}{1-x}W\right],$$
$$Y = 4(p_1e)^2 - \frac{x}{1-x}e^2\chi_1, \quad W=(p_1e)^2. \tag{1.158}$$

The contribution of vertex insertion, vacuum polarization* and G_1-type (see Fig. 7(1, 2, 5)) has the following form:

$$\Delta|M|^2_{\Pi,\Gamma,\Gamma_a} = 2^{10}\pi^2\alpha^4\left[\Pi_t+\Gamma_t+\frac{1}{4}\Gamma_a\right]\frac{s_1^3-u_1^3}{t^2s_1\chi_1^2}Y,$$
$$\Pi_t = \frac{1}{3}L_t - \frac{5}{9}, \quad \Gamma_t = (L_\lambda-1)(1-L_t) - \frac{1}{4}L_t - \frac{1}{4}L_t^2 + \frac{\pi^2}{12}, \tag{1.159}$$
$$\Gamma_a = -3L_t^2 + 4L_tL_\rho + 3L_t + 4L_\lambda - 2\ln(1-\rho) - \frac{\pi^2}{3} + 2\mathrm{Li}_2(1-\rho) - 4,$$
$$L_\lambda = \ln\frac{m}{\lambda}, \quad L_t = \ln\frac{-t}{m^2}.$$

Here λ is as usual the IR cut-off parameter to be cancelled at the end of calculus against a soft photon contribution.

For the contribution of G_2-type FD (see Fig. 7(6)) with four denominators we obtain

$$\Delta|M|^2_G =$$
$$= 2^9\alpha^4\pi^2\frac{s_1^3-u_1^3}{ts_1\chi_1(1-x)}\left[(J-J_1)Y + \frac{2(2-x)}{1-x}W(J_{11}-J_1+xJ_{1k}-xJ_k)\right].$$

It turns out that only the scalar integral and the coefficients before p_1, k_1 in the vector and tensor integrals give nonvanishing contribution in the limit $\theta_0 \to 0$:

$$\int\frac{d^4k}{i\pi^2}\frac{(1,k^\mu,k^\mu k^\nu)}{(0)(1)(2)(q)} = (J, J_1p_1^\mu + J_kk_1^\mu, J_{11}p_1^\mu p_1^\nu + J_{kk}k_1^\mu k_1^\nu + J_{1k}(p_1k_1)^{\mu\nu}),$$
$$(0) = k^2-\lambda^2, \quad (1) = k^2-2p_1k, \quad (2) = k^2-2p_1'k, \tag{1.160}$$
$$(q) = k^2 - 2k(p_1-k_1) - \chi_1, \quad (ab)^{\mu\nu} = a^\mu b^\nu + a^\nu b^\mu,$$

and the terms having no p_1 or k_1 momentum in the decomposition have been omitted for their unimportance.

*For realistic applications one should also add to Π the contributions due to μ and τ leptons and hadrons.

The B-type FD shown in Fig. 7(7) with uncrossed legs gives

$$\Delta|M|^2_B = 2^9\pi^2\alpha^4 Y \frac{1}{s_1 t \chi_1^2}\Bigg[(u_1^3 - s_1^3)s_1(B + a - b) - u_1^3 s_1 \times \\ \times \left(c + a_{1'2'} + a_{1'2} + \frac{2}{s_1}a_g\right) + s_1^3(c[t - u_1] + 2J_0)\Bigg], \quad (1.161)$$

where the coefficients are associated with scalar, vector, and tensor integrals over the loop momentum

$$\int \frac{d^4k}{i\pi^2}\frac{(1, k^\mu, k^\mu k^\nu)}{b_0 b_1 b_2 b_3} = (B, B^\mu, B^{\mu\nu}), \quad J_0 = \int \frac{d^4k}{i\pi^2}\frac{1}{b_1 b_2 b_3},$$

$$b_0 = k^2 - \lambda^2, \quad b_1 = k^2 + 2p_1'k, \quad b_2 = k^2 - 2p_2'k,$$

$$b_3 = k^2 - 2qk + t, \quad q = p_2' - p_2, \quad B^\mu = (ap_1' + bp_2' + cp_2)^\mu,$$

$$B^{\mu\nu} = a_g g^{\mu\nu} + a_{1'1'}p_1'^\mu p_1'^\nu + a_{22}p_2^\mu p_2^\nu + a_{2'2'}p_2'^\mu p_2'^\nu + a_{1'2}(p_1'p_2)^{\mu\nu} + \\ + a_{1'2'}(p_1'p_2')^{\mu\nu} + a_{22'}(p_2 p_2')^{\mu\nu}.$$

For (the so-called «pentagon type») P-type FD (see Fig. 7(8)) with uncrossed photon legs we have

$$\Delta|M|^2_P = 2^9\pi^2\alpha^4 \frac{s_1^3 - u_1^3}{t\chi_1(1-x)} \times \\ \times \left[Y(E - E_1) + \frac{2(2-x)}{1-x}W(E_{11} - E_1 + xE_{1k} - xE_k)\right]. \quad (1.162)$$

Here we are using the definition (with tensor structures giving no contributions in the limit $\theta_0 \to 0$ dropped)

$$\int \frac{d^4k}{\mathrm{i}\pi^2}\frac{(1, k^\mu, k^\mu k^\nu)}{a_0 a_1 a_2 a_3 a_4} = \\ = (E, E_1 p_1^\mu + E_k k_1^\mu, E_{11}p_1^\mu p_1^\nu + E_{kk}k_1^\mu k_1^\nu + E_{1k}(p_1^\mu k_1^\nu + p_1^\nu k_1^\mu)),$$

$$a_0 = k^2 - \lambda^2, \quad a_1 = k^2 - 2p_1 k, \quad a_2 = k^2 - 2k(p_1 - k_1) - \chi_1, \\ a_3 = k^2 + 2p_2 k, \quad a_4 = k^2 - 2qk + t. \quad (1.163)$$

Note that in the evaluating of P-type FD we are allowed to put $k_1 = xp_1$, thus keeping only p_1 momentum containing terms in the decomposition.

Collecting all the contributions (for the explicit expressions of all the coefficients see Subsec. 2.4) given above we arrive at the general expression for the virtual corrections with $\rho = x[1 + (\varepsilon\theta/m)^2] \ll s/m^2$

$$2\,\mathrm{Re}\sum (M_0^* M)_{\mathbf{k}_1\|\mathbf{p}_1} = \frac{2^{11}\alpha^4\pi^2}{\chi_1} F\Upsilon \Bigg\{ \frac{2-x}{1-x}\frac{w}{\Upsilon}\Phi + 2L_\lambda(2 - L_t - L_{t_1} - L_s - \\ - L_{s_1} + L_u + L_{u_1}) + \frac{\pi^2}{3} + \mathrm{Li}_2(x) - \frac{101}{18} + \ln\left|\frac{\rho}{1-\rho}\right| + L_{u_1}^2 - L_t^2 - \\ - L_{s_1}^2 + L_\rho \ln(1-x) + \frac{11}{3}L_t - \vartheta + \ln^2\frac{s_1}{t} + \frac{1}{F}\Bigg[\Pi + 3\frac{t^3 - u_1^3}{s_1^2 t}\ln\frac{s_1}{t} + \\ + \frac{2u_1(u_1^2 + s_1^2) - ts_1^2}{4t^2 s_1}\ln^2\frac{u_1}{t} + \frac{2u_1(u_1^2 + t^2) - t^2 s_1}{4ts_1^2}\ln^2\frac{u_1}{s_1} + \\ + \frac{s_1}{2t}\ln\frac{u_1}{t} + \frac{t}{2s_1}\ln\frac{u_1}{s_1} - \frac{3}{4}\pi^2\left(\frac{s_1}{t} + \frac{t}{s_1}\right)\Bigg]\Bigg\}, \quad (1.164)$$

where we have used the following definitions:

$$\vartheta = \frac{x}{\rho - x}\left[\mathrm{Li}_2(1-\rho) - \frac{\pi^2}{6} + \mathrm{Li}_2(x) + L_\rho \ln(1-x)\right],$$

$$\Pi = \frac{s_1^3 - u_1^3}{s_1 t^2}\left[\frac{\pi}{\alpha}\left(\frac{1}{1-\Pi_t} - 1\right) - \frac{1}{3}L_t + \frac{5}{9}\right] + \\ + \frac{t^3 - u_1^3}{s_1^2 t}\left[\frac{\pi}{\alpha}\,\mathrm{Re}\left(\frac{1}{1-\Pi_{s_1}} - 1\right) - \frac{1}{3}L_{s_1} + \frac{5}{9}\right],$$

$$\Pi_{s_1} = \frac{1}{3}(L_{s_1} - \mathrm{i}\pi) - \frac{5}{9}, \quad \Phi = \chi_1 A_2 + t_1\chi_1(J_{11} - J_1 + xJ_{1k} - xJ_k),$$
$$w = \frac{1}{x} - \frac{1}{\rho}, \quad L_{s_1} = \ln\frac{s_1}{m^2}, \quad L_u = \ln\frac{-u}{m^2}, \quad L_{u_1} = \ln\frac{-u_1}{m^2},$$
$$L_t = \ln\frac{-t}{m^2}, \quad L_{t_1} = \ln\frac{-t_1}{m^2}.$$

After integration over χ_1 one gets additional large logs of the form $L_0 = L_s + \ln(\theta_0^2/4)$. Terms containing the last factor have to be cancelled against a contribution coming from the emission of hard photon outside a narrow cone $\theta < \theta_0 \ll 1$ (and supplied by the same set of virtual and soft corrections). In the case of two hard photon emission it is necessary to consider four kinematical regions, namely when both are emitted inside/outside a cone and one inside/another outside.

Fortunately enough, the w structure, which obviously violates factorization feature, does not contribute in LLA due to a cancellation of large logs in Φ. What for a correction to the above structure coming from P-type graph, it vanishes in the sum of FD with crossed and uncrossed photon legs.

The total expression can be obtained by summing virtual photon emission corrections and those arising from the emission of additional soft photon with energy not exceeding $\Delta\varepsilon \ll \varepsilon$. The soft correction can be written as

$$\sum |M|^2_{\text{hard+soft}} = \sum |M|^2_B w_{\text{soft}}(\mathbf{k}_1 \parallel \mathbf{p}_1),$$
$$w_{\text{soft}}(\mathbf{k}_1 \parallel \mathbf{p}_1) = -\frac{\alpha}{4\pi^2} \int\limits_{\omega<\Delta\varepsilon} \frac{d^3\mathbf{k}}{\sqrt{\mathbf{k}^2+\lambda^2}} \left(-\frac{p_1}{p_1 k} + \frac{p_1'}{p_1' k} + \frac{p_2}{p_2 k} - \frac{p_2'}{p_2' k}\right)^2, \tag{1.165}$$

where M_B denotes the matrix element of the hard photon emission at the Born level and in the kinematics $\mathbf{k}_1 \parallel \mathbf{p}_1$ it reads

$$\sum |M|^2_B = \frac{2^{11}\alpha^3\pi^3}{\chi_1} \Upsilon F. \tag{1.166}$$

Now let us check the cancellation of the terms containing L_λ. Indeed it takes place in the sum of contributions arising from emission of virtual and soft real photons. To show that we put the soft correction into the form:

$$w_{\text{soft}}(\mathbf{k}_1 \parallel \mathbf{p}_1) = \frac{\alpha}{\pi}\Bigg\{2\left(\ln\frac{\Delta\varepsilon}{\varepsilon} + L_\lambda\right)(-2+L_s+L_{s_1}+L_t+L_{t_1}-L_u-L_{u_1})+$$
$$+\frac{1}{2}(L_s^2+L_{s_1}^2+L_t^2+L_{t_1}^2-L_u^2-L_{u_1}^2) + \ln y_1(L_{u_1}-L_{s_1}-L_{t_1})+$$
$$+\ln y_2(L_u-L_t-L_{s_1}) + \ln(y_1 y_2) - \frac{2\pi^2}{3} - \frac{1}{2}\ln^2\frac{y_1}{y_2} + \mathrm{Li}_2\left(\frac{1+c_{1'2'}}{2}\right)+$$
$$+\mathrm{Li}_2\left(\frac{1+c_{1'}}{2}\right) + \mathrm{Li}_2\left(\frac{1-c_{2'}}{2}\right) - \mathrm{Li}_2\left(\frac{1-c_{1'}}{2}\right) - \mathrm{Li}_2\left(\frac{1+c_{2'}}{2}\right)\Bigg\}, \tag{1.167}$$

where c_i are the cosines of emission angles of ith particle with respect to the beam direction ($\mathbf{p}_1$ in cms); $c_{1'2'}$ is the cosine of the angle between scattered fermions in cms of the colliding particles and y_i are their energy fractions, and in the case $\mathbf{k}_1 \parallel \mathbf{p}_1$ we have

$$c_1' = c, \quad \frac{1+c_{1'2'}}{2} = 1 - \frac{1-x}{y_1 y_2}, \quad \frac{1-c_2'}{2} = \frac{y_1(1+c)}{2y_2(1-x)}. \tag{1.168}$$

Then the cancellation of infrared singularities in the sum is evident from comparison of Eqs. (1.164), (1.167). The terms with $\ln(\Delta\varepsilon/\varepsilon)$ should be cancelled when adding a contribution of a second hard photon having energy above the registration threshold $\Delta\varepsilon$.

The complete expression for the correction in the case $\mathbf{k}_1 \parallel \mathbf{p}_1$ reads

$$R = 2\,\mathrm{Re}\sum(M_0^* M) + |M|^2_{\mathrm{soft}} = \frac{2^{11}\alpha^4\pi^2}{\chi_1} F\Upsilon\Bigg\{\frac{2-x}{1-x}\frac{w}{\Upsilon}\Phi +$$
$$+ 4\ln\left(\frac{\Delta\varepsilon}{\varepsilon}\right)\left[-1 + L_{t_1} + \frac{1}{2}\left(-\ln(1-x) + 2\ln\frac{s}{-u}\right)\right] + \frac{11}{3}L_t +$$
$$+ (L_\rho - L_t)\ln(1-x) - L_t\ln(y_1y_2) + \ln^2\frac{s_1}{-t} + \ln y_1\ln(1-x) +$$
$$+ \ln(y_1y_2)\left(1 + \ln\frac{-u}{s}\right) - \frac{\pi^2}{3} + \mathrm{Li}_2(x) - \frac{101}{18} - \vartheta + \ln\left|\frac{\rho}{1-\rho}\right| -$$
$$-\frac{1}{2}\ln^2\frac{y_1}{y_2} + \ln(1-x)\ln\frac{-u}{s} + \mathrm{Li}_2\left(\frac{1+c_{1'2'}}{2}\right) + \mathrm{Li}_2\left(\frac{1+c_{1'}}{2}\right) + \mathrm{Li}_2\left(\frac{1-c_{2'}}{2}\right) -$$
$$-\mathrm{Li}_2\left(\frac{1-c_{1'}}{2}\right) - \mathrm{Li}_2\left(\frac{1+c_{2'}}{2}\right) + \frac{1}{F}\Bigg[\Pi + 3\frac{t^3-u_1^3}{s_1^2 t}\ln\frac{s_1}{-t} +$$
$$+ \frac{2u_1(u_1^2+s_1^2) - ts_1^2}{4t^2 s_1}\ln^2\frac{u_1}{t} + \frac{2u_1(u_1^2+t^2) - t^2 s_1}{4ts_1^2}\ln^2\frac{-u}{s} +$$
$$+ \frac{s_1}{2t}\ln\frac{u_1}{t} + \frac{t}{2s_1}\ln\frac{-u}{s} - \frac{3}{4}\pi^2\left(\frac{s_1}{t} + \frac{t}{s_1}\right)\Bigg]\Bigg\}, \quad (1.169)$$

$$d\sigma(\mathbf{k}_1 \parallel \mathbf{p}_1) = \frac{1}{2^{11}\pi^5 s} R\,d\Gamma.$$

The similar expression can be obtained for the case $\mathbf{k}_1 || \mathbf{p}_1'$.

1.5.4. The Results in LLA. It should be noted that all the terms quadratic in *large* logarithms $L_{t_1} \sim L_{s_1} \sim L_u \gg L_\rho$ are mutually cancelled out in the total sum of virtual and real photon contribution. From formula (1.169) it immediately follows that (upon doing an integration over a hard-photon angular (within a narrow cone) phase space) the w term that is not proportional to Υ, which is in fact the kernel of the nonsinglet electron structure function, is not dangerous in the sense of a feasible violation of the expected Drell–Yan form of the cross section, because it does contribute only at next-to-leading order.

Performing the above-mentioned integration and confining ourselves to LLA we get for the sum of virtual and soft photons

$$\frac{d\sigma^{\gamma(S+V)}}{dx\,dc} = \frac{d\sigma_0^\gamma}{dx\,dc}\frac{\alpha}{\pi}L_s\left[4\ln\frac{\Delta\varepsilon}{\varepsilon} + \frac{11}{3} - \frac{1}{2}\ln(1-x) - \ln(y_1y_2)\right]. \quad (1.170)$$

The LLA contribution coming from the emission of second hard photon with total energy exceeding $\Delta\varepsilon$ consists of a part corresponding to the case in which both hard photons (with total energy εx) are emitted by initial electron

$$\frac{d\sigma^{2\gamma}}{dx\,dc} = \frac{d\sigma_0^{\gamma}}{dx\,dc}\frac{\alpha}{\pi}L_s\left[\frac{xP_{\Theta}^{(2)}(1-x)}{4(1+(1-x)^2)} + \frac{1}{2}\ln(1-x) - \ln\frac{\Delta\varepsilon}{\varepsilon} - \frac{3}{4}\right], \tag{1.171}$$

$$P_{\Theta}^{(2)}(z) = 2\left[\frac{1+z^2}{1-z}\left(2\ln(1-z) - \ln z + \frac{3}{2}\right) + \frac{1+z}{2}\ln z - 1 + z\right],$$

and the remaining part which describes the emission of second hard photon along scattered electron and positrons. The latter, upon combining with the part of contributions of soft and virtual photons to our process

$$\frac{d\sigma_0^{\gamma}}{dx\,dc}\frac{3\alpha}{\pi}L_s\left[\ln\frac{\Delta\varepsilon}{\varepsilon} + \frac{3}{4}\right],$$

may be represented via electron structure function in the spirit of the Drell–Yan approach

$$\left(\frac{d\sigma_0^{\gamma}}{dx\,dc\,dy_3\,dy_4}\right)\Bigg|_{\mathbf{k}_1||\mathbf{p}_1} = \\ = \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}L_s\int\limits_0^1 dz_2\; D(z_2,\beta)\,\frac{d\sigma_0\,(p_1(1-x), z_2p_2, q_1, q_2)}{dc}\times \\ \times\frac{1}{z_3}D\left(\frac{y_3}{z_3},\beta\right)\frac{1}{z_4}D\left(\frac{y_4}{z_4},\beta\right), \tag{1.172}$$

$$\beta = \frac{\alpha}{2\pi}(L_s - 1),$$

where $y_{3,4}$ is the energy fraction of the scattered leptons and the nonsinglet structure function $D(z,\beta)$ could be found in [17], $z_1 = q_{10}/E$, $z_2 = q_{20}/E$ are determined by $2 \to 2$ subprocess.

