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THE CAUCHY PROBLEM FOR BBGKY HIERARCHY OF QUANTUM KINETIC EQUATIONS WITH COULOMB POTENTIAL

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The existence of a unique solution, in terms of initial data of the BBGKY hierarchy of quantum kinetic equations with Coulomb potential, is proved. This is based on nonrelativistic quantum mechanics and utilizing a semigroup method.

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INTRODUCTION

Since the time, when it was formulated in 1946, the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) $[1, 2]$ hierarchy of kinetic equations was the object of investigation for physicists as well as mathematicians.

Well known, that charged particles interact via the Coulomb potential. Until present, there is no solution of the BBGKY hierarchy of quantum kinetic equations in the case when the particles interact via a Coulomb potential. This is an important problem for many researchers. The present paper addresses the solution of this problem.

1. FORMULATION OF THE PROBLEM

We consider the hierarchy BBGKY of quantum kinetic equations, which describes the evolution of a system of identical particles with mass m and charge q interacting via a Coulomb potential $[1, 3, 4]$ $\phi(x_i, x_j) = q^2/|x_i - x_j|$, which depends on the distance between particles $|x_i - x_j|$ and charges q. We assume that the charge is a real constant.

In the present work, the Cauchy problem is solved for a quantum system of finite number paricles contained in the finite bounded region (vessel) with volume

$V = |\Lambda|$. The BBGKY hierarchy is given by $[1,2]$

$$i\frac{\partial \rho_s^\Lambda(t,x_1,\ldots,x_s;x_1',\ldots,x_s')}{\partial t} = [H_s^\Lambda, \rho_s^\Lambda](t,x_1,\ldots,x_s;x_1',\ldots,x_s') +$$
$$+\frac{N}{V}\left(1-\frac{s}{N}\right)\mathrm{Tr}_{x_{s+1}}\sum_{1\leqslant i\leqslant s}\left(\phi_{i,s+1}(|x_i - x_{s+1}|) - \phi_{i,s+1}(|x_i' - x_{s+1}|)\right),$$

$$\rho_{s+1}^\Lambda(t,x_1,\ldots,x_s,x_{s+1};x_1',\ldots,x_s',x_{s+1}), \tag{1}$$

with the initial condition

$$\rho_s^\Lambda(t,x_1,\ldots,x_s;x_1',\ldots,x_s')|_{t=0} = \rho_s^\Lambda(0,x_1,\ldots,x_s;x_1',\ldots,x_s'). \tag{2}$$

In the problem given by equation (1) and (2) the vector represented by x_i gives the position of ith particle in the 3-dimensional Euclidean space R^3, $x_i = (x_i^1, x_i^2, x_i^3)$, $i = 1,2,\ldots,s$, and $x_i^\alpha, \alpha = 1,2,3$ are coordinates of a vector x_i. The length of the vector x_i is denoted by

$$|x_i| = ((x_i^1)^2 + (x_i^2)^2 + (x_i^3)^2)^{1/2}.$$

In (1) $\hbar = 1$ is the Plank constant and $[,]$ denotes the Poisson bracket.

The reduced statistical operator of s particles is $\rho_s^\Lambda(x_1,\ldots,x_s;x_1',\ldots,x_s')$ related by positive symmetric density matrix of N particles by $[1,2]$

$$\rho_s^\Lambda(x_1,\ldots,x_s;x_1',\ldots,x_s') =$$
$$= V^s\,\mathrm{Tr}_{x_{s+1},\ldots,x_N} D_N^\Lambda(x_1,\ldots,x_s,x_{s+1},\ldots,x_N;x_1',\ldots,x_s',x_{s+1},\ldots,x_N),$$

where $s \in N$, N is the number of particles, V — the volume of the system of particles. The trace is defined in terms of the kernel $\rho^\Lambda(x,x')$ by the formula

$$\mathrm{Tr}_x \rho^\Lambda = \int_\Lambda \rho^\Lambda(x,x)\,dx.$$

The Hamiltonian of the system is defined as

$$H_s^\Lambda(x_1,\ldots,x_s) = \sum_{1\leqslant i\leqslant s}\left(-\frac{1}{2m}\triangle_{x_i} + u^\Lambda(x_i)\right) + \sum_{1\leqslant i<j\leqslant s}\phi_{i,j}(|x_i - x_j|),$$

where $\triangle_i$ is the Laplacian

$$\triangle_i = \frac{\partial^2}{\partial(x_i^1)^2} + \frac{\partial^2}{\partial(x_i^2)^2} + \frac{\partial^2}{\partial(x_i^3)^2}, \quad \phi_{i,j}(|x_i - x_j|) = \frac{q^2}{|x_i - x_j|},$$

and $u^\Lambda(x)$ is an external field which keeps the system in the region Λ ($u^\Lambda(x) = 0$ if $x \in \Lambda$ and $u^\Lambda(x) = +\infty$ if $x \notin \Lambda$). Here $\phi_{i,j}(|x_i - x_j|)$ is symmetric.