These functions describe the emission of (real and virtual) photons both by final electron and by positrons. The multiplier before the integral stands for the emission of a hard photon by the initial electron. Thus Eq. (1.172) actually represents the partially integrated Drell–Yan form of the cross section. Quite the same arguments are applicable to the second case in which a hard photon is emitted by the final electron.

The cross section of the hard subprocess $e(p_1z_1) + \bar{e}(p_2z_2) \to e(q_1) + \bar{e}(q_2)$ entering Eq. (1.172) has the form

$$\frac{d\sigma_0(z_1p_1, z_2p_2; q_1, q_2)}{dc} = \\ = \frac{8\pi\alpha^2}{s}\left[\frac{z_1^2 + z_2^2 + z_1z_2 + 2c(z_2^2 - z_1^2) + c^2(z_1^2 + z_2^2 - z_1z_2)}{z_1(1-c)(z_1 + z_2 + c(z_2 - z_1))^2}\right]^2.$$

The momenta of scattered electron q_1 and positron q_2 are completely determined by the energy-momentum conservation law

$$q_1^0 = \varepsilon \frac{2z_1 z_2}{z_1 + z_2 + c(z_2 - z_1)}, \quad q_1^0 + q_2^0 = \varepsilon(z_1 + z_2),$$
$$c = \cos \widehat{\mathbf{q}_1, \mathbf{p}}_1, \quad z_1 \sin \widehat{\mathbf{q}_1, \mathbf{p}}_1 = z_2 \sin \widehat{\mathbf{q}_2, \mathbf{p}}_1.$$

In general, their energies differ from those detected in experiment $\varepsilon_1', \varepsilon_2'$, namely

$$\varepsilon_1' = q_1^0 z_3, \quad \varepsilon_2' = q_2^0 z_4,$$

whereas the emission angles are the same in LLA.

Collecting the two expressions presented in Eqs. (1.170), (1.171) one can rewrite the result in LLA as

$$\left.\frac{d\sigma^\gamma}{dx\,dc\,dy_3\,dy_4}\right|_{\mathbf{k}_1 \| \mathbf{p}_1} = \\ = \left(\frac{d\sigma_0^\gamma}{dx\,dc\,dy_3\,dy_4}\right)_{\mathbf{k}_1 \| \mathbf{p}_1} \left\{1 + \frac{\alpha}{\pi} L_s \left[\frac{2}{3} - \ln(z_3 z_4) + \frac{x P_\Theta^{(2)}(1-x)}{4(1 + (1-x)^2)}\right]\right\},$$

and $\dfrac{d\sigma_0^\gamma}{dx\,dc\,dy_3\,dy_4}$ is given in (1.172). For the case $\mathbf{k}_1 \parallel \mathbf{p}_1'$, the correction is found to be

$$\left.\frac{d\sigma^\gamma}{dx\,dc\,dy_4}\right|_{\mathbf{k}_1 \| \mathbf{p}_1'} = \\ = \left(\frac{d\sigma_0^\gamma}{dx\,dc\,dy_4}\right)_{\mathbf{k}_1 \| \mathbf{p}_1'} \left\{1 + \frac{\alpha}{\pi} L_s \left[\frac{2}{3} + \frac{x P_\Theta^{(2)}(1-x)}{4(1 + (1-x)^2)}\right]\right\}. \quad (1.173)$$

In terms of $2 \to 2$ hard subprocess in LLA we have

$$\left.\left(\frac{d\sigma_0^\gamma}{dx\,dc\,dy_4}\right)\right|_{\mathbf{k}_1 || \mathbf{p}_1} = \\ = \frac{\alpha}{2\pi} \frac{1 + (1-x)^2}{x} L_s \int_0^1 dz_1 \int_0^1 dz_2 D(z_1, \beta) D(z_2, \beta) \times \\ \times \frac{d\sigma_0(z_1 p_1, z_2 p_2, q_1, q_2)}{dc} \frac{1}{z_4} D\left(\frac{y_4}{z_4}, \beta\right), \quad (1.174)$$

where $z_4 = q_{20}/E$ is determined by the kinematics of $e_-(p_1 z_1) + e_+(p_2 z_2) \to e_-(q_1) + e_+(q_2)$ subprocess.

1.6. Emission of Two Hard Photons in Large-Angle Bhabha Scattering. Consider the emission of two real hard photons:

$$e^+(p_+) + e^-(p_-) \to e^+(q_+) + e^-(q_-) + \gamma(k_1) + \gamma(k_2). \tag{1.175}$$

The relevant contribution to the *experimental* cross section has the following form:

$$\sigma_{\rm exp} = \int d\sigma \Theta_+ \Theta_-, \tag{1.176}$$

where Θ_+ and Θ_- are the experimental restrictions providing the simultaneous detection of both the scattered electron and positron. First, this means that their energy fractions should be larger than a certain (small) quantity $\varepsilon_{\rm th}/\varepsilon$, $\varepsilon_{\rm th}$ is the energy threshold of the detectors. The second condition restricts their angles with respect to the beam axes. They should be larger than a certain finite value ψ_0 ($\psi_0 \sim 35^\circ$ in the experimental conditions accepted in [18]):

$$\pi - \psi_0 > \theta_-, \quad \theta_+ > \psi_0, \quad \theta_\pm = \widehat{\mathbf{q}_\pm \mathbf{p}}_-, \tag{1.177}$$

where $\theta_\pm$ are the polar angles of the scattered leptons with respect to the beam axes ($\mathbf{p}_-$). We accept the condition on the energy threshold of the charged-particle registration $q^0_\pm > \varepsilon_{\rm th}$. Both photons are assumed to be hard. Their minimal energy

$$\omega_{\rm min} = \Delta\varepsilon, \quad \Delta \ll 1 \tag{1.178}$$

could be considered as the threshold of the photon registration.

The main ($\sim (\alpha L/\pi)^2$, $L = \ln(s/m^2)$) contribution to the total cross section (1.176) comes from the collinear region: when both the emitted photons move within narrow cones along the charged particle momenta (they may go along the same particle). So we will distinguish 16 kinematical regions:

$$\begin{gathered} \widehat{\mathbf{a}\mathbf{k}}_1 \ \text{ and } \ \widehat{\mathbf{a}\mathbf{k}}_2 < \theta_0, \quad \widehat{\mathbf{a}\mathbf{k}}_1 \ \text{ and } \ \widehat{\mathbf{b}\mathbf{k}}_2 < \theta_0, \\ \frac{m}{\varepsilon} \ll \theta_0 \ll 1, \quad a \neq b, \quad a,\, b = p_-, p_+, q_-, q_+. \end{gathered} \tag{1.179}$$

The matrix element module square summed over spin states in the regions (1.179) is of the form of the Born matrix element multiplied by the so-called collinear factors. The contribution to the cross section of each region has also the form of $2 \to 2$ Bhabha cross sections in the Born approximation multiplied by the factors of the form

$$d\sigma_i^{\rm coll} = d\sigma_{0i}\left[a_i(x_j, y_j)\ln^2\left(\frac{\varepsilon^2\theta_0^2}{m^2}\right) + b_i(x_j, y_j)\ln\left(\frac{\varepsilon^2\theta_0^2}{m^2}\right)\right], \tag{1.180}$$

where $x_j = \omega_j/\varepsilon$, $y_1 = q_-^0/\varepsilon$, $y_2 = q_+^0/\varepsilon$ are the energy fractions of the photons and of the scattered electron and positron. The dependence on the auxiliary parameter θ_0 will be cancelled in the sum of the contributions of the collinear and semicollinear regions. The last region corresponds to the kinematics, when only one photon is emitted inside the narrow cone $\theta_1 < \theta_0$ along one of the charged particle momenta. And the second photon is emitted outside any cone of that sort along charged particles ($\theta_2 > \theta_0$):

$$d\sigma_i^{\rm sc} = \frac{\alpha}{\pi} L d\sigma_{0i}^{\gamma}(k_2), \qquad (1.181)$$

where $\mathrm{d}\sigma_{0i}^{\gamma}$ has the known form of the single hard bremsstrahlung cross section in the Born approximation.

Below we show explicitly that the result of the integration over the single hard photon emission in Eq. (1.181) in the kinematical region $\theta_2^i > \theta_0$ (θ_2^i is the emission angle of the second hard photon with respect to the direction of one of the four charged particles) has the following form:

$$\int d\sigma_{0i}^{\gamma}(k_2) = -2\ln\left(\frac{\theta_0^2}{4}\right) a_i(x,y)\, d\sigma_{0i} + d\tilde{\sigma}_i. \qquad (1.182)$$

The collinear factors in the double bremsstrahlung process were first considered in papers of the CALCUL collaboration. Unfortunately, they have a rather complicated form which is less convenient for further analytical integration in comparison with the expressions given below. The method of calculation of the collinear factors may be considered as a generalization of the quasi-real electron method to the case of multiple bremsstrahlung. Another generalization is required for the calculations of the cross section of process $e^+e^- \to 2e^+2e^-$ [24].

It is interesting that the collinear factors for the kinematical region of the two hard photon emission along the projectile and the scattered electron are found the same as for the electron–proton scattering process considered in paper [21].

There are 40 Feynman diagrams of the tree type which describe the double bremsstrahlung process in e^+e^- collisions. The differential cross section in terms of helicity amplitudes has a very complicated form. We note that the contribution from the kinematical region in which the angles (in the c. m. s.) between any two final particles are large compared with m/ε is of the order:

$$\frac{\alpha^2 r_0^2 m^2}{\pi^2 \varepsilon^2} \sim 10^{-36}\ \mathrm{cm}^2, \quad \sqrt{s} = 2\varepsilon \sim 1\ \mathrm{GeV} \qquad (1.183)$$

(r_0 is the classical electron radius). So, the corresponding events will possess poor statistics at the colliders with the luminosity $\mathcal{L} \sim 10^{31}$–10^{32} $\mathrm{cm}^{-2}\cdot\mathrm{s}^{-1}$. More probable are the cases of double bremsstrahlung imitating the processes $e^+e^- \to e^+e^-$ or $e^+e^- \to e^+e^-\gamma$, which corresponds to the emission of one or two photons along charged-particle momenta.

1.6.1. Kinematics in the Collinear Region. It is convenient to introduce, in the collinear region, new variables and transform the phase volume of the final state in the following way [29] (from now on we work in the c.m.s.):

$$\int d\Gamma = \int \frac{d^3q_-\, d^3q_+\, d^3k_1\, d^3k_2}{16q_-^0 q_+^0 \omega_1\omega_2(2\pi)^8}\delta^{(4)}(p_- + p_+ - q_- - q_+ - k_1 - k_2) =$$
$$= \frac{m^4\pi^2}{4(2\pi)^6}\int\limits_\Delta^1 dx_1 \int\limits_\Delta^1 dx_2 x_1 x_2 \int\limits_0^{2\pi}\frac{d\varphi}{2\pi}\int\limits_0^{z_0} dz_1 \int\limits_0^{z_0} dz_2 \int d\Gamma_q,$$
$$\int d\Gamma_q = \int \frac{d^3q_-\, d^3q_+}{4q_-^0 q_+^0(2\pi)^2}\delta^{(4)}(\eta_1 p_- + \eta_2 p_+ - \lambda_1 q_- - \lambda_2 q_+), \tag{1.184}$$
$$z_{1,2} = \left(\frac{\theta_{1,2}\varepsilon}{m}\right)^2,\quad \varphi = \widehat{\mathbf{k}_{1\perp}\mathbf{k}_{2\perp}},\quad x_i = \frac{\omega_i}{\varepsilon},\quad z_0 = \left(\frac{\theta_0\varepsilon}{m}\right)^2 \gg 1,\quad \Delta = \frac{\omega_{\min}}{\varepsilon},$$

where θ_i $(i = 1, 2)$ is the polar angle of the i-photon emission with respect to the momentum of the charged particle that emits the photon; $\eta_\pm$ and $\lambda_\pm$ depend on the specific emission kinematics, they are given in Table 3.

Table 3. **η_i and λ_i for different collinear kinematics**

η,λ	p_-p_-	q_-q_-	p_+p_+	q_+q_+	p_-p_+	q_-q_+	p_-q_-	p_+q_+	p_-q_+	p_+q_-
η_1	y	1	1	1	$1-x_1$	1	$1-x_1$	1	$1-x_1$	1
η_2	1	1	y	1	$1-x_2$	1	1	$1-x_1$	1	$1-x_1$
λ_1	1	$\frac{1}{y}$	1	1	1	$\frac{1}{1-x_1}$	$1+\frac{x_2}{y_1}$	1	1	$1+\frac{x_2}{y_1}$
λ_2	1	1	1	$\frac{1}{y}$	1	$\frac{1}{1-x_2}$	1	$1+\frac{x_2}{y_2}$	$1+\frac{x_2}{y_2}$	1

The columns of Table 3 correspond to a certain choice of the kinematics in the following way: p_-p_- means the emission of both the photons along the projectile electron, p_+q_- means that the first of the photons goes along the projectile positron; the second — along the scattered electron, and so on. The contributions from 6 remaining kinematical regions (when the photons in the last 6 columns are interchanged) could be found by the simple substitution $x_1 \leftrightarrow x_2$. We will use the momentum conservation law

$$\eta_1 p_- + \eta_2 p_+ = \lambda_1 q_- + \lambda_2 q_+, \tag{1.185}$$

and the following relations coming from it:

$$\eta_1 + \eta_2 = \lambda_1 y_1 + \lambda_2 y_2,\quad \lambda_1 y_1 \sin\theta_- = \lambda_2 y_2 \sin\theta_+,\quad y_{1,2} = \frac{q_{1,2}^0}{\varepsilon},$$
$$\lambda_2 y_2 = \frac{\eta_1^2 + \eta_2^2 + (\eta_2^2 - \eta_1^2)c}{\eta_1 + \eta_2 + (\eta_2 - \eta_1)c},\quad c = \cos\theta_-,\quad y = 1 - x_1 - x_2. \tag{1.186}$$

Each of 16 contributions to the cross section of process (1.175) can be expressed in terms of the corresponding Born-like cross section multiplied by its collinear factor:

$$d\sigma_{\rm coll} = \frac{1}{2!}\left(\frac{\alpha}{2\pi}\right)^2 \frac{x_1 x_2}{2} \sum_{(\eta,\lambda)} \overline{K}(\eta,\lambda) d\tilde{\sigma}_0(\eta,\lambda) dx_1\, dx_2,$$

$$d\tilde{\sigma}_0(\eta,\lambda) = \frac{2\alpha^2}{s}\, B(\eta,\lambda)\, dI(\eta,\lambda), \quad B(\eta,\lambda) = \left(\frac{\tilde{s}^2+\tilde{t}^2+\tilde{s}\tilde{t}}{\tilde{s}\tilde{t}}\right)^2,$$

$$dI_i(\eta,\lambda) = \int \frac{d^3q_- d^3q_+}{q_-^0 q_+^0} \delta^{(4)}(\eta_1 p_- + \eta_2 p_+ - \lambda_1 q_- - \lambda_2 q_+) = \\ = \frac{4\pi\eta_1\eta_2\, dc}{\lambda_1^2\lambda_2^2[c(\eta_2-\eta_1)+\eta_1+\eta_2]^2}, \quad (1.187)$$

$$\overline{K}(\eta,\lambda) = m^4 \int\limits_0^{z_0} dz_1 \int\limits_0^{z_0} dz_2 \int\limits_0^{2\pi} \frac{d\varphi}{2\pi} \mathcal{K}(\eta,\lambda),$$

$$\tilde{t} = (\eta_1 p_- - \lambda_1 q_-)^2 = -\tilde{s}\frac{\eta_1(1-c)}{\eta_1+\eta_2+(\eta_2-\eta_1)c},$$

$$\tilde{s} = (\eta_1 p_- + \eta_2 p_+)^2 = 4\varepsilon^2\eta_1\eta_2 = s\eta_1\eta_2, \quad \tilde{s}+\tilde{t}+\tilde{u}=0.$$

The sum over (η,λ) means the sum over 16 collinear kinematical regions. The corresponding (η,λ) could be found in Table 3. The quantities $\mathcal{K}_i(\eta,\lambda)$ are as follows:

$$\mathcal{K}(p_-p_-) = \frac{2}{y}\mathcal{A}(A_1,A_2,A,x_1,x_2,y),$$

$$\mathcal{K}(q_-q_-) = 2y\mathcal{A}\left(B_1,B_2,B,\frac{-x_1}{y},\frac{-x_2}{y},\frac{1}{y}\right),$$

$$\mathcal{K}(p_+p_+) = \frac{2}{y}\mathcal{A}(C_1,C_2,C,x_1,x_2,y),$$

$$\mathcal{K}(q_+q_+) = 2y\mathcal{A}\left(D_1,D_2,D,\frac{-x_1}{y},\frac{-x_2}{y},\frac{1}{y}\right),$$

$$\mathcal{A}(A_1,A_2,A,x_1,x_2) = -\frac{yA_2}{A^2A_1} - \frac{yA_1}{A^2A_2} + \frac{1+y^2}{x_1x_2A_1A_2} + \frac{r_1^3+yr_2}{AA_1x_1x_2} + \\ + \frac{r_2^3+yr_1}{AA_2x_1x_2} + \frac{2m^2(y^2+r_1^2)}{AA_1^2x_2} + \frac{2m^2(y^2+r_2^2)}{AA_2^2x_1}, \quad (1.188)$$

$$\mathcal{K}(p_-p_+) = 2K_1K_2, \quad \mathcal{K}(p_-q_+) = -2K_1K_3, \quad \mathcal{K}(p_+q_-) = -2K_4K_5,$$
$$\mathcal{K}(q_-q_+) = 2K_6K_7, \quad \mathcal{K}(p_-q_-) = -2K_1K_5, \quad \mathcal{K}(p_+q_+) = -2K_4K_3,$$

$$K_1 = \frac{g_1}{A_1x_1r_1} + \frac{2m^2}{A_1^2}, \quad K_2 = \frac{g_2}{C_2x_2r_2} + \frac{2m^2}{C_2^2}, \quad K_3 = \frac{g_4}{D_2x_2t_2} - \frac{2m^2}{D_2^2},$$
$$K_4 = \frac{g_1}{C_1x_1r_1} + \frac{2m^2}{C_1^2}, \quad K_5 = \frac{g_3}{B_2x_2t_1} - \frac{2m^2}{B_2^2}, \quad K_6 = \frac{g_1}{B_1x_1} - \frac{2m^2}{B_1^2},$$
$$K_7 = \frac{g_2}{D_2x_2} - \frac{2m^2}{D_2^2}, \tag{1.189}$$
$$r_1 = 1 - x_1, \quad r_2 = 1 - x_2,$$
$$g_1 = 1 + r_1^2, \quad g_2 = 1 + r_2^2,$$
$$g_3 = y_1^2 + t_1^2, \quad g_4 = y_2^2 + t_2^2,$$
$$t_1 = y_1 + x_2, \quad t_2 = y_2 + x_2,$$

y_1, , y_2 are the energy fractions of the scattered electron and positron defined in Eq. (1.186).

Expressions (1.189) agree with the results of paper [22] except for a simpler form of $\mathcal{K}(q_-q_+)$. As for Eq. (1.188) it has an evident advantage in comparison to the corresponding formulae given in paper [22]. Let us note that the remaining factors $\mathcal{K}(p, q)$ could be obtained from the ones given in Eq. (1.189) using relations of the following type:

$$\mathcal{K}(p_-q_-)(x_1, x_2, A_1, B_2) = \mathcal{K}(q_-p_-)(x_2, x_1, A_2, B_1). \tag{1.190}$$

Note also that terms of the kind $m^4/(B_2^2C_1^2)$ do not give logarithmic enhanced contributions, and we will omit them below. The denominators of the propagators entering into Eqs. (1.188), (1.189) are:

$$\begin{aligned} A_i &= (p_- - k_i)^2 - m^2, & A &= (p_- - k_1 - k_2)^2 - m^2, \\ B_i &= (q_- + k_i)^2 - m^2, & B &= (q_- + k_1 + k_2)^2 - m^2, \\ C_i &= (k_i - p_+)^2 - m^2, & C &= (k_1 + k_2 - p_+)^2 - m^2, \\ D_i &= (q_+ + k_i)^2 - m^2, & D &= (q_+ + k_1 + k_2)^2 - m^2. \end{aligned} \tag{1.191}$$

For further integration it is useful to rewrite the denominators in terms of the photon energy fractions $x_{1,2}$ and their emission angles. In the case of the emission of both the photons along p_- we would have

$$\begin{aligned} \frac{A}{m^2} &= -x_1(1 + z_1) - x_2(1 + z_2) + x_1x_2(z_1 + z_2) + 2x_1x_2\sqrt{z_1z_2}\cos\varphi, \\ \frac{A_i}{m^2} &= -x_i(1 + z_i), \end{aligned} \tag{1.192}$$

where $z_i = (\varepsilon\theta_i/m)^2$, φ is the azimuthal angle between the planes containing the space vector pairs $(\mathbf{p}_-, \mathbf{k}_1)$ and $(\mathbf{p}_-, \mathbf{k}_2)$. In the same way one can obtain in the case $k_1, k_2 \| q_-$:

$$\frac{B}{m^2} = \frac{x_1}{y_1}(1 + y_1^2 z_1) + \frac{x_2}{y_1}(1 + y_1^2 z_2) + x_1 x_2 (z_1 + z_2) + 2x_1 x_2 \sqrt{z_1 z_2}\cos\varphi,$$
$$\frac{B_i}{m^2} = \frac{x_i}{y_1}(1 + y_1^2 z_i). \tag{1.193}$$

Then we perform the elementary azimuthal angle integration and the integration over z_1, z_2 within the logarithmical accuracy [29]:

$$\overline{a} = m^4 \int\limits_0^{z_0} dz_1 \int\limits_0^{z_0} dz_2 \int\limits_0^{2\pi} \frac{d\varphi}{2\pi} a. \tag{1.194}$$

By using the relevant integrals

$$\overline{\frac{A_2}{A^2 A_1}} = \frac{L_0}{x_1 x_2 r_1^2}\left[\frac{1}{2}L_0 + \ln\frac{x_2 r_1^2}{x_1 y} - 1 + \frac{x_1 x_2}{y}\right],$$
$$\overline{\frac{1}{A A_1}} = \frac{L_0}{x_1 x_2 r_1}\left[\frac{1}{2}L_0 + \ln\frac{x_2 r_1^2}{x_1 y}\right], \quad \overline{\frac{m^2}{A A_1^2}} = -\frac{L_0}{x_1^2 x_2 r_1},$$
$$\overline{\frac{1}{A_1 A_2}} = \frac{L_0^2}{x_1 x_2}, \quad \overline{\frac{1}{A_1 B_2}} = -\frac{L_0}{y_1 x_1 x_2}(L_0 + 2\ln y_1), \tag{1.195}$$
$$L_0 = \ln z_0 \equiv L + l, \quad l = \ln\left(\frac{\theta_0^2}{4}\right), \quad L = \ln\left(\frac{4\varepsilon^2}{m^2}\right),$$

where θ_0 is the collinear parameter, we obtain the differential cross section in the collinear region (the remaining integrals could be obtained by simple substitutions defined in Eq. (1.191):

$$d\sigma_{\text{coll}} = \frac{\alpha^4 L}{4\pi^2 s}\frac{d^3 q_+\, d^3 q_-}{q_+^0 q_-^0}\frac{dx_1\, dx_2}{x_1 x_2}(1 + \mathcal{P}_{1,2})\left\{\frac{1}{y r_1^2}\left[\frac{1}{2}(L + 2l) g_1 g_5 + \right.\right.$$
$$\left. + (y^2 + r_1^4)\ln\frac{x_2 r_1^2}{x_1 y} + x_1 x_2 (y - x_1 x_2) - 2 r_1 g_5\right][B_{p_- p_-}\delta_{p_- p_-} + B_{p_+ p_+}\delta_{p_+ p_+}] +$$
$$+ \frac{1}{y r_1^2}\left[\frac{1}{2}(L + 2l + 4\ln y) g_1 g_5 + (y^2 + r_1^4)\ln\frac{x_1 r_1^2}{x_2 y} + x_1 x_2 (y - x_1 x_2) - 2 r_1 g_1\right] \times$$

$$\times [B_{q_-q_-}\delta_{q_-q_-} + B_{q_+q_+}\delta_{q_+q_+}] + B_{p_-p_+}\delta_{p_-p_+}\left[(L+2l)\frac{g_1g_2}{r_1r_2} - 2\frac{g_1}{r_1} - 2\frac{g_2}{r_2}\right] +$$
$$+ B_{q_-q_+}\delta_{q_-q_+}\left[(L+2l+2\ln(r_1r_2))\frac{g_1g_2}{r_1r_2} - 2\frac{g_1}{r_1} - 2\frac{g_2}{r_2}\right] +$$
$$+ [B_{p_-q_-}\delta_{p_-q_-} + B_{p_+q_-}\delta_{p_+q_-}]\left[(L+2l+2\ln y_1)\frac{g_1g_3}{r_1y_1t_1} - 2\frac{g_1}{r_1} - 2\frac{g_3}{y_1t_1}\right] +$$
$$+ [B_{p_+q_+}\delta_{p_+q_+} + B_{p_-q_+}\delta_{p_-q_+}]\left[(L+2l+2\ln y_2)\frac{g_1g_4}{r_1y_2t_2} - 2\frac{g_1}{r_1} - 2\frac{g_4}{y_2t_2}\right]\Bigg\}. \quad (1.196)$$

We used the symbol $\mathcal{P}_{1,2}$ for the interchange operator ($\mathcal{P}_{1,2}f(x_1,x_2) = f(x_2,x_1)$) and $g_5 = y^2 + r_1^2$. Other variables are defined in Eq. (1.189). Delta function $\delta_{p,q}$ corresponds to the specific conservation law of the kinematical situation defined by the pair p, q (see Table 1): $\delta_{p,q} = \delta^{(4)}(\eta_2 p_+ + \eta_1 p_- - \lambda_1 q_- - \lambda_2 q_+)$. Besides, we imply that the first photon is emitted along the momentum p; and the second, along the momentum q (p, $q = p_-$, p_+, q_-, q_+). These δ functions could be taken into account in the integration as is made in the expression for $dI(\eta,\lambda)$ (see Eq. (1.187)). Finally, we define

$$B_{p,q} = \left(\frac{\eta_2 s}{\lambda_1 t} + \frac{\lambda_1 t}{\eta_2 s} + 1\right)^2, \quad t = (p_- - q_-)^2. \quad (1.197)$$

1.6.2. Contribution of the Semicollinear Region. We will suggest for definiteness that the photon with momentum k_2 moves inside a narrow cone along the momentum direction of one of the charged particles, while the other photon moves in any direction outside that cone along any charged particle. This choice allows us to omit the statistical factor $1/2!$. The quasi-real electron method [6, 7] may be used to obtain the cross section:

$$d\sigma^{\rm sc} = \frac{\alpha^4}{32s\pi^4}\frac{d^3q_-\,d^3q_+\,d^3k_1}{q_-^0q_+^0k_1^0}V\frac{d^3k_2}{k_2^0}\left\{\frac{\mathcal{K}_{p_-}}{p_-k_2}\delta_{p_-}R_{p_-} + \right.$$
$$\left. + \frac{\mathcal{K}_{p_+}}{p_+k_2}\delta_{p_+}R_{p_+} + \frac{\mathcal{K}_{q_-}}{q_-k_2}\delta_{q_-}R_{q_-} + \frac{\mathcal{K}_{q_+}}{q_+k_2}\delta_{q_+}R_{q_+}\right\}. \quad (1.198)$$

We omitted the terms of the kind $m^2/(p_-k_2)^2$ in Eq. (1.198) because their contribution does not contain the large logarithm L. The quantities entering into Eq. (1.198) are given by

$$V = \frac{s}{k_1p_+k_1p_-} + \frac{s'}{k_1q_+k_1q_-} - \frac{t'}{k_1p_+k_1q_+} - \frac{t}{k_1p_-k_1q_-} +$$
$$+ \frac{u'}{k_1p_+k_1q_-} + \frac{u}{k_1q_+k_1p_-}. \quad (1.199)$$

V is the known accompanying radiation factor; $\mathcal{K}_i$ are the single-photon emission collinear factors:

$$\mathcal{K}_{p_-}=\mathcal{K}_{p_+}=\frac{g_2}{x_2 r_2},\quad \mathcal{K}_{q_-}=\frac{y_1^2+(y_1+x_2)^2}{x_2(y_1+x_2)},\quad \mathcal{K}_{q_+}=\frac{y_2^2+(y_2+x_2)^2}{x_2(y_2+x_2)}. \tag{1.200}$$

Quantities R_i read:

$$R_{p_-}=R[sr_2,tr_2,ur_2,s',t',u'],\qquad R_{p_+}=R[sr_2,t,u,s',t'r_2,u'r_2], \tag{1.201}$$
$$R_{q_-}=R\left[s,t\frac{t_1}{y_1},u,s'\frac{t_1}{y_1},t',u'\frac{t_1}{y_1}\right],\qquad R_{q_+}=R\left[s,t,u\frac{t_2}{y_2},s'\frac{t_2}{y_2},t'\frac{t_2}{y_2},u'\right],$$

with R:

$$R[s,t,u,s',t',u']=\frac{1}{ss'tt'}\left[ss'(s^2+s'^2)+tt'(t^2+t'^2)+uu'(u^2+u'^2)\right],$$
$$s=(p_++p_-)^2,\quad s'=(q_++q_-)^2,\quad t=(p_--q_-)^2, \tag{1.202}$$
$$t'=(p_+-q_+)^2,\quad u=(p_--q_+)^2,\quad u'=(p_+-q_-)^2.$$

Finally, we define

$$\begin{aligned}
\delta_{p_-}&=\delta^{(4)}(p_-r_2+p_+-q_+-q_--k_1),\\
\delta_{p_+}&=\delta^{(4)}(p_-+p_+r_2-q_+-q_--k_1),\\
\delta_{q_-}&=\delta^{(4)}\left(p_-+p_+-q_+-q_-\frac{y_1+x_2}{y_1}-k_1\right),\\
\delta_{q_+}&=\delta^{(4)}\left(p_-+p_+-q_+\frac{y_2+x_2}{y_2}-q_--k_1\right).
\end{aligned} \tag{1.203}$$

Performing the integration over angular variables of the collinear photon we obtain

$$d\sigma^{\rm sc}=\frac{\alpha^4 L}{16s\pi^3}\,\frac{d^3q_-\,d^3q_+\,d^3k_1}{q_-^0\,q_+^0\,k_1^0}\,dx_2 V\Bigg\{\mathcal{K}_{p_-}[R_{p_-}\delta_{p_-}+R_{p_+}\delta_{p_+}]+ + \frac{1}{y_2}\mathcal{K}_{q_+}R_{q_+}\delta_{q_+}+\frac{1}{y_1}\mathcal{K}_{q_-}R_{q_-}\delta_{q_-}\Bigg\}. \tag{1.204}$$

To see that the sum of cross sections (1.196) and (1.204)

$$d\sigma^{\gamma\gamma}=d\sigma^{\rm coll}+\int dO_1\left(\frac{d\sigma^{\rm sc}}{dO_1}\right) \tag{1.205}$$

does not depend on the auxiliary parameter θ_0. We verify that terms Ll from Eq. (1.196) cancel out with the terms

$$L\frac{k_1^0 q_i^0}{2\pi}\int\frac{dO_1}{k_1 q_i}\approx -Ll, \tag{1.206}$$

which arise from 16 regions in the semicollinear kinematics.

1.7. Second-Order Contributions to Elastic Large-Angle Bhabha Scattering. In this Subsection we put some NLO results concerning 2-loop contributions to the vertex functions, the contributions coming from the squares of 1-loop Feynman amplitudes which correspond to both the vertex- and box-type Feynman diagrams and the expressions of contributions of two soft photons emission processes. In this Subsection we do not consider the effects of vacuum polarization inserted into the virtual photon Green function since it was examined earlier in [15], see also Subsec. 1.2.

NLO virtual photonic corrections were also found earlier in [40]. Recently the complete calculation of two-loop photonic corrections to elastic Bhabha scattering was fulfilled in [39, 41, 42].

1.7.1. Two-Loop Vertex Contribution. The corresponding Feynman diagrams up to two-loop level are depicted in Fig. 8 (there are four more diagrams coming from cross channels to Fig. 8, g, h, i, j). We use the following asymptotes of the fermion vertex function in the case of space-like and time-like 4-vectors of virtual photons [10]:

$$\Gamma_\mu(q^2) = \gamma_\mu\left[1 + \frac{\alpha}{\pi}\Gamma^{(2)}(q^2) + \left(\frac{\alpha}{\pi}\right)^2\Gamma^{(4)}(q^2)\right],$$

where

$$q^2 = s > 0 \quad \text{or} \quad q^2 = t < 0,$$

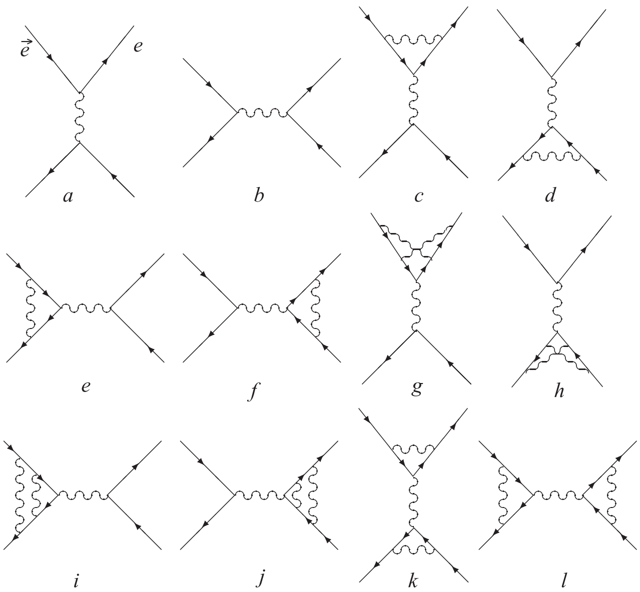

Fig. 8. Vertex diagrams up to 2-loop level

$$\Gamma^{(2)}(s) = \frac{L-1}{2}L_\lambda - \frac{1}{4}L^2 + \frac{3}{4}L + \frac{\pi^2}{3} - 1 + i\pi\left(\frac{1}{2}L - \frac{1}{2}L_\lambda - \frac{3}{4}\right),$$

$$\Gamma^{(2)}(t) = (L_t - 1)\left(\frac{1}{2}L_\lambda + 1\right) - \frac{1}{4}L_t^2 - \frac{1}{4}L_t + \frac{\pi^2}{12},$$

$$\mathrm{Re}\,\Gamma^{(4)}(s) = \frac{1}{8}(L^2 - 2L + 1 - \pi^2)L_\lambda^2 + \frac{1}{2}L_\lambda\Big[(L-1)\Big(-\frac{1}{4}L^2 + \frac{3}{4}L - 1 + \frac{\pi^2}{3}\Big) + \pi^2\Big(\frac{1}{2}L - \frac{3}{4}\Big)\Big] + \frac{1}{32}L^4 - \frac{3}{16}L^3 + \left(\frac{17}{32} - \frac{5\pi^2}{24}\right)L^2 + \left(-\frac{21}{32} + \frac{3}{2}\zeta(3) + \frac{17\pi^2}{36}\right)L + \mathcal{O}(1), \quad (1.207)$$

$$\Gamma^{(4)}(t) = \frac{1}{32}L_t^4 - \frac{3}{16}L_t^3 + \left(\frac{17}{32} - \frac{\pi^2}{48}\right)L_t^2 + \left(-\frac{21}{32} - \frac{\pi^2}{16} + \frac{3}{2}\zeta(3)\right)L_t + \frac{1}{8}L_\lambda^2(L_t-1)^2 + \frac{1}{2}L_\lambda(L_t - 1)\left(-\frac{1}{4}L_t^2 + \frac{3}{4}L_t - 1 + \frac{\pi^2}{12}\right) + \mathcal{O}(1),$$

$$L = \ln\frac{s}{m^2}, \quad L_t = \frac{-t}{m^2}, \quad L_\lambda = \ln\frac{\lambda^2}{m^2}. \quad \zeta(3) \approx 1.2020569.$$

In these formulae we have retained only the Dirac form factor of electron and dropped the Pauli one, since its contribution is suppressed by the factor of m^2/s.

The second order PT contribution to the matrix element squared reads

$$\Delta|M|^2 = 2(M_a + M_b)^*(M_{2\text{-vertex}}) + |M_c + M_d + M_e + M_f + M_m + M_n + M_p + M_q|^2, \quad (1.208)$$

where $M_{2\text{-vertex}}$ is the matrix element of ten 2-loop vertex Feynman diagrams. The matrix element of elastic Bhabha scattering, including relevant contributions up to the 2-loop level, can be written in the following form:

$$M = M_{0t}(1 + \delta_t^{(1)} + \delta_t^{(2)}) - M_{0s}(1 + \delta_s^{(1)} + \delta_s^{(2)}) + B_1^{(1)} + B_2^{(1)} - B_3^{(1)} - B_4^{(1)} + B^{(2)}, \quad (1.209)$$

where

$$M_{0t} = \frac{4\pi\alpha i}{t}\bar{u}(p_1')\gamma_\mu u(p_1)\bar{v}(p_2)\gamma_\mu v(p_2'),$$
$$M_{0s} = \frac{4\pi\alpha i}{t}\bar{v}(p_2')\gamma_\mu u(p_1)\bar{u}(p_1')\gamma_\mu v(p_2).$$

The quantities $B_i^{(1)}$ correspond to the 1-loop box-type diagrams (see Fig. 9, *m–q*), whereas $B^{(2)}$ comes from the 2-loop ones (some representatives

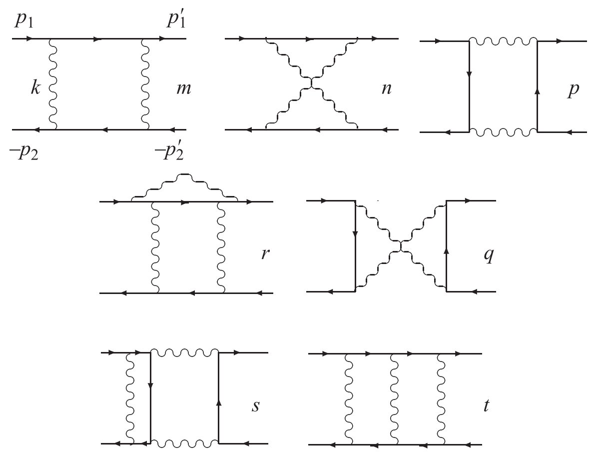

Fig. 9. Box diagrams up to 2-loop level

are drawn in Fig. 9, *r–t*). At the Born level we have

$$\sum |M_{0t}^2| = (4\pi\alpha)^2 \frac{8}{t^2}(s^2+u^2), \quad \sum |M_{0s}^2| = (4\pi\alpha)^2 \frac{8}{s^2}(t^2+u^2),$$
$$\sum M_{0s}M_{0t}^* = -(4\pi\alpha)^2 8\frac{u^2}{st}, \ \sum |M_{0t}-M_{0s}|^2 = 16(4\pi\alpha)^2\left(\frac{s}{t}+\frac{t}{s}+1\right)^2, \tag{1.210}$$
$$s=(p_1+p_2)^2, \qquad t=(p_1-p_1')^2, \qquad u=(p_1-p_2')^2,$$
$$p_1+p_2=p_1'+p_2', \qquad p_{1,2}^2 = p_{1,2}'^2 = m^2.$$

The quantities $\delta_t^{(1)}, \delta_t^{(2)}$ are real. They read

$$\delta_t^{(1)} = \frac{\alpha}{\pi}\left[2\Gamma^{(2)}(t) + \Pi^{(2)}(t)\right],$$
$$\delta_t^{(2)} = \left(\frac{\alpha}{\pi}\right)^2 [(\Gamma^{(2)}(t))^2 + 2\Pi^{(2)}(t)\Gamma^{(2)}(t) + (\Pi^{(2)}(t))^2 + \Pi^{(4)}(t) + 2\Gamma^{(4)}(t)]. \tag{1.211}$$

Here $\Pi^{(2,4)}(t)$ are the vacuum polarization insertions. Similar expressions are held for $\delta_s^{(1)}, \delta_s^{(2)}$ and can be derived from (1.211) by using crossing relations (relevant quantities are, in general, complex valued).

The relevant second order PT contribution to the matrix element, squared and summed over spin states, can be rewritten as follows:

$$\Delta \sum |M|^2 = \alpha^2 \left(\frac{\alpha}{\pi}\right)^2 (\Delta_1+\Delta_2+\Delta_3+\Delta_4), \tag{1.212}$$

where

$$\frac{\alpha^4}{\pi^2}\Delta_1 = \sum |M_{0t}|^2(|\delta_t^{(1)}|^2 + 2\delta_t^{(2)}) + \sum |M_{0s}|^2((\delta_s^{(1)})^2 + 2\,\mathrm{Re}\,\delta_s^{(2)}) - \\ - 2\,\mathrm{Re}\sum M_{0s}^* M_{0t}(\delta_t^{(1)}\delta_s^{(1)} + \delta_s^{(2)} + \delta_t^{(2)}),$$

$$\frac{\alpha^4}{\pi^2}\Delta_2 = 2\,\mathrm{Re}\sum (M_{0t}\delta_t^{(1)} - M_{0t}\delta_t^{(1)})^*(B_1^{(1)} + B_2^{(1)} - B_3^{(1)} - B_4^{(1)}), \quad (1.213)$$

$$\frac{\alpha^4}{\pi^2}\Delta_3 = \sum |B_1^{(1)} + B_2^{(1)} - B_3^{(1)} - B_4^{(1)}|^2,$$

$$\frac{\alpha^4}{\pi^2}\Delta_4 = 2\mathrm{Re}\sum (M_{0t} - M_{0s})^* B^{(2)}.$$

Quantity $B^{(2)}$, which enters into the definition of Δ_4, in this Subsection is not calculated. Here we put explicit expressions only for Δ_1, Δ_2, and Δ_3. The first term Δ_1 was given above. As for Δ_2, using the usual notation $\hat{a} = \gamma_\mu a^\mu$, it can be cast down in the form

$$\Delta_2 = (1 + \mathcal{P}_{st})\,\mathrm{Re}\,\frac{\Gamma^{(2)}(t)}{t} \times \\ \times \int \frac{d^4k}{i\pi^2}\left\{ \frac{\mathrm{Tr}\,(\gamma_\mu(\hat{p}_1 + \hat{k})\gamma_\nu \hat{p}_1 \gamma_\rho \hat{p}_1')\,\mathrm{Tr}\,(\gamma_\mu(-\hat{p}_2 + \hat{k})\gamma_\nu \hat{p}_2 \gamma_\rho \hat{p}_2')}{A(p_1+k)A(k)A(k-q)A(-p_2+k)} + \right. \\ + \frac{\mathrm{Tr}\,(\gamma_\mu(\hat{p}_1 + \hat{k})\gamma_\nu \hat{p}_1 \gamma_\rho \hat{p}_1')\,\mathrm{Tr}\,(\gamma_\mu(-\hat{p}_2' - \hat{k})\gamma_\nu \hat{p}_2' \gamma_\rho \hat{p}_2)}{A(p_1+k)A(k)A(k-q_1)A(p_2'+k)} + \\ + \frac{2\mathrm{Tr}\,(\hat{p}_2\hat{p}_1'(\hat{p}_1 + \hat{k})\hat{p}_2'\hat{p}_1(\hat{p}_1' + \hat{k}))}{A(p_1+k)A(k)A(k-q_1)A(p_1'+k)} + \\ \left. + \frac{2u^2(p_1+k)(-p_2'-k)}{A(p_1+k)A(k)A(k-q_1)A(p_2'+k)} \right\} + \delta\Delta_2,$$

$$A(p_{1,2} \pm k) = (p_{1,2} \pm k)^2 - m^2, \qquad A(p_{1,2}' \pm k) = (p_{1,2}' \pm k)^2 - m^2,$$
$$A(q - k) = (q - k)^2 - \lambda^2, \qquad A(q_1 - k) = (q_1 - k)^2 - \lambda^2,$$

$$q = p_1' - p_1, \quad q_1 = -p_1 - p_2,$$

where the permutation operator $\mathcal{P}_{st}$ acts as follows:

$$\mathcal{P}_{st}A(s,t,u,L,L_t) = A(t,s,u,L_t,L).$$

Calculating the first term in Δ_2, we have to put

$$L_s = L, \qquad \psi_{1s} = \frac{1}{s}\left(\frac{1}{2}L^2 + \frac{\pi^2}{6}\right). \quad (1.214)$$

The second term in the right-hand side (rhs) of Eq. (1.214) $\delta\Delta_2$ arises from the product of imaginary parts of $(\Gamma^{(2)}(s))^*$ and box structures (see Eq. (1.213)). It can be obtained by applying the following rules:

$$\begin{gathered}\delta\,\mathrm{Re}\,\Gamma^{(2)}(s)^* L_s = -\pi^2\left(\frac{1}{2}L - \frac{1}{2}L_\lambda - \frac{3}{4}\right),\\ \delta\,\mathrm{Re}\,\Gamma^{(2)}(s)^*\psi_{1s} = -\frac{\pi^2}{s}L\left(\frac{1}{2}L - \frac{1}{2}L_\lambda - \frac{3}{4}\right),\\ \delta\,\mathrm{Re}\,L_s^* L_s = \pi^2, \quad \delta\,\mathrm{Re}\,L_s^*\psi_{1s} = \frac{\pi^2 L}{s}, \quad \delta\,\mathrm{Re}\,\psi_{1s}^*\psi_{1s} = \frac{\pi^2 L^2}{s^2}.\end{gathered} \tag{1.215}$$

By performing the loop-momentum integration one arrives at the result (see Eq. (1.218)).

Consider now Δ_3. The symmetry properties permit us to express it in the form

$$\begin{gathered}\Delta_3 = |B_1 + B_2 - B_3 - B_4|^2 = [1 + \mathcal{P}_{su} + (1 + \mathcal{P}_{tu})\mathcal{P}_{st}]|B_1|^2 + \\ + 2(1 + \mathcal{P}_{st})\left[B_1 B_2^* - B_2 B_3^* - \frac{1}{2}B_1 B_3^* - \frac{1}{2}B_2 B_4^*\right] + \delta\Delta_3,\end{gathered} \tag{1.216}$$

$$\begin{gathered}\mathcal{P}_{su}A(s,t,u,L,L_t) = A(u,t,s,L_u,L_t),\\ \mathcal{P}_{tu}A(s,t,u,L,L_t) = A(s,u,t,L,L_u).\end{gathered}$$

The quantity $\delta\Delta_3$ is to be written according to the rules mentioned earlier (1.215). In calculations of the first two terms in Δ_3 we have to take L_s and ψ_{1s} as in (1.214). The remaining contributions to Δ_3 are

$$\begin{gathered}|B_1|^2 = \int \frac{d^4k_1}{i\pi^2}\int \frac{d^4k}{i\pi^2}\frac{\mathrm{Tr}\,[\hat{p}_1'\gamma_\mu(\hat{p}_1 + \hat{k})\gamma_\nu\hat{p}_1\gamma_\xi(\hat{p}_1 + \hat{k}_1)\gamma_\eta]}{A(p_1 + k_1)A(k_1)A(k_1 - q)A(-p_2 + k_1)} \times \\ \times \frac{\mathrm{Tr}\,[\hat{p}_2\gamma_\nu(-\hat{p}_2 - \hat{k})\gamma_\mu\hat{p}_2'\gamma_\xi(-\hat{p}_2' + \hat{k}_1)\gamma_\eta]}{A(p_1 + k)A(k)A(k - q)A(p_2 - k)}\end{gathered} \tag{1.217}$$

and similar expressions for $B_i B_j^*$ (see [27]). Standard but rather tedious computation gives the following result:

$$\begin{gathered}\Delta_i = L_\lambda^2(a_{i1}L^2 + a_{i2}L) + L_\lambda(a_{i3}L^3 + a_{i4}L^2 + a_{i5}L) + \\ + a_{i6}L^4 + a_{i7}L^3 + a_{i8}L^2 + a_{i9}L, \quad i = 1,2,3.\end{gathered} \tag{1.218}$$

Coefficients a_{ij} are functions of $\theta = \widehat{\mathbf{p}_1, \mathbf{p}_1'}$. They look somewhat cumbersome and are not written here.

1.7.2. Emission of Soft Photons. Second order corrections to the 1-loop virtual photon emission corrected cross section, which arise from emission of a single real soft photon having energy less than $\Delta\varepsilon$, can be written down in the factorized form:

$$\frac{d\sigma^{SV}}{d\sigma_0} = \frac{\alpha}{\pi}\delta_S\frac{\alpha}{\pi}\delta_V = \left(\frac{\alpha}{\pi}\right)^2\Delta_{SV}, \quad d\sigma_0 = \frac{\alpha^2}{s}\left(\frac{1-\chi+\chi^2}{\chi}\right)^2,$$
$$\chi = \frac{1}{2}(1-\cos\theta) = \sin^2\frac{\theta}{2}, \quad \theta = (\widehat{\mathbf{p}_1, \mathbf{p}_1'}), \quad \chi = \frac{-t}{s}, \; 1-\chi = \frac{-u}{s},$$

$$\delta_S = 4\ln\frac{m\Delta\varepsilon}{\lambda\varepsilon}\left(L - 1 + \ln\frac{\chi}{1-\chi}\right) + L^2 + 2L\ln\frac{\chi}{1-\chi} + \ln^2\chi - \ln^2(1-\chi) - \\ - \frac{2\pi^2}{3} + 2\mathrm{Li}_2(1-\chi) - 2\mathrm{Li}_2(\chi),$$

$$\delta_V = 4\ln\frac{m}{\lambda}\left(1 - L + \ln\frac{1-\chi}{\chi}\right) - L^2 + 2L\ln\frac{1-\chi}{\chi} - \ln^2\chi + \ln^2(1-\chi) + \\ + 3L - 4 + f(\chi), \quad (1.219)$$

$$f(\chi) = (1-\chi+\chi^2)^{-2}\left[\frac{\pi^2}{12}(-4+8\chi+3\chi^2-10\chi^3+8\chi^4) + \frac{1}{2}(-2+5\chi - \right. \\ - 7\chi^2 + 5\chi^3 - 2\chi^4)\ln^2(1-\chi) + \frac{1}{4}\chi(3-\chi-3\chi^2+4\chi^3)\ln^2\chi + \\ + \frac{1}{6}(22-30\chi+33\chi^2-11\chi^3)\ln\chi - \frac{1}{2}\chi(1+\chi^2)\ln(1-\chi) + \\ \left. + \frac{1}{2}(4-8\chi+7\chi^2-2\chi^3)\ln\chi\,\ln(1-\chi)\right].$$

The virtual corrections due to vacuum polarization are not taken into account in the expression for δ_V. They give an additional contribution to the latter that looks like

$$\delta_{V\Pi} = \frac{2}{3}L - \frac{10}{9} - \frac{1}{3}(1-\chi+\chi^2)^{-2}(2-3\chi+3\chi^2-\chi^3)\ln\chi. \quad (1.220)$$

Consideration of emission of two soft photons having total energy $\omega_1 + \omega_2 \leqslant \Delta\varepsilon$ requires some caution. The final result has the following form:

$$\frac{d\sigma^{SS}}{d\sigma_0} = \frac{1}{2!}\left(\frac{\alpha}{\pi}\right)^2\left[\delta_S^2 - \frac{8}{3}\pi^2\left(L - 1 + \ln\frac{\chi}{1-\chi}\right)^2\right] \equiv \left(\frac{\alpha}{\pi}\right)^2\Delta_{SS}. \quad (1.221)$$

Note that in the case photons are emitted independently, i.e., $\omega_1 \leqslant \Delta\varepsilon$ and $\omega_2 \leqslant \Delta\varepsilon$, the second term in square brackets will be absent. The multiplier $1/2!$ is due to the identity of the photons.

1.7.3. Total Sum of Two-Loop Corrections. What one would expect from the real 2-box amplitudes contribution is that the total correction must be free of infrared divergences supplying by cancellation of the fourth and third power of large logarithms. Here we put the 2-loop real and virtual photon contribution:

$$\Delta_{SV} + \Delta_{SS} = F^2 \Bigg\{ L_\lambda^2 L^2(-2) + L_\lambda^2 L 4\left(1 - \ln\frac{\chi}{1-\chi}\right) + \\ + L_\lambda L^3 2 + L_\lambda L^2\left(-8 + 6\ln\frac{\chi}{1-\chi}\right) + L_\lambda L a_1 - \\ - L^4\frac{1}{2} + L^3\left(3 - 2\ln\frac{\chi}{1-\chi}\right) + L^2 a_2 + Lb + \quad (1.222) \\ + \ln\frac{\Delta\varepsilon}{\varepsilon}\left(12L^2 + z_1 L\right) + \ln^2\frac{\Delta\varepsilon}{\varepsilon}\left(8L^2 + z_2 L\right)\Bigg\} \quad (1.223)$$

and

$$\sum_{i=1}^{3}\Delta_i = F^2\Bigg[L_\lambda^2 L^2 2 + L_\lambda^2 L 4(L_{su} - L_{st} - 1) - L_\lambda L^3 2 + \\ + L_\lambda L^2\left(6L_{st} - 6L_{su} + \frac{28}{3}\right) + L^4\frac{1}{2} + \\ + L^3\left(2L_{su} - 2L_{st} - \frac{11}{3}\right)\Bigg] + L_\lambda L c_1 + L^2 c_2 + Ld, \quad (1.224)$$

with $F = \frac{s}{t} + \frac{t}{s} + 1$ and coefficients a_i, b, c_i, d, z_1 are given in [27].

Here Δ_{SS} and Δ_{SV} denote quasi-elastic contributions, coming from double soft and soft-virtual photons emission. It immediately follows that all the terms proportional to L^4, $L_\lambda^2 L^2$, $L_\lambda L^3$, and $L_\lambda^2 L$ disappear in the total sum. One must expect the cancellation of the third power of large logarithms as well as the rest of infrared singularities when contribution of 2-box diagrams will be taken into account. As for the terms containing $L^2\ln^2(\Delta\varepsilon/\varepsilon)$ and $L^2\ln(\Delta\varepsilon/\varepsilon)$, they are explicitly seen to agree with those which could be derived in the renormalization group approach. To show this, let us write down the expression for cross section according to the renormalization group:

$$\frac{d\sigma}{d\sigma_0} = \left(1 + \frac{\alpha}{2\pi}L\mathcal{P}_\Delta^{(1)} + \frac{1}{2!}\left(\frac{\alpha}{2\pi}L\right)^2\mathcal{P}_\Delta^{(2)}\right)^4, \\ \mathcal{P}_\Delta^{(1)} = 2\ln\Delta + \frac{3}{2}, \quad \mathcal{P}_\Delta^{(2)} = \left(2\ln\Delta + \frac{3}{2}\right)^2 - 4\frac{\pi^2}{6}, \quad \Delta = \frac{\Delta\varepsilon}{\varepsilon}, \quad (1.225)$$

and somewhat rewrite the main result of this paper:

$$\Delta_{SS} + \Delta_{SV} + \sum_{i=1}^{3} \Delta_i = F^2 L^2 \left[\frac{1}{2}\mathcal{P}_\Delta^{(2)} + \frac{3}{2}\mathcal{P}_\Delta^{(1)} \right] + F^2 \frac{4}{3}\left[L_\lambda L^2 - L^3 \right] + \\ + L_\lambda L[F^2 a_1 + c_1] + L^2 \left[F^2 \left(a_2 + 2\frac{\pi^2}{6} - \frac{9}{2} \right) + c_2 \right] + \\ + L\left[F^2(b + z_1 + z_2) + d \right]. \quad (1.226)$$

Then, one can immediately be convinced that indeed an agreement takes place. We expect the 2-boxes contribution to compensate the second, third, and fourth terms on rhs of Eq. (1.226) and to modify the fifth one.

2. TABLE OF INTEGRALS. ONE-LOOP FEYNMAN INTEGRALS

2.1. Integrals for Bhabha Scattering with Virtual and Soft Real Pair Production. The quantities b_i which enter the integral over the virtual 4-momentum in box diagrams (see Eq. (1.26)) reads:

$$b = \frac{2}{st}\tilde{l}_s(l_t - 2l_\lambda), \qquad b_1 = -\frac{2}{u}\left(\frac{1}{s}\tilde{l}_s(l_t - l_s) + \Psi_s + \Psi_t \right),$$
$$b_2 = \frac{1}{2}(b - b_1), \qquad b_3 = -\frac{1}{u}\left((t - s)b_1 + 4\left(\Psi_t - \frac{1}{t}l_t \right) - \frac{4}{s}\tilde{l}_s \right),$$
$$b_4 = b_1 - \frac{4}{st}l_t, \qquad b_5 = -\frac{s}{u}b_1 + \frac{2}{u}\left(\Psi_t - \frac{1}{t}l_t \right) - \frac{2}{su}\tilde{l}_s, \quad (2.1)$$
$$b_6 = \frac{s-u}{2u}b_1 + \frac{1}{2}b + \frac{t-u}{tus}\tilde{l}_s - \frac{1}{u}\left(\Psi_t - \frac{1}{t}l_t \right), \quad b_7 = -\frac{s}{4}b_1 + \frac{1}{2}\Psi_t,$$
$$\Psi_t = \frac{1}{t}\left(\frac{2\pi^2}{3} + \frac{1}{2}l_t^2 \right), \qquad \Psi_s = \frac{1}{s}\left(\frac{2\pi^2}{3} + \frac{1}{2}l_s^2 \right).$$

The integrals for the box diagrams with a vacuum polarization insertion in one-photon propagator (see notations of Eq. (1.32)) are:

$$I(ba_1a_2) = \frac{1}{3s}\,\mathrm{Re}\left\{ \frac{1}{6}\tilde{l}_s^3 - \frac{5}{6}\tilde{l}_s^2 + \tilde{l}_s\left(\frac{28}{9} + \frac{\pi^2}{6} \right) + \mathcal{O}(1) \right\},$$
$$I(aba_1) = \frac{1}{3t}\,\mathrm{Re}\left\{ \frac{1}{3}l_t^3 - \frac{5}{6}l_t^2 + \frac{\pi^2}{3}l_t + \mathcal{O}(1) \right\}. \quad (2.2)$$

The other integrals with three denominators can be obtained by the following substitutions:

$$I(aba_2) = I(aba_3) = I(aba_1),$$
$$I(ba_1a_3) = I(ba_1a_2)\big(\tilde{l}_s \to l_u,\ s \to u\big). \tag{2.3}$$

The integral with four denominators is given in Eq. (1.33).

Fortunately, within the logarithmic accuracy one can present the square of the matrix element in the form, where only coefficients f_0 and f_4 of the tensor integral with four denominators enter:

$$f_0 = -\frac{1}{2}\int\limits_0^1 \frac{dv\varphi(v)}{1-v^2}\int\limits_0^1 z\,dz\int\limits_0^1 y\,dy\int\limits_0^1 \frac{dx}{z(y^2P_x^2+(1-y)t)-(1-y)(t-M^2)},$$
$$f_4 = 2\frac{\partial f_0}{\partial s}. \tag{2.4}$$

One can see that both f_0 and f_4 have only the first power of large logarithm:

$$f_0 \approx -\frac{1}{6}\int\limits_{1-\sigma}^1 dv\iiint\frac{zy\,dz\,dy\,dx}{(1-v)[zy^2P_x^2-(1-y)(1-z)t]+2m_e^2(1-y)} =$$
$$= \frac{l_t}{6t}M\left(\frac{s}{t}\right),$$
$$f_4 \approx -\frac{l_t}{3t^2}M_1\left(\frac{s}{t}\right), \qquad \sigma \ll 1, \tag{2.5}$$

where we defined

$$M(\xi) = \int\limits_0^1\int\limits_0^1\int\limits_0^1\frac{zy\,dz\,dy\,dx}{zy^2x(1-x)\xi+(1-y)(1-z)},$$
$$M_1(\xi) = \int\limits_0^1\int\limits_0^1\int\limits_0^1\frac{z^2y^3x(1-x)\,dz\,dy\,dx}{[zy^2x(1-x)\xi+(1-y)(1-z)]^2}. \tag{2.6}$$

These quantities are convenient for numerical integration for $\xi > 0$; for $\xi < 0$ it is better to use the following expressions:

$$M(\xi) = \int\limits_0^1 f(x)\varphi(x,\xi)\,dx, \quad \xi < 0,$$
$$f(x) = \left(1-\frac{1}{\sqrt{1+4x}}\right)\ln x + \frac{2}{\sqrt{1+4x}}\ln\left(1+\frac{2}{1+\sqrt{1+4x}}\right),$$

$$\varphi(x,\xi) = \frac{\xi}{\Delta}\left[\frac{1}{x} + \frac{2}{\sqrt{\Delta}}\ln\left|\frac{-\xi+\sqrt{\Delta}}{-\xi-\sqrt{\Delta}}\right|\right], \quad \Delta = \xi^2 - 4\xi x > 0,$$

$$M_1(\xi) = \int\limits_0^1 f(x)\psi(x,\xi)\,dx, \quad \xi < 0, \tag{2.7}$$

$$\psi(x,\xi) = \frac{1}{\Delta^2}\left[\frac{\xi^2}{x} + 2\xi + \frac{4(\xi^2 - x\xi)}{\sqrt{\Delta}}\ln\left|\frac{-\xi+\sqrt{\Delta}}{-\xi-\sqrt{\Delta}}\right|\right].$$

Also we put here the explicit expression for function $H(c)$ entering Eq. (1.62):

$$H(c) = \left(1 + \frac{s}{t} + \frac{t}{s}\right)^{-1}\left\{\frac{u^3}{3t^3}f_{4p} + \frac{s^2u}{3t^3}f_{4up} - \frac{t^2}{3s^2}h_{0up} + \frac{t^2u}{3s^3}h_{4up} - \right.$$
$$- \frac{s^2}{3t^2}f_{0up} - \frac{u^3}{3st^2}f_{0p} - \frac{u^3}{3s^2t}h_{0p} + \frac{u^3}{3s^3}h_{4p} + \frac{\pi^2}{6}\left(-\frac{4t^2}{s^2} + \frac{10t}{3s} + \frac{125}{18} - \right.$$
$$\left. - \frac{2s^2}{t^2} + \frac{7s}{3t}\right) + l_{st}^2\left(-\frac{11s}{12t} + \frac{7t}{12s} + \frac{t^2}{s^2} - \frac{5}{12}\right) + l_{su}^2\left(-\frac{s^2}{t^2} - \frac{5s}{2t} - \right.$$
$$\left. - \frac{5t}{2s} - \frac{t^2}{s^2} - \frac{19}{6}\right) + l_{st}l_{su}\left(\frac{2s^2}{t^2} + \frac{4s}{t}\frac{t}{s} + \frac{19}{6}\right) +$$
$$\left. + l_{st}\left(-\frac{17s^2}{3t^2} - \frac{25s}{3t} - \frac{17t}{6s} - \frac{17}{2}\right) + l_{su}\left(-\frac{s}{6t} - \frac{t}{6s}\right)\right\} -$$
$$- 2\mathrm{Li}_2\left(\frac{1-c}{2}\right) + 2\mathrm{Li}_2\left(\frac{1+c}{2}\right) - \frac{2\pi^2}{9}, \tag{2.8}$$

where

$$f_{0p} = M\left(\frac{s}{t}\right), \quad f_{4p} = M_1\left(\frac{s}{t}\right), \quad h_{0p} = M\left(\frac{t}{s}\right), \quad h_{4p} = M_1\left(\frac{t}{s}\right),$$
$$f_{0up} = M\left(\frac{u}{t}\right), \quad f_{4up} = M_1\left(\frac{u}{t}\right), \quad h_{0up} = M\left(\frac{t}{u}\right), \quad h_{4up} = M_1\left(\frac{t}{u}\right),$$
$$\frac{t}{s} = -\frac{1-c}{2}, \quad \frac{u}{s} = -\frac{1+c}{2}, \quad \frac{t}{u} = \frac{1-c}{1+c}, \tag{2.9}$$
$$l_{st} = \ln\left(\frac{2}{1-c}\right), \quad l_{su} = \ln\left(\frac{2}{1+c}\right).$$

Functions M and M_1 are given in Eq. (2.6) above. For an illustration in Table 1 we give function $H(c)$ for different values of c.

2.2. The Scwinger Substitution. J. Schwinger suggests [13] a representation for the photon Green function in the second order of perturbation theory which takes into account the fermion–antifermion intermediate state (we use Feynman gauge):

$$G_{\mu\nu}(q) = g_{\mu\nu}\frac{\alpha}{\pi}\int\limits_0^1 \frac{dv\,\varphi_e(v)}{(1-v^2)(q^2-M^2(v))},$$
$$\varphi_e(v) = \varphi(v) = \frac{1}{3}[2-(1-v^2)(2-v^2)], \quad M^2(v) = \frac{4m^2}{1-v^2}, \tag{2.10}$$

where m is the fermion mass.

We have the case when in the intermediate state the point-like charged scalar particle and antiparticle are ([12] of 1959, § 61 and [43]):

$$\varphi_\pi(v) = \frac{1}{3}v^4. \tag{2.11}$$

The known asymptotic of polarization operator in the scattering channel $q^2 = t$ follows

$$\int\limits_0^1 \frac{dv\,\varphi(v)}{(1-v^2)(t-M^2(v))} = \frac{1}{3t}\left[\ln\frac{-t}{m^2} - \frac{5}{3}\right] \tag{2.12}$$

and

$$\int\limits_0^1 \frac{d\nu\,\varphi(\nu)}{1-v^2}\frac{1}{t-M^2(v)}\ln\frac{-t+M^2(v)}{m^2} = \frac{1}{3t}\left[\ln^2\frac{-t}{m^2} - \frac{5}{3}\ln\frac{-t}{m^2} + \zeta_2\right],$$
$$\int\limits_0^1 \frac{d\nu\varphi_\pi(\nu)}{1-v^2}\frac{1}{t-M^2(v)} = \frac{1}{3t}\left[\frac{1}{2}\ln\frac{-t}{m^2} - \frac{4}{3}\right]. \tag{2.13}$$

Consider first the vertex function for the scattering channel $e(p_1) + \gamma^*(q) \to e(p_2)$ in two-loop approximation, with the photon polarization inserted. We will consider the case when the fermion in the loop is an electron. The relevant vertex function is (we are interested in asymptotic behavior $-q^2 = -t \gg m^2$):

$$F_1(q^2)\gamma_\mu = \frac{\alpha^2}{4\pi^2}\int\limits_0^1 \frac{dv\,\varphi(v)}{(1-v^2)(q^2-M^2(v))}\times$$
$$\times\int\frac{d^4k}{i\pi^2}\frac{\gamma_\nu(\hat{p}_2-\hat{k})\gamma_\mu(\hat{p}_1-\hat{k})\gamma^\nu}{(k^2-M^2(v))(k^2-2p_1k)(k^2-2p_2k)}.$$

Performing the standard procedure of joining the denominators, loop momentum integration, we obtain

$$F_1(t) = \frac{\alpha^2}{2\pi^2} \int_0^1 \frac{dv\,\varphi(v)}{1-v^2} \int_0^1 dx \int_0^1 y\,dy[f_1 + f_2], \tag{2.14}$$

with

$$f_1 = \frac{t}{D}(1 - y + y^2 x(1-x)), \quad f_2 = \ln\frac{D+\Lambda^2}{D} - \frac{3\Lambda^4 + 2\lambda^2 D}{2(\Lambda^2 + D)^2}, \tag{2.15}$$

Λ is the ultraviolet cut-off parameter and

$$D = y^2 p_x^2 + (1-y)M^2(v), \quad p_x = xp_1 + (1-x)p_2, \quad p_x^2 = m^2 - x(1-x)t. \tag{2.16}$$

Integration of the f_2 leads to

$$\frac{\alpha^2}{6\pi^2}\left[-\frac{1}{4}\ln^2\frac{\Lambda^2}{m^2} - \frac{5}{6}\ln\frac{\Lambda^2}{m^2} - \frac{1}{4}l_t^2 + \frac{19}{12}l_t + \mathcal{O}(\infty)\right], \quad l_t = \ln\frac{-t}{m^2}. \tag{2.17}$$

Terms containing the logarithms of the cut-off parameter Λ will be removed by applying the renormalization procedure.

The integration of other terms of f_1 leads to

$$\frac{\alpha^2}{6\pi^2}\left[\int_0^1 \frac{t dx}{p_x^2}\left[\frac{1}{4}\ln\frac{p_x^2}{m^2} - \frac{11}{6}\ln\frac{p_x^2}{m^2} + \frac{\pi^2}{6} + \frac{38}{9}\right] - \frac{1}{2}l_t + \mathcal{O}(\infty)\right]. \tag{2.18}$$

Using the relations

$$t\int_0^1 \frac{dx}{p_x^2}\left[1; \ln\frac{p_x^2}{m^2}; \ln^2\frac{p_x^2}{m^2}\right] = \left[-2l_t; -l_t^2 + \frac{\pi^2}{3}; -\frac{2}{3}l_t^3 + \frac{2\pi^2}{3}l_t - 8\xi_3\right], \tag{2.19}$$

and adding the contribution of the f_2 we obtain

$$F_1(t) = \frac{\alpha^2}{4\pi^2}\left[-\frac{1}{9}l_t^3 + \frac{19}{18}l_t^2 - \left(\frac{\pi^2}{18} + \frac{265}{54}\right)l_t + \mathcal{O}(\infty)\right], \tag{2.20}$$

in agreement with the more general result obtained in paper of Barbieri, Mignaco, and Remiddi [10].

Let us consider, for example, the vacuum polarization insertion to Box-type Feynman diagrams in the Bhabha scattering process. The relevant 3- and 4-denominator Feynman integrals are:

$$\left[J(aba_1a_2); J(aba_1); J(ba_1a_2)\right] = \\ = \int \frac{dv\,\varphi(v)}{1-v^2} \int \frac{d^4k}{i\pi^2}\left[\frac{1}{aba_1a_2}; \frac{1}{aba_1}; \frac{1}{ba_1a_2}\right] \tag{2.21}$$

with

$$a = k^2 - \lambda^2, \quad a_1 = (k+q_1)^2 - m^2, \quad a_2 = (k-q_2)^2 - m^2,$$
$$b = (k-q)^2 - M^2(v), \quad s = (q_1+q_2)^2, \quad q_1^2 = q_2^2 = m^2, \quad q^2 = (q_1-p_1)^2 = t, \tag{2.22}$$

see also designations (1.31), (1.32). The standard procedure leads:

$$J(aba_1a_2) = \int\limits_0^1 \frac{dv\varphi(v)}{1-v^2} \int\limits_0^1 \frac{dx}{p_x^2(t-M^2(v))} \times$$
$$\times \left[-\frac{1}{2} \ln \frac{(t-M^2(v))^2}{p_x^2\lambda^2} - \int\limits_0^1 \frac{(yp_x^2 - M^2(v))\,dy}{y^2p_x^2 + (1-y)M^2(v)} \right],$$

and p_x is defined in (2.16). Further extraction of the asymptotics is standard. The resulting expressions are (see also (1.33)):

$$I(aba_1a_2) = \frac{1}{3st}\Bigg\{ -\frac{1}{6}\tilde{l}_s^3 + \frac{1}{2}\tilde{l}_s^2 l_t + \tilde{l}_s l_t^2 - \frac{10}{3}\tilde{l}_s l_t - \frac{28}{9} l_s - \frac{\pi^2}{6} l_t +$$
$$+ \tilde{l}_s \ln\frac{m^2}{\lambda^2}\left(l_t - \frac{5}{3}\right) + \mathcal{O}(1)\Bigg\},$$
$$J(aba_1) = \frac{1}{3t}\left\{ \frac{1}{3}l_t^3 - \frac{5}{6}l_t^2 + 2\zeta_2 l_t + \mathcal{O}(1)\right\}, \tag{2.23}$$
$$J(ba_1a_2) = \frac{1}{3s}\left\{ \frac{1}{6}\tilde{l}_s^3 - \frac{5}{6}\tilde{l}_s^2 + \left(\frac{28}{9} + \zeta_2\right)\tilde{l}_s + \mathcal{O}(\infty)\right\},$$
$$\tilde{l}_s = \ln\frac{s}{m^2} - i\pi, \quad l_t = \ln\frac{-t}{m^2}.$$

2.3. Radiative Bhabha Scattering Process. In this Subsection we put the relevant integrals for radiative Bhabha scattering process with one-loop RC. All kinematics are defined in Subsec. 1.4. Here we used partially the results of previous works [12, 31] and refer to it for further details.

2.3.1. Integrals for G-Type Feynman Diagrams. For the set of FD, labelled as *glasses* (G), only three independent external momenta are relevant due to the conservation law: $p_1 + q = p_1' + k_1, q = p_2 - p_2'$. Choosing p_1, p_1', q as independent 4-vectors, we use the notation:

$$J_{ijk} = \int \frac{d^4k}{i\pi^2}\frac{1}{(i)(j)(k)},$$
$$J_{012q} = \int \frac{d^4k}{i\pi^2}\frac{1}{(0)(1)(2)(q)},$$

$$
\begin{aligned}
J^{\mu}_{ijk} &= \int \frac{d^4k}{i\pi^2}\frac{k^\mu}{(i)(j)(k)} = a_{ijk}p_1^\mu + b_{ijk}p_1'^\mu + c_{ijk}q^\mu,\\
J^{\mu\nu}_{ij\dots} &= \int \frac{d^4k}{i\pi^2}\frac{k^\mu k^\nu}{(i)(j)\dots} = g^T_{ij\dots}g^{\mu\nu} + a^T_{ij\dots}p_1^\mu p_1^\nu + b^T_{ij\dots}p_1'^\mu p_1'^\nu + c^T_{ij\dots}q^\mu q^\nu +\\
&\qquad + \alpha^T_{ij\dots}(p_1p_1')^{\mu\nu} + \beta^T_{ij\dots}(p_1q)^{\mu\nu} + \gamma^T_{ij\dots}(p_1'q)^{\mu\nu},\\
J^{\mu\nu\lambda}_{012q} &= \int \frac{d^4k}{i\pi^2}\frac{k^\mu k^\nu k^\lambda}{(0)(1)(2)(q)} = K_{g1}(gp_1)^{\mu\nu\lambda} + K_{g2}(gp_1')^{\mu\nu\lambda} + K_{gq}(gq)^{\mu\nu\lambda} +\\
&+ K_{111}p_1^\mu p_1^\nu p_1^\lambda + K_{222}p_1'^\mu p_1'^\nu p_1'^\lambda + K_{qqq}q^\mu q^\nu q^\lambda + K_{112}(p_1^2p_1')^{\mu\nu\lambda} +\\
&+ K_{122}(p_1p_1'^2)^{\mu\nu\lambda} + K_{11q}(p_1^2q)^{\mu\nu\lambda} + K_{1qq}(p_1q^2)^{\mu\nu\lambda} + K_{22q}(p_1'^2q)^{\mu\nu\lambda} +\\
&+ K_{2qq}(p_1'q^2)^{\mu\nu\lambda} + K_{12q}(p_1p_1'q)^{\mu\nu\lambda},
\end{aligned}
\tag{2.24}
$$

where the inverse propagators are

$$
\begin{aligned}
&(0) = k^2 - \lambda^2,\\
&(1) = (p_1 - k)^2 - m^2,\\
&(2) = (p_1' - k)^2 - m^2, \quad (q) = (p_1' - q - k)^2 - m^2,
\end{aligned}
\tag{2.25}
$$

λ is a fictitious photon mass. The symmetrized tensor structures are defined as follows:

$$
\begin{aligned}
(pq)^{\mu\nu} &= p^\mu q^\nu + p^\nu q^\mu,\\
(p^2q)^{\mu\nu\lambda} &= p^\mu p^\nu q^\lambda + p^\mu p^\lambda q^\nu + p^\nu p^\lambda q^\mu,\\
(gp)^{\mu\nu\rho} &= g^{\mu\nu}p^\rho + g^{\mu\rho}p^\nu + g^{\rho\nu}p^\mu,\\
(pqr)^{\mu\nu\lambda} &= p^\mu q^\nu r^\lambda + p^\mu q^\lambda r^\nu + p^\nu q^\mu r^\lambda + p^\nu q^\lambda r^\mu + p^\lambda q^\mu r^\nu + p^\lambda q^\nu r^\mu.
\end{aligned}
$$

The vector and tensor integrals can be calculated by multiplying both sides of expression (2.24) by vectors p_1^μ, $p_1'^\mu$, and q^μ. Then one has to use the relations

$$
2p_1k = (0) - (1), \quad 2k_1k = (q) - (1) + \chi_1, \quad 2p_1'k = (0) - (2) \tag{2.26}
$$

and compare the coefficients before vector components on both sides.

Considering the vector and tensor integrals with three denominators, we use ultraviolet divergent integrals with two denominators. Using the Feynman trick to join denominators, they can be expressed as

$$
\begin{aligned}
&\int \frac{d^4k}{i\pi^2}\frac{1}{[(k-b)^2 - d]^2} = \ln\frac{\Lambda^2}{d} - 1,\\
&\int \frac{d^4k}{i\pi^2}\frac{k^\mu}{[(k-b)^2 - d]^2} = b^\mu\left(\ln\frac{\Lambda^2}{d} - \frac{3}{2}\right),
\end{aligned}
\tag{2.27}
$$

where Λ is the cut-off parameter supposed to be large $\Lambda \gg s$.

We put here the complete list of these integrals:

$$\begin{aligned}
&J_{01} = L_\Lambda + 1, \quad J_{1q} = L_\Lambda - 1, \quad J_{2q} = L_\Lambda - L_t + 1, \\
&J_{0q} = L_\Lambda - L_{\chi_1} + 1, \quad J_{12} = L_\Lambda - L_{t_1} + 1, \quad J_{02} = L_\Lambda + 1, \\
&J_{01}^\mu = \frac{1}{2}p_1^\mu\left(L_\Lambda - \frac{1}{2}\right), \quad J_{1q}^\mu = (p_1^\mu - \frac{1}{2}k_1^\mu)\left(L_\Lambda - \frac{3}{2}\right), \\
&J_{2q}^\mu = \frac{1}{2}(p_1^\mu - k_1^\mu + p_1'^\mu)\left(L_\Lambda - L_t + \frac{1}{2}\right), \\
&J_{0q}^\mu = (p_1^\mu - k_1^\mu)\left(\frac{1}{2}L_\Lambda - \frac{1}{2}L_{\chi_1} + \frac{1}{4}\right), \\
&J_{12}^\mu = (p_1^\mu + p_1'^\mu)\left(\frac{1}{2}L_\Lambda - \frac{1}{2}L_{t_1} + \frac{1}{4}\right), \quad J_{02}^\mu = p_1'^\mu\left(\frac{1}{2}L_\Lambda - \frac{1}{4}\right),
\end{aligned} \tag{2.28}$$

where

$$L_{t_1} = \ln\frac{-t_1}{m^2}, \quad L_q = L_t = \ln\frac{-t}{m^2}, \quad L_{\chi_1} = \ln\frac{\chi_1}{m^2},$$
$$L_{\chi_1'} = \ln\frac{\chi_1'}{m^2} - i\pi, \quad L_\Lambda = \ln\frac{\Lambda^2}{m^2}.$$

The scalar integrals with three denominators read

$$\begin{aligned}
&J_{012} = \frac{1}{2t_1}\left[-2L_\lambda L_{t_1} + L_{t_1}^2 - \frac{\pi^2}{3}\right], \qquad J_{12q} = \frac{1}{2(\chi_1' - \chi_1)}(L_t^2 - L_{t_1}^2), \\
&J_{02q} = \frac{1}{t+\chi_1}\left[L_t(L_t - L_{\chi_1}) + \frac{1}{2}(L_t - L_{\chi_1})^2 + 2\mathrm{Li}_2\left(1 + \frac{\chi_1}{t}\right)\right], \\
&J_{01q} = -\frac{1}{2\chi_1}L_{\chi_1}^2 - \frac{\pi^2}{3\chi_1}, \quad L_\lambda = \ln\frac{\lambda^2}{m^2}.
\end{aligned} \tag{2.29}$$

The coefficients for vector integrals with three denominators are

$$\begin{aligned}
&a_{012} = b_{012} = \frac{1}{t_1}L_{t_1}, \quad c_{012} = 0, \\
&a_{01q} = J_{01q} + \frac{2}{\chi_1}(L_{\chi_1} - 1), \quad b_{01q} = -c_{01q} = \frac{1}{\chi_1}(-L_{\chi_1} + 2), \\
&a_{02q} = 0, \quad b_{02q} = \frac{\chi_1}{\chi_1 + t}J_{02q} + \frac{2tL_t}{(\chi_1+t)^2} + \frac{(\chi_1 - t)L_{\chi_1}}{(\chi_1+t)^2}, \quad c_{02q} = \frac{L_{\chi_1} - L_t}{\chi_1 + t}, \\
&a_{12q} = \frac{t}{t - t_1}J_{12q} + \frac{(t+t_1)L_{t_1} - 2tL_t}{(t-t_1)^2} + \frac{2}{t - t_1}, \quad b_{12q} = J_{12q} - a_{12q}, \\
&c_{12q} = \frac{t_1}{t - t_1}J_{12q} + \frac{-(t+t_1)L_t + 2t_1L_{t_1}}{(t-t_1)^2} + \frac{2}{t - t_1}.
\end{aligned} \tag{2.30}$$

The tensor integrals for G-type FD (see Eq. (2.24)) have the following form:

$$
\begin{aligned}
&g^T_{012} = \frac{1}{4}(L_\Lambda - L_{t_1}) + \frac{3}{8},\\
&a^T_{012} = b^T_{012} = \frac{1}{2t_1}(L_{t_1} - 1), \quad \alpha^T_{012} = \frac{1}{2t_1},\\
&c^T_{012} = \beta^T_{012} = \gamma^T_{012} = 0,\\
&g^T_{01q} = \frac{1}{4}(L_\Lambda - L_{\chi_1}) + \frac{3}{8}, \quad a^T_{01q} = J_{01q} + \frac{3}{\chi_1}L_{\chi_1} - \frac{9}{2\chi_1},\\
&b^T_{01q} = c^T_{01q} = -\gamma^T_{01q} = -\frac{1}{2\chi_1}(L_{\chi_1} - 2),\\
&\beta^T_{01q} = -\alpha^T_{01q} = \frac{1}{2\chi_1}(L_{\chi_1} - 3),\\
&g^T_{02q} = \frac{1}{4}L_\Lambda - \frac{\chi_1}{4(t+\chi_1)}L_{\chi_1} - \frac{t}{4(t+\chi_1)}L_t + \frac{3}{8},\\
&b^T_{02q} = \frac{3\chi_1^2 - 4t\chi_1 - t^2}{2(t+\chi_1)^3}L_{\chi_1} + \frac{t(t+4\chi_1)}{(t+\chi_1)^3}L_t + \frac{t-\chi_1}{2(t+\chi_1)^2} + \frac{\chi_1^2}{(t+\chi_1)^2}J_{02q},\\
&c^T_{02q} = \frac{L_t - L_{\chi_1}}{2(t+\chi_1)}, \quad \gamma^T_{02q} = \frac{t+2\chi_1}{2(t+\chi_1)^2}(L_{\chi_1} - L_t) - \frac{1}{2(t+\chi_1)},\\
&a^T_{02q} = \alpha^T_{02q} = \beta^T_{02q} = 0,
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
&g^T_{12q} = \frac{1}{4}L_\Lambda + \frac{t_1 L_{t_1} - tL_t}{4(t-t_1)} + \frac{3}{8},\\
&a^T_{12q} = \frac{3t^2 + 4tt_1 - t_1^2}{2(t-t_1)^3}L_{t_1} - \frac{3t^2}{(t-t_1)^3}L_t + \frac{4t-t_1}{(t-t_1)^2} + \frac{t^2}{(t-t_1)^2}J_{12q},\\
&b^T_{12q} = \frac{-t^2 + 4tt_1 + 3t_1^2}{2(t-t_1)^3}L_{t_1} + \frac{t(t-4t_1)}{(t-t_1)^3}L_t + \frac{3t_1}{(t-t_1)^2} + \frac{t_1^2}{(t-t_1)^2}J_{12q},\\
&c^T_{12q} = \frac{3t_1^2}{(t-t_1)^3}L_{t_1} + \frac{t^2 - 4tt_1 - 3t_1^2}{2(t-t_1)^3}L_t + \frac{4t_1 - t}{(t-t_1)^2} + \frac{t_1^2}{(t-t_1)^2}J_{12q},\\
&\alpha^T_{12q} = -\frac{t^2 + 4tt_1 + t_1^2}{2(t-t_1)^3}L_{t_1} + \frac{t(t+2t_1)}{(t-t_1)^3}L_t - \frac{2t+t_1}{(t-t_1)^2} - \frac{tt_1}{(t-t_1)^2}J_{12q},\\
&\beta^T_{12q} = \frac{t_1(5t+t_1)}{2(t-t_1)^3}L_{t_1} - \frac{t(t+5t_1)}{2(t-t_1)^3}L_t + \frac{3(t+t_1)}{2(t-t_1)^2} + \frac{tt_1}{(t-t_1)^2}J_{12q},\\
&\gamma^T_{12q} = -\frac{t_1(t+5t_1)}{2(t-t_1)^3}L_{t_1} + \frac{-t^2 + 5tt_1 + 2t_1^2}{2(t-t_1)^3}L_t + \frac{t-7t_1}{2(t-t_1)^2} - \frac{t_1^2}{(t-t_1)^2}J_{12q}.
\end{aligned}
\tag{2.32}
$$

Four-denominator scalar integral reads:

$$J_{012q} = -\frac{1}{t_1\chi_1}\left[-L_\lambda L_{t_1} + 2L_{t_1}L_{\chi_1} - L_t^2 - 2Li\left(1-\frac{t}{t_1}\right) - \frac{\pi^2}{6}\right]. \qquad (2.33)$$

Vector 4-denominator integrals are:

$$\begin{aligned} a_{012q} &= \frac{1}{d}\big[-(t\chi_1' + t_1\chi_1)J_{12q} + (t+\chi_1)^2 J_{02q} - \\ &\qquad - \chi_1(\chi_1' - t_1)J_{01q} - t_1(t+\chi_1)Y\big], \\ b_{012q} &= \frac{1}{d}\big[(t_1\chi_1' + t\chi_1)J_{12q} - (tt_1 + \chi_1'\chi_1)J_{02q} + \\ &\qquad + \chi_1(\chi_1 - t_1)J_{01q} + t_1(t_1 - \chi_1)Y\big], \\ c_{012q} &= \frac{1}{d}\big[-t_1(\chi_1' + \chi_1)J_{12q} + t_1(t+\chi_1)J_{02q} + \chi_1 t_1 J_{01q} - t_1^2 Y\big]. \end{aligned} \qquad (2.34)$$

$$Y = J_{012} + \chi_1 J_{012q}, \quad d = -2t_1\chi_1\chi_1'. \qquad (2.35)$$

Two-rank 4-denominator tensors are:

$$\begin{aligned} g^T_{012q} &= \frac{1}{2}(J_{12q} - \chi_1 c_{012q}), \\ a^T_{012q} &= \frac{1}{d}\big[(t+\chi_1)^2(J_{12q} - \chi_1 c_{012q}) - (\chi_1 t_1 + \chi_1' t)a_{12q} + \\ &\qquad + \chi_1(t_1 - \chi_1')a_{01q} - t_1(t+\chi_1)(a_{012} + \chi_1 a_{012q})\big], \\ b^T_{012q} &= \frac{1}{d}\big[(t_1-\chi_1)^2(J_{12q} - \chi_1 c_{012q}) + (\chi_1' t_1 + \chi_1 t)b_{12q} + \\ &\qquad + \chi_1(\chi_1 - t_1)b_{01q} - (t_1 t + \chi_1\chi_1')b_{02q} + t_1(t_1 - \chi_1)(a_{012} + \chi_1 b_{012q})\big], \\ \gamma^T_{012q} &= \frac{1}{d}\big[-t_1(t_1 - \chi_1)(J_{12q} - 2\chi_1 c_{012q}) + \\ &\qquad + (\chi_1' t_1 + \chi_1 t)c_{12q} - (tt_1 + \chi_1\chi_1')c_{02q} + \chi_1(t_1 - \chi_1)b_{01q}\big], \\ \alpha^T_{012q} &= \frac{1}{d}\big[-(tt_1 + \chi_1\chi_1')(J_{12q} - \chi_1 c_{012q}) + (\chi_1' t_1 + \chi_1 t)a_{12q} + \\ &\qquad + \chi_1(\chi_1 - t_1)a_{01q} + t_1(t_1 - \chi_1)(a_{012} + \chi_1 a_{012q})\big], \\ \beta^T_{012q} &= \frac{1}{d}\big[t_1(t_1 + \chi_1')(J_{12q} - 2\chi_1 c_{012q}) - \\ &\qquad - (\chi_1 t_1 + \chi_1' t)c_{12q} + (\chi_1 + t)^2 c_{02q} + \chi_1(\chi_1' - t_1)b_{01q}\big], \\ c^T_{012q} &= \frac{1}{t}\big[J_{12q} - 4g^T_{012q} + t_1\alpha^T_{012q} + (\chi_1' - t_1)\beta^T_{012q} + t\gamma^T_{012q}\big]. \end{aligned} \qquad (2.36)$$

We put now the coefficients of 3-rank tensor structures:

$$\begin{aligned}
K_{1g} &= \frac{1}{d}[-(t+\chi_1)^2 A_1 - t_1(t+\chi_1)A_8 + (tt_1+\chi_1\chi_1')A_{18}],\\
K_{2g} &= \frac{1}{d}[(tt_1+\chi_1\chi_1')A_1 + t_1(t_1-\chi_1)A_8 - (t_1-\chi_1)^2 A_{18}],\\
K_{qg} &= \frac{1}{d}[-t_1(t+\chi_1)A_1 - t_1^2 A_8 + t_1(t_1-\chi_1)A_{18}],\\
K_{111} &= \frac{1}{d}[-(t+\chi_1)^2 A_2 - t_1(t+\chi_1)A_9 + (tt_1+\chi_1\chi_1')A_{19}],\\
K_{112} &= \frac{1}{d}[(tt_1+\chi_1\chi_1')A_2 + t_1(t_1-\chi_1)A_9 - (t_1-\chi_1)^2 A_{19}],\\
K_{11q} &= \frac{1}{d}[-t_1(t+\chi_1)A_2 - t_1^2 A_9 + t_1(t_1-\chi_1)A_{19}],\\
K_{12q} &= \frac{1}{t+\chi_1}[t_1 K_{112} + \alpha^T_{12q} - \alpha^T_{01q} - 2K_{1g}],\\
K_{1qq} &= \frac{1}{t+\chi_1}[t_1 K_{11q} + \beta^T_{12q} - \beta^T_{01q}],\\
K_{qqq} &= \frac{1}{t+\chi_1}[t_1 K_{1qq} + c^T_{12q} - c^T_{01q}],\\
K_{122} &= -\frac{1}{t_1}[(t_1-\chi_1)K_{12q} + \alpha^T_{12q} - \alpha^T_{01q} - 2K_{2g}],\\
K_{2qq} &= -\frac{1}{t_1}[(t_1-\chi_1)K_{qqq} + c^T_{12q} - c^T_{02q}],\\
K_{22q} &= -\frac{1}{t_1}[(t_1-\chi_1)K_{2qq} + \gamma^T_{12q} - \gamma^T_{02q}],\\
K_{222} &= -\frac{1}{t_1}[(t_1-\chi_1)K_{22q} + b^T_{12q} - b^T_{02q}],
\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}
&A_1 = g^T_{12q} - g^T_{02q}, \quad A_{18} = g^T_{012} - g^T_{01q} + \chi_1 g^T_{012q}, \quad A_8 = g^T_{12q} - g^T_{01q},\\
&A_2 = a^T_{12q} - 4K_{1g}, \quad A_{19} = a^T_{012} - a^T_{01q} + \chi_1 a^T_{012q}, \quad A_9 = a^T_{12q} - a^T_{01q}.
\end{aligned} \tag{2.38}$$

We give below some checking equations for coefficients before tensor structures of G-type integrals. The complete checking system can be obtained by contraction of general tensor expansion with relevant vectors, simplifying the numerators of the integrand and using a set of vector integrals given above. Additional check can be inferred by contraction with metric tensor. In this case the scalar integrals should be used. The complete set of 10 equations for the 2-rank tensor and 24 equations for the 3-rank 4-denominator tensor integrals for the G-type was convinced to be fulfilled. For definiteness we give four equations of such a type, obtained by contraction with metric tensor. They are:

$$\begin{aligned}
4g^T_{012q} + tc^T_{012q} - t_1\alpha^T_{012q} + (\chi_1 - t_1)\beta^T_{012q} + (t+\chi_1)\gamma^T_{012q} = J_{12q},\\
6K_{1g} - t_1K_{112} + (\chi_1 - t_1)K_{11q} + tK_{1qq} + (t+\chi_1)K_{12q} = a_{12q},\\
6K_{2g} - t_1K_{122} + (\chi_1 + t)K_{22q} + tK_{2qq} + (\chi_1 - t_1)K_{12q} = b_{12q},\\
6K_{qg} + tK_{qqq} + (\chi_1 - t_1)K_{1qq} + (t+\chi_1)K_{2qq} - t_1K_{12q} = c_{12q}.
\end{aligned} \tag{2.39}$$

Another indirect check is the absence of infrared divergence containing terms in all the vector and tensor integrals.

2.3.2. Integrals for B-Type Feynman Diagrams. We use here the following set of denominators:

$$\begin{aligned}
&(1) = (p_1 - k)^2 - m^2,\\
&(2) = (p_1 - k_1 - k)^2 - m^2,\\
&(3) = (p_2 + k)^2 - m^2,\\
&(4) = (p_1 - k_1 - p_1' - k)^2 - \lambda^2, \quad (5) = k^2 - \lambda^2.
\end{aligned} \tag{2.40}$$

Four-momentum conservation law we use reads $p_1 + p_2 = p_1' + p_2' + k_1$. Scalar products of the loop momentum k with the external four-vectors can be expressed in terms of the denominators:

$$\begin{aligned}
&2p_1k = (5) - (1), \quad 2p_2k = (3) - (5),\\
&2p_1'k = (4) - (2) - t - \chi_1,\\
&2k_1k = (2) - (1) + \chi_1, \quad 2p_2'k = (3) - (4) + t.
\end{aligned} \tag{2.41}$$

Using these relations one can consider only one type of integrals with five denominators (the so-called «pentagon») namely the scalar one. Using the elegant technique developed in the paper of Van-Neerven and Vermasseren [31], it can be expressed in the form:

$$J_{12345} = -\frac{1}{D}[D_1J_{2345} + D_2J_{1345} + D_3J_{1245} + D_4J_{1235} + D_5J_{1234}],$$

$$\begin{aligned}
D &= 2ss_1t\chi_1\chi_1',\\
D_1 &= s_1t[-t(s-s_1) - s\chi_1 - s_1\chi_1' - \chi_1\chi_1'],\\
D_2 &= st[t(s-s_1) + s\chi_1 + s_1\chi_1' - \chi_1\chi_1'],\\
D_3 &= \chi_1\chi_1'[-t(s+s_1) - s\chi_1 + s_1\chi_1' + \chi_1\chi_1'],\\
D_4 &= s\chi_1[t(s-s_1) + s\chi_1 - s_1\chi_1' - \chi_1\chi_1'],\\
D_1 &= s_1\chi_1'[t(s-s_1) - s\chi_1 + s_1\chi_1' + \chi_1\chi_1'].
\end{aligned} \tag{2.42}$$

It is interesting to note that the method described above to calculate the coefficients of the tensor structures cannot be applied to the tensor integrals with 5 denominators given above. Some additional information is needed to close the system of algebraic equations.

We mention a trick which permits one to obtain additional equations for vector and tensor integrals whose denominators do not contain the term $k^2 - \lambda^2$. It consists in shifting a loop momentum. Thus, for J^{μ}_{1234} we have

$$\int \frac{d^4k}{i\pi^2} \frac{k}{(1)(2)(3)(4)}\bigg|_{k=p_1-\tilde{k}} =$$

$$\int \frac{d^4\tilde{k}}{i\pi^2} \frac{(p_1 - \tilde{k})}{(\tilde{1})(\tilde{2})(\tilde{3})(\tilde{4})} = p_1 J_{1234} + \tilde{a}(p_1 + p_2) + \tilde{c}k_1 + \tilde{d}p'_1,$$

$$(\tilde{1}) = \tilde{k}^2 - m^2, \qquad (\tilde{2}) = (\tilde{k} - k_1)^2 - m^2,$$

$$(\tilde{3}) = (p_1 + p_2 + \tilde{k})^2 - m^2, \quad (\tilde{4}) = (\tilde{k} - p'_1 - k_1)^2.$$

The comparison of right-hand side of this equation with the standard expansion

$$J^{\mu}_{1234} = (ap_1 + bp_2 + ck_1 + dp'_1)^{\mu}_{1234}$$

leads to the new relation:

$$a_{1234} = J_{1234} + b_{1234}.$$

Analogous useful relations can be obtained for tensor integrals as well. We put below the relevant scalar, vector, and tensor integrals with 3 and 4 denominators from (2.40) and introduce the parameterization:

$$J_{ij\ldots} = \int \frac{d^4k}{i\pi^2} \frac{1}{(i)(j)\ldots},$$

$$J^{\mu}_{ij\ldots} = \int \frac{d^4k}{i\pi^2} \frac{k^{\mu}}{(i)(j)\ldots} = (a_{ij\ldots}p_1 + b_{ij\ldots}p_2 + c_{ij\ldots}k_1 + d_{ij\ldots}p'_1)^{\mu},$$

$$J^{\mu\nu}_{ij\ldots} = \int \frac{d^4k}{i\pi^2} \frac{k^{\mu}k^{\nu}}{(i)(j)\ldots} = (g^T g + a^T p_1 p_1 + b^T p_2 p_2 + \quad (2.43)$$

$$+ c^T k_1 k_1 + d^T p'_1 p'_1 + \alpha^T (p_1 p_2) + \beta^T (p_1 k_1) + \gamma^T (p_1 p'_1) +$$

$$+ \rho^T (p'_1 p_2) + \sigma^T (k_1 p_2) + \tau^T (p'_1 k_1))^{\mu\nu}_{ij\ldots}.$$

Vector 3-denominator integrals are:

$$a_{245} = -c_{245} = J_{245} + \frac{L_{\chi_1} - L_t}{t + \chi_1}, \; b_{245} = 0,$$

$$d_{245} = -\frac{\chi_1}{t + \chi_1} J_{245} - \frac{2\chi_1 L_{\chi_1}}{(t + \chi_1)^2} + \frac{(\chi_1 - t_1) L_t}{(t + \chi_1)^2},$$

$$
\begin{aligned}
&a_{145} = -\frac{t}{\chi_1 - t_1} J_{145} + \frac{2\chi_1' L_{\chi_1'}}{(t_1 - \chi_1)^2} - \frac{t + \chi_1'}{(\chi_1 - t_1)^2} L_t, \\
&b_{145} = 0, \quad c_{145} = d_{145} = \frac{L_t - L_{\chi_1'}}{\chi_1' - t}, \\
&a_{345} = -c_{345} = -d_{345} = \frac{L_t}{t}, \quad b_{345} = -J_{345} + \frac{2L_t}{t}, \\
&a_{125} = J_{125} + \frac{L_{\chi_1}}{\chi_1}, \quad b_{125} = d_{125} = 0, \quad c_{125} = \frac{L_{\chi_1} - 2}{\chi_1}, \\
&a_{235} = -c_{235} = \frac{L_{s_1} - L_{\chi_1}}{s - \chi_2}, \; d_{235} = 0, \\
&b_{235} = -\frac{\chi_1}{s - \chi_2} J_{235} - \frac{2\chi_1 L_{\chi_1}}{(s - \chi_2)^2} + \frac{\chi_1 - s_1}{(s - \chi_2)^2} L_{s_1}, \\
&a_{135} = -b_{135} = \frac{L_s}{s}, \; c_{135} = d_{135} = 0, \qquad\qquad (2.44) \\
&a_{234} = -c_{234} = J_{234} - \frac{L_{s_1}}{s_1}, \quad b_{234} = -\frac{L_{s_1}}{s_1}, \quad d_{234} = -J_{234} + \frac{2L_{s_1}}{s_1}, \\
&a_{123} = J_{123} + b_{123}, \quad b_{123} = \frac{L_{s_1} - L_s}{s - s_1}, \quad d_{123} = 0, \\
&c_{123} = -\frac{s}{s - s_1} J_{123} - \frac{2}{s - s_1} + \frac{2sL_s}{(s - s_1)^2} - \frac{(s + s_1) L_{s_1}}{(s - s_1)^2}, \\
&a_{124} = J_{124}, \quad b_{124} = 0, \quad c_{124} = -J_{124} + \frac{L_{\chi_1'} - 2}{\chi_1'}, \quad d_{124} = -\frac{L_{\chi_1'}}{\chi_1'}, \\
&a_{134} = \frac{s}{s - \chi_1'} J_{134} + \frac{2\chi_1' L_{\chi_1'} - (s + \chi_1') L_s}{(s - \chi_1')^2}, \quad b_{134} = a_{134} - J_{134}, \\
&c_{134} = d_{134} = -\frac{s}{s - \chi_1'} J_{134} + \frac{-(\chi_1' + s) L_{\chi_1'} + 2sL_s}{(s - \chi_1')^2}.
\end{aligned}
$$

Vector integrals with 4 denominators read:

$$
\begin{aligned}
a_{1245} = \frac{\Delta_{3a}}{\Delta_3}, \quad b_{1245} = 0, \quad c_{1245} = \frac{\Delta_{3c}}{\Delta_3}, \\
d_{1245} = \frac{\Delta_{3d}}{\Delta_3}, \quad \Delta_3 = 2t_1 \chi_1 \chi_1',
\end{aligned}
\qquad (2.45)
$$

$$
\begin{aligned}
\Delta_{3a} = \chi_1' \big[&\chi_1 (2t_1 + \chi_1') J_{1245} + \chi_1' J_{124} - \chi_1 J_{125} - \\
&- (t + \chi_1) J_{245} + (t_1 + \chi_1) J_{145} \big], \\
\Delta_{3c} = t_1 \big[&-\chi_1 \chi_1' J_{1245} + \chi_1' J_{124} + \chi_1 J_{125} - (t + \chi_1) J_{245} + (t - \chi_1') J_{145} \big], \\
\Delta_{3d} = \chi_1 \big[&-\chi_1 \chi_1' J_{1245} - \chi_1' J_{124} + \chi_1 J_{125} + (\chi_1' - t_1) J_{245} + (t - \chi_1') J_{145} \big].
\end{aligned}
$$

$$a_{1235} = \frac{\Delta_{4a}}{\Delta_4}, \quad b_{1235} = \frac{\Delta_{4b}}{\Delta_4}, \quad c_{1235} = \frac{\Delta_{4c}}{\Delta_4}, \quad d_{1235} = 0, \quad \Delta_4 = 2s\chi_1\chi_2,$$

$$\begin{aligned} \Delta_{4a} &= \chi_2\bigl[s\chi_1 J_{1235} - (s-s_1)J_{123} - (s-\chi_2)J_{235} + \chi_1 J_{125} + sJ_{135}\bigr], \\ \Delta_{4b} &= \chi_1\bigl[s\chi_1 J_{1235} + (s-s_1)J_{123} - (s+\chi_2)J_{235} - \chi_1 J_{125} + sJ_{135}\bigr], \\ \Delta_{4c} &= s\bigl[-s\chi_1 J_{1235} + (\chi_2-\chi_1)J_{123} + (s-\chi_2)J_{235} + \chi_1 J_{125} - sJ_{135}\bigr]. \end{aligned} \tag{2.46}$$

$$a_{1345} = \frac{\Delta_{2a}}{\Delta_2}, \quad b_{1345} = \frac{\Delta_{2b}}{\Delta_2}, \quad c_{1345} = d_{1345} = \frac{\Delta_{2c}}{\Delta_2}, \quad \Delta_2 = 2stu,$$

$$\begin{aligned} \Delta_{2a} &= -st(s+t)J_{1345} + t(s+t)J_{345} + s(s+t)J_{135} + \\ &\qquad + (ut - s\chi_1')J_{145} + (us - t\chi_1')J_{134}, \\ \Delta_{2b} &= -st(s+u)J_{1345} + t(s-u)J_{345} + s(s+u)J_{135} - \\ &\qquad - (s+u)^2 J_{145} + (u\chi_1' - st)J_{134}, \\ \Delta_{2c} &= s\bigl[stJ_{1345} - tJ_{345} - sJ_{135} + (s+u)J_{145} + (t-u)J_{134}\bigr]. \end{aligned} \tag{2.47}$$

$$a_{2345} = -c_{2345} = \frac{\Delta_{1a}}{\Delta_1}, \quad b_{2345} = \frac{\Delta_{1b}}{\Delta_1}, \quad d_{2345} = \frac{\Delta_{1d}}{\Delta_1}, \quad \Delta_1 = -2s_1u_1t,$$

$$\begin{aligned} \Delta_{1a} &= -s_1u_1tJ_{2345} - u_1(t+\chi_1)J_{245} - u_1s_1J_{234} + \\ &\qquad + u_1(s-\chi_2)J_{235} + tu_1J_{345}, \\ \Delta_{1b} &= -s_1t(t+\chi_1)J_{2345} + (t+\chi_1)^2 J_{245} + s_1(t+\chi_1)J_{234} + \\ &\qquad + (u_1\chi_1 + s_1t)J_{235} + t(u_1-s_1)J_{345}, \\ \Delta_{1c} &= -s_1t(s-\chi_2)J_{2345} + (u_1\chi_1 + s_1t)J_{245} + s_1(u_1-t)J_{234} + \\ &\qquad + (s-\chi_2)^2 J_{235} + t(s-\chi_2)J_{345}. \end{aligned} \tag{2.48}$$

$$a_{1234} = J_{1234} + \frac{\Delta_{5b}}{\Delta_5}, \quad b_{1234} = \frac{\Delta_{5b}}{\Delta_5}, \quad c_{1234} = -J_{1234} - \frac{\Delta_{5b}}{\Delta_5} + \frac{\Delta_{5c}}{\Delta_5},$$

$$d_{1234} = -J_{1234} + \frac{\Delta_{5a}}{\Delta_5} - \frac{\Delta_{5b}}{\Delta_5}, \quad \Delta_5 = 2s_1\chi_1'\chi_2', \quad \chi_2' = s - s_1 - \chi_1',$$

$$\begin{aligned} \Delta_{5a} &= \chi_2'\bigl[-(s-s_1)J_{123} + (s-\chi_1')J_{134} + \chi_1' J_{124} - s_1J_{234} + s_1\chi_1' J_{1234}\bigr], \\ \Delta_{5b} &= \chi_1'\bigl[(s-s_1)J_{123} + (2s_1 - s + \chi_1')J_{134} - \chi_1' J_{124} - s_1J_{234} + s_1\chi_1' J_{1234}\bigr], \\ \Delta_{5c} &= s_1\bigl[(\chi_2' - \chi_1')J_{123} - (s-\chi_1')J_{134} + \chi_1' J_{124} + s_1J_{234} - s_1\chi_1' J_{1234}\bigr]. \end{aligned} \tag{2.49}$$

We put now the tensor coefficients for B-type integrals with 4 denominators.

$$\begin{aligned}
g^T_{1245} &= \frac{1}{2}\left[2J_{124} - a_{124} - \chi_1 c_{1245} + (t+\chi_1)d_{1245}\right],\\
a^T_{1245} &= \frac{1}{t_1\chi_1}\big[\chi_1'(-J_{124} + a_{124} - c_{145}) + t_1 a_{145} - (t+\chi_1)a_{245} + \\
&\qquad + t_1\chi_1 a_{1245} - \chi_1'(t+\chi_1)d_{1245}\big],\\
c^T_{1245} &= \frac{1}{\chi_1\chi_1'}\left[t_1(-J_{124} + a_{124}) + \chi_1 c_{125} + (t_1-\chi_1)c_{145} - \chi_1\chi_1' c_{1245}\right],\\
d^T_{1245} &= \frac{1}{t_1\chi_1'}\left[\chi_1(-J_{124} + a_{124} - a_{245}) + (t_1-\chi_1)c_{145} - t_1 d_{245} - \chi_1\chi_1' d_{1245}\right],\\
\beta^T_{1245} &= \frac{1}{\chi_1}\left[-J_{124} + a_{124} + c_{145} + \chi_1 c_{1245}\right],\\
\gamma^T_{1245} &= \frac{1}{t_1}\left[J_{124} - a_{124} + a_{245} + c_{145} + (t+\chi_1)d_{1245}\right],\\
\tau^T_{1245} &= \frac{1}{\chi_1'}\left[-J_{124} + a_{245} + \chi_1 c_{1245} - (t+\chi_1)d_{1245}\right],\\
b^T_{1245} &= \alpha^T_{1245} = \rho^T_{1245} = \sigma^T_{1245} = 0.
\end{aligned} \tag{2.50}$$

As a check one can use the result of contraction by the metric tensor:

$$4g^T_{1245} + \chi_1\beta^T_{1245} - t_1\gamma^T_{1245} + \chi_1'\tau^T_{1245} = J_{124}. \tag{2.51}$$

$$\begin{aligned}
g^T_{1235} &= \frac{1}{2}[2J_{123} - a_{123} + b_{123} - \chi_1 c_{1235}],\\
a^T_{1235} &= \frac{1}{s\chi_1}[\chi_2 J_{123} - (\chi_1+\chi_2)a_{123} + \chi_1 a_{125} - \chi_1\chi_2 c_{1235}],\\
b^T_{1235} &= \frac{1}{s\chi_2}[\chi_1(J_{123} - a_{235}) + (\chi_1+\chi_2)b_{123} - \chi_2 b_{235} - \chi_1^2 c_{1235}],\\
c^T_{1235} &= \frac{1}{\chi_1\chi_2}[s(J_{123} + b_{123}) - (s-\chi_2)a_{235} + \chi_2 c_{123} - s\chi_1 c_{1235}],\\
\alpha^T_{1235} &= \frac{1}{s}[-J_{123} + a_{123} - a_{235} - b_{123}],\\
\beta^T_{1235} &= \frac{1}{\chi_1}[-J_{123} + a_{123} + \chi_1 c_{1235}],\\
\sigma^T_{1235} &= \frac{1}{\chi_2}\left[-J_{123} + a_{235} - b_{123} + \chi_1 c_{1235}\right],\\
d^T_{1235} &= \gamma^T_{1235} = \rho^T_{1235} = \tau^T_{1235} = 0.
\end{aligned} \tag{2.52}$$

One of the checking relations here has the form

$$4g^T_{1235} + s\alpha^T_{1235} + \chi_1\beta^T_{1235} + \chi_2\sigma^T_{1235} = J_{123}. \tag{2.53}$$

$$\begin{aligned}
g^T_{1345} &= \frac{1}{2}[J_{134} + tc_{1345}],\\
a^T_{1345} &= \frac{1}{st(\chi'_1 - s - t)}[(s+t)^2 J_{134} + t(\chi'_1 - s - t)a_{145} -\\
&\quad - (s(s+t) + t\chi'_1)a_{134} + \chi'_1(s+t)(c_{145} - c_{134}) + t(s+t)^2 c_{1345}],\\
b^T_{1345} &= \frac{1}{s}[b_{134} - b_{345} - (\chi'_1 - t)\rho^T_{1345}],\\
c^T_{1345} &= d^T_{1345} = \tau^T_{1345} = \frac{1}{t(\chi'_1 - s - t)}\times\\
&\quad \times [(\chi'_1 - t)(c_{145} - c_{134}) - s(b_{134} - tc_{1345})], \qquad (2.54)\\
\alpha^T_{1345} &= \frac{1}{st(\chi'_1 - s - t)}[-t(\chi'_1 - s - t)a_{345} + \chi'_1(\chi'_1 - t)(c_{145} - c_{134}) -\\
&\quad - s\chi'_1(a_{134} - J_{134}) + st\chi'_1 c_{1345}],\\
\beta^T_{1345} &= \gamma^T_{1345} = \frac{1}{t(\chi'_1 - s - t)}[(s+t)(b_{134} - tc_{1345}) - \chi'_1(c_{145} - c_{134})],\\
\rho^T_{1345} &= \sigma^T_{1345} = \frac{1}{st(\chi'_1 - s - t)}[-(\chi'_1 - t)^2 c_{145} + t(\chi'_1 - s - t)a_{345} +\\
&\quad + (\chi'_1(\chi'_1 - t) - st)c_{134} + s(\chi'_1 - t)b_{134} - st(\chi'_1 - t)c_{1345}].
\end{aligned}$$

The relation of the same type for the above coefficients reads:

$$4g^T_{1345} + \chi'_1 c^T_{1345} + s\alpha^T_{1345} + (\chi_1 - t_1)\beta^T_{1345} + (\chi_2 - u_1)\sigma^T_{1345} = J_{134}. \tag{2.55}$$

$$\begin{aligned}
g^T_{2345} &= \frac{1}{2}\big[J_{234} + \chi_1 a_{2345} + (t + \chi_1)d_{1345}\big],\\
a^T_{2345} &= c^T_{2345} = -\beta^T_{2345} = \frac{1}{s_1 t}\big[-ta_{345} - (s_1 + \chi_1)a_{235} + s_1 t a_{2345}\big],\\
b^T_{2345} &= \frac{1}{s_1 t(\chi_1 + s_1 + t)}\big[s_1 t(b_{235} - b_{345}) - \chi_1(\chi_1 + t)a_{235} -\\
&\quad - t(t + \chi_1)a_{345} - s_1 t(\chi_1 + t)b_{2345}\big],\\
d^T_{2345} &= \frac{1}{\chi_1 + s_1 + t}\left[d_{245} - d_{234} - \frac{\chi_1 + s_1}{s_1 t(\chi_1 + s_1 + t)}\big(s_1 t(a_{245} - a_{234}) +\right.\\
&\quad \left. + t(\chi_1 + s_1)a_{345} + (\chi_1 + s_1)^2 a_{235} - s_1 t(\chi_1 + s_1)a_{2345}\big)\right],
\end{aligned}$$

$$\alpha^T_{2345} = -\sigma^T_{2345} = \frac{1}{s_1 t}\big[-\chi_1 a_{235} - t a_{345}\big],$$

$$\gamma^T_{2345} = -\tau^T_{2345} = \frac{1}{s_1 t(\chi_1 + s_1 + t)}[s_1 t(a_{245} - a_{234}) + t(\chi_1 + s_1)a_{345} + \\ + (\chi_1 + s_1)^2 a_{235} - s_1 t(\chi_1 + s_1)a_{2345}], \quad (2.56)$$

$$\rho^T_{2345} = \frac{1}{s_1 t(\chi_1 + s_1 + t)}[-s_1 t a_{234} + \chi_1(\chi_1 + s_1)a_{235} + \\ + t(\chi_1 + s_1)a_{345} - s_1 t \chi_1 a_{2345} - s_1 t(\chi_1 + t)d_{2345}].$$

The above coefficients have to satisfy the relation

$$4g^T_{2345} - \chi_1 a^T_{2345} + (s - \chi_2)\alpha^T_{2345} - (t + \chi_1)\gamma^T_{2345} - u_1 \rho^T_{2345} = J_{234};$$

$$g^T_{1234} = \frac{1}{2}\left[J_{123} - \chi_1' \frac{\Delta^{(3)}}{\Delta}\right],$$

$$a^T_{1234} = 2\frac{\Delta^{(2)}}{\Delta} + J_{1234} + \tilde{b}_{1234},$$

$$b^T_{1234} = \tilde{b}_{1234},$$

$$c^T_{1234} = 2\frac{\Delta^{(2)}}{\Delta} - 2\frac{\Delta^{(3)}}{\Delta} + J_{1234} + \tilde{b}_{1234} + \tilde{c}_{1234} - 2\tilde{\gamma}_{1234},$$

$$d^T_{1234} = 2\frac{\Delta^{(2)}}{\Delta} - 2\frac{\Delta^{(1)}}{\Delta} + J_{1234} + \tilde{b}_{1234} + \tilde{a}_{1234} - 2\tilde{\alpha}_{1234},$$

$$\alpha^T_{1234} = \frac{\Delta^{(2)}}{\Delta} + \tilde{b}_{1234}, \quad (2.57)$$

$$\beta^T_{1234} = \frac{\Delta^{(3)}}{\Delta} - 2\frac{\Delta^{(2)}}{\Delta} - J_{1234} - \tilde{b}_{1234} + \tilde{\gamma}_{1234},$$

$$\gamma^T_{1234} = \frac{\Delta^{(1)}}{\Delta} - 2\frac{\Delta^{(2)}}{\Delta} - J_{1234} - \tilde{b}_{1234} + \tilde{\alpha}_{1234},$$

$$\rho^T_{1234} = -\frac{\Delta^{(2)}}{\Delta} - \tilde{b}_{1234} + \tilde{\alpha}_{1234},$$

$$\sigma^T_{1234} = -\frac{\Delta^{(2)}}{\Delta} - \tilde{b}_{1234} + \tilde{\gamma}_{1234},$$

$$\tau^T_{1234} = 2\frac{\Delta^{(2)}}{\Delta} - \frac{\Delta^{(1)}}{\Delta} - \frac{\Delta^{(3)}}{\Delta} + J_{1234} + \tilde{b}_{1234} + \tilde{\beta}_{1234} - \tilde{\alpha}_{1234} - \tilde{\gamma}_{1234},$$

where the quantities with the sign tilde are defined as follows:

$$\tilde{a}_{1234} = \frac{1}{s\chi_1'}(L_s - L_{s_1} - L_{\chi_1'}),$$

$$\tilde{b}_{1234} = \frac{1}{\chi_2'}\left[\chi_1'\frac{\Delta^{(2)}}{\Delta} + \frac{\chi_1'}{s-\chi_1'}J_{134} + \frac{L_{s1}}{s_1} - \frac{L_s}{s-\chi_1'} + \right.$$
$$\left. + \frac{\chi_1'(s_1-\chi_2')}{s_1(s-\chi_1')^2}(L_{\chi_1'} - L_s)\right],$$

$$\tilde{c}_{1234} = \frac{1}{\chi_2'}\left[\frac{s_1^2}{\chi_2'}\frac{\Delta^{(2)}}{\Delta} - \frac{s_1}{\chi_2'}J_{124} + \left(\frac{s_1}{\chi_2'} + \frac{s_1}{s-s_1}\right)J_{123} + \frac{2-L_s}{s-s_1} - \right. \tag{2.58}$$
$$\left. - \frac{2-L_{\chi_1'}}{\chi_1'} - \frac{2s_1}{(s-s_1)^2}(L_s - L_{s_1})\right],$$

$$\tilde{\alpha}_{1234} = \frac{L_{\chi_1'} - L_s}{s_1(s-\chi_1')}, \quad \tilde{\beta}_{1234} = \frac{\Delta^{(3)}}{\Delta} - \frac{L_s - L_{s_1}}{\chi_1'(s-s_1)},$$

$$\tilde{\gamma}_{1234} = \frac{1}{\chi_2'}\left[\chi_1'\frac{\Delta^{(3)}}{\Delta} - J_{123} + \frac{L_s - L_{s_1}}{s-s_1} + \frac{L_s - L_{\chi_1'}}{s-\chi_1'}\right].$$

One of the checking relations takes the form

$$2g_{1234}^T + \chi_1\beta_{1234}^T + \chi_2\sigma_{1234}^T + \chi_1'\tau_{1234}^T = a_{134} - a_{234} + \chi_1 a_{1234}.$$

Scalar integrals with two, three, and four denominators have a form (we imply the real part everywhere and the ultraviolet asymptotics is assumed as well):

$$\begin{aligned}
&J_{12} = -1 + L_\Lambda, && J_{13} = 1 + L_\Lambda - L_s,\\
&J_{14} = 1 + L_\Lambda - L_{\chi_1'}, && J_{15} = J_{24} = J_{34} = J_{35} = L_\Lambda + 1,\\
&J_{23} = 1 + L_\Lambda - L_{s_1}, && J_{25} = 1 + L_\Lambda - L_{\chi_1},\\
&J_{45} = 1 + L_\Lambda - L_t,
\end{aligned} \tag{2.59}$$

where

$$\begin{aligned}
&L_\Lambda = \ln\frac{\Lambda^2}{m^2}, \quad L_s = \ln\frac{s}{m^2}, \quad L_\lambda = \ln\frac{\lambda^2}{m^2},\\
&L_{s_1} = \ln\frac{s_1}{m^2}, \quad L_{\chi_1'} = \ln\frac{\chi_1'}{m^2}, \quad L_{\chi_1} = \ln\frac{\chi_1}{m^2},\\
&L_t = \ln\frac{-t}{m^2}.
\end{aligned} \tag{2.60}$$

Three-denominator scalar integrals are

$$J_{123} = \frac{1}{2(s-s_1)}(L_s^2 - L_{s_1}^2), \quad J_{345} = \frac{1}{t}\left[\frac{1}{2}L_t^2 + \frac{2\pi^2}{3}\right],$$
$$J_{124} = \frac{1}{\chi_1'}\left[\frac{1}{2}L_{\chi_1'}^2 - \frac{\pi^2}{6}\right], \quad J_{125} = \frac{1}{\chi_1}\left[-\frac{1}{2}L_{\chi_1}^2 - \frac{\pi^2}{3}\right],$$
$$J_{134} = \frac{1}{s-\chi_1'}\left[\frac{3}{2}L_s^2 + \frac{1}{2}L_{\chi_1'}^2 - 2L_sL_{\chi_1'} + 2\mathrm{Li}_2\left(1-\frac{\chi_1'}{s}\right)\right],$$
$$J_{235} = \frac{1}{s_1+\chi_1}\left[\frac{3}{2}L_{s_1}^2 + \frac{1}{2}L_{\chi_1}^2 - 2L_{s_1}L_{\chi_1} + 2\mathrm{Li}_2\left(1+\frac{\chi_1}{s_1}\right) - \frac{3\pi^2}{2}\right], \quad (2.61)$$
$$J_{135} = \frac{1}{s}\left[\frac{1}{2}L_s^2 - L_sL_\lambda - \frac{2\pi^2}{3}\right], \; J_{234} = \frac{1}{s_1}\left[\frac{1}{2}L_{s_1}^2 - L_{s_1}L_\lambda - \frac{2\pi^2}{3}\right],$$
$$J_{245} = \frac{1}{t+\chi_1}\left[\frac{1}{2}L_t^2 - \frac{1}{2}L_{\chi_1}^2 + 2\mathrm{Li}_2\left(1+\frac{\chi_1}{t}\right)\right],$$
$$J_{145} = \frac{1}{-t+\chi_1'}\left[\frac{1}{2}L_{\chi_1'}^2 - \frac{1}{2}L_t^2 - \frac{\pi^2}{2} - 2\mathrm{Li}_2\left(1-\frac{\chi_1'}{t}\right)\right].$$

Four-denominator scalar integrals read:

$$J_{1245} = \frac{1}{\chi_1\chi_1'}\left[-L_{\chi_1}^2 - L_{\chi_1'}^2 - L_t^2 - 2L_{\chi_1}L_{\chi_1'} + 2L_{\chi_1}L_t + 2L_{\chi_1'}L_t + \frac{2\pi^2}{3}\right],$$
$$J_{2345} = \frac{1}{s_1t}\left[L_{s_1}^2 - L_{s_1}L_\lambda - 2L_{s_1}L_{\chi_1} + 2L_{s_1}L_t - \frac{5\pi^2}{6}\right],$$
$$J_{1345} = \frac{1}{st}\left[L_s^2 - L_sL_\lambda - 2L_sL_{\chi_1'} + 2L_sL_t + 7\frac{\pi^2}{6}\right], \quad (2.62)$$
$$J_{1235} = \frac{1}{s\chi_1}\left[L_{s_1}^2 + L_sL_\lambda - 2L_sL_{\chi_1} + 2\mathrm{Li}_2\left(1-\frac{s_1}{s}\right) - 5\frac{\pi^2}{6}\right],$$
$$J_{1234} = \frac{1}{s_1\chi_1'}\left[-L_s^2 - L_{s_1}L_\lambda + 2L_{s_1}L_{\chi_1'} - 2\mathrm{Li}_2\left(1-\frac{s}{s_1}\right) - 7\frac{\pi^2}{6}\right].$$

The results given above are valid with power accuracy, since we omit only terms of order of m^2/s compared with ones of order unity.

2.4. Integrals for Collinear Radiative Bhabha Scattering. Here we give the expressions for the quantities associated with G-type integrals (see (1.160)):

$$J = -\frac{1}{\chi_1 t_1}\left[-2L_\lambda L_{t_1} + 2L_{t_1}L_\rho - L_t^2 - 2\mathrm{Li}_2(x) - \frac{\pi^2}{6}\right],$$

$$J_1 = \frac{1}{t_1\chi_1}\int\limits_0^\rho \frac{dz}{1-z}\frac{\ln z}{1-\lambda z} = \frac{A}{t_1\chi_1}\left(1 + \frac{x}{\rho - x}\right) = \frac{A+\vartheta}{t_1\chi_1}, \quad J_k = -\frac{1}{\rho}J_1, \tag{2.63}$$

$$J_{11} = -\frac{1}{t_1\chi_1}\int\limits_0^\rho \frac{dz}{(1-z)(1-\lambda z)}\left(1 + \frac{z\ln z}{1-z}\right), \quad J_{1k} = \frac{1}{\rho},$$

$$A = \mathrm{Li}_2(1-\rho) - \frac{\pi^2}{6} + \mathrm{Li}_2(x) + L_\rho \ln(1-x), \quad \lambda = \frac{x}{\rho}, \quad \rho = \frac{\chi_1}{m^2}.$$

In the limit $\rho \gg 1$ we have

$$\Phi = \chi_1 A_2 + t_1\chi_1(J_{11} - J_1 + xJ_{1k} - xJ_k) = -\frac{1}{2} + \mathcal{O}(\rho^{-1}),$$

and that is the reason why w structure does contribute only to next-to-leading terms.

In general the expression for 5-denominator one-loop scalar, vector, and tensor integrals are some complicate functions of five independent kinematical invariants (in the derivation we use the same technique as, for example, in Subsec. 2.3). In the limit $m^2 \ll \chi_1 \ll s \sim -t$ they may be considerably simplified because of singular $1/\chi_1$ terms only kept (see (1.163)):

$$\begin{aligned}
E &= \frac{1}{s_1}D_{0124} + \frac{1}{t}D_{0123},\\
E_1 &= -xE_k = \frac{1}{2\chi_1}\left(D_{0134} - (1-x)D_{0234} - xD_{1234} + \chi_1 E\right),\\
D_{0124} &= \frac{1}{xt_1\chi_1}\left[L_\rho^2 + 2L_\rho \ln\frac{x}{1-x} - \ln^2\frac{x}{1-x} - \frac{2\pi^2}{3}\right],\\
\mathrm{Re}\,D_{0123} &= \frac{1}{s\chi_1}\left[L_{s_1}^2 - 2L_{s_1}L_\rho - 2L_sL_\lambda + \frac{\pi^2}{6} + 2\mathrm{Li}_2(x)\right],\\
\mathrm{Re}\,D_{0234} &= \frac{1}{s_1 t}\left[L_{s_1}^2 + 2L_{s_1}L_\lambda - 2L_\rho L_{s_1} + 2L_{s_1}L_t - \frac{5\pi^2}{6}\right],\\
\mathrm{Re}\,D_{0134} &= \frac{1}{st}\left[L_s^2 + 2L_sL_\lambda - 2(L_{t_1} + \ln(x))L_s + 2L_sL_t + \frac{7\pi^2}{6}\right],\\
\mathrm{Re}\,D_{1234} &= -\frac{1}{s_1xt_1}\left[-L_s^2 + 2L_s(L_{t_1} + \ln(x)) + 2L_{s_1}L_\lambda - \frac{7\pi^2}{6}\right].
\end{aligned} \tag{2.64}$$

The structure $E_{11}+xE_{1k}$ has the form $1/(s\chi_1)f(x,\chi_1)$ and will vanish after performing the operation $(1+Q_2)s_1tP$ given in (1.157) which yields a contribution of P-type graphs with crossed and uncrossed photon legs.

The following coefficient for the scalar integral is obtained in the calculation of B-type FD (see (1.161) and below):

$$B = \frac{1}{s_1 t}\left[L_{s_1}^2 + 2L_{s_1}L_\lambda - 2L_{s_1}L_\rho + 2L_{s_1}L_t + \frac{\pi^2}{6}\right]. \qquad (2.65)$$

For the vector integral coefficients we get

$$\begin{aligned}
a &= -\frac{1}{2s_1u_1t}\left[-\pi^2 s_1 + 2u_1\mathrm{Li}_2(1-\rho) - s_1L_t^2 + tL_{s_1}^2 - 2tL_{s_1}L_t\right],\\
b &= -\frac{1}{2s_1t}\left[\frac{2\pi^2}{3} + 2\mathrm{Li}_2(1-\rho) - 2L_{s_1}^2 + 4L_{s_1}L_\rho - 2L_{s_1}L_t\right],\\
c &= \frac{1}{2s_1u_1t}\Big[2u_1\mathrm{Li}_2(1-\rho) + \frac{\pi^2}{6}(4u_1+6t) + (t-2u_1)L_{s_1}^2\\
&\qquad - s_1L_t^2 + 4u_1L_{s_1}L_\rho + 2s_1L_{s_1}L_t\Big].
\end{aligned} \qquad (2.66)$$

The relevant quantities for tensor B-type integrals are:

$$\begin{aligned}
a_{1'2'} &= \frac{1}{s_1t}\left(\frac{\rho}{\rho-1}L_\rho - L_t\right),\\
a_g &= -\frac{1}{4u_1}[(L_{s_1}-L_t)^2+\pi^2],\\
a_{1'2} &= -\frac{1}{2u_1^2}[(L_t-L_{s_1})^2+\pi^2] + \frac{1}{tu_1}(L_{s_1}-L_t) - \frac{1}{s_1t}\left(\frac{\rho}{\rho-1}L_\rho - L_{s_1}\right),\\
J_0 &= \frac{1}{s_1}\left[\frac{3}{2}L_{s_1}^2 - 2L_{s_1}L_\rho - \mathrm{Li}_2(1-\rho) - \frac{4\pi^2}{3}\right].
\end{aligned} \qquad (2.67)$$

As has been mentioned in the text, the physical gauge exploited provides a direct extraction of the kernel of the structure function out of the traces both in the tree- and loop-level amplitudes. The pattern emerging (see the text after (1.152)):

$$\begin{aligned}
&(\hat p_1 - \hat k_1 + m)\,\hat e\,(\hat p_1 + m)\,\hat e\,(\hat p_1 - \hat k_1 + m) =\\
&\qquad = 4(p_1e)^2(\hat p_1 - \hat k_1) - e^2\chi_1\hat k_1 \approx (1-x)Y\hat p_1, \qquad (2.68)\\
&\hat k_1\hat e(\hat p_1+m)\hat e(\hat p_1 - \hat k_1 + m) \approx (1-x)\left(2\frac{2-x}{1-x}W - Y\right)\hat p_1
\end{aligned}$$

shows this clearly.

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