2. SOLUTION OF THE CAUCHY PROBLEM FOR BBGKY HIERARCHY OF QUANTUM KINETIC EQUATIONS WITH COULOMB POTENTIAL

To obtain the solution of the Cauchy problem defined by (1) and (2), we use a semigroup method [5–9].

Let $L_2^s(\Lambda)$ be the Hilbert space of functions $\psi_s^\Lambda(x_1,\dots,x_s)$, $x_i \in R^3(\Lambda)$, and B_s^Λ be the Banach space of positive-definite, self-adjoint nuclear operators $\rho_s^\Lambda(x_1,\dots,x_s;x_1',\dots,x_s')$ on $L_2^s(\Lambda)$

$$(\rho_s^\Lambda\psi_s^\Lambda)(x_1,\dots,x_s)=\int\limits_\Lambda \rho_s^\Lambda(x_1,\dots,x_s;x_1',\dots,x_s')\psi_s^\Lambda(x_1',\dots,x_s')\,dx_1'\cdots dx_s',$$

with norm

$$|\rho_s^\Lambda|_1=\sup\sum_{1\leqslant i\leqslant\infty}|(\rho_s^\Lambda\psi_i^s,\varphi_i^s)|,$$

where the upper bound is taken over all orthonormalized systems of finite, twice differentiable functions with compact support $\{\psi_i^s\}$ and $\{\varphi_i^s\}$ in $L_2^s(\Lambda)$, $s\geqslant 1$. We'll suppose that the operators ρ_s^Λ and H_s^Λ act in the space $L_2^s(\Lambda)$ with zero boundary conditions.

Let B^Λ be the Banach space of sequences of nuclear operators

$$\rho^\Lambda=\{\rho_0^\Lambda,\rho_1^\Lambda(x_1;x_1'),\dots,\rho_s^\Lambda(x_1,\dots,x_s;x_1',\dots,x_s'),\dots\},$$

where ρ_0^Λ are complex numbers, $\left|\rho_0^\Lambda\right|_1=\left|\rho_0^\Lambda\right|$ and $\rho_s^\Lambda\subset B_s^\Lambda$,

$$\rho_s^\Lambda(x_1,\dots,x_s;x_1',\dots,x_s')=0,\quad\text{when}\quad s>s_0,$$

where s_0 is finite and the norm is

$$|\rho^\Lambda|_1=\sum_{s=0}^\infty|\rho_s^\Lambda|_1.$$

The Coulomb potential $\phi_{i,j}=q^2/|r_{i,j}|$ can be represented as

$$\phi_{i,j}=\phi_{i,j}^1+\phi_{i,j}^2,$$

where

$$\phi_{i,j}^1=\frac{q^2}{|r_{i,j}|}\left(\frac{1}{1+|r_{i,j}|}\right)\subset L_2(R^3),\quad \phi_{i,j}^2=\frac{q^2}{1+|r_{i,j}|}\subset L_\infty(R^3),$$

$$r_{i,j}=\left((x_i^1-x_j^1)^2+(x_i^2-x_j^2)^2+(x_i^3-x_j^3)^2\right)^{1/2}.$$

Therefore the Coulomb potential satisfies the conditions of Theorem X.15 [10], and the Hamiltonian with Coulomb potential

$$H_s^\Lambda(x_1,\dots,x_i,\dots,x_j,\dots,x_s)=-\sum_{1\leqslant i\leqslant s}\frac{1}{2}\triangle_{x_i}+\sum_{1\leqslant i<j\leqslant s}\frac{q^2}{|x_i-x_j|}$$

is a self-adjoint operator on the set $D(H_s^\Lambda)=D\Bigg(-\sum\limits_{1\leqslant i\leqslant s}\triangle_i\Bigg)$.

Let $\tilde{B}_s^\Lambda$ be a dense set of «good» elements of B_s^Λ of type $B_s^\Lambda\cap D(H_s^\Lambda)\otimes D(H_s^\Lambda)$, where $D(H_s^\Lambda)$ is the domain of the operator H_s^Λ [4] and $\otimes$ denotes the algebraic tensor product.

We introduce the operators $\omega^\Lambda(t)$, $\Omega(\Lambda)$, and $U^\Lambda(t)$ on the space B^Λ by

$$(\omega^\Lambda(t)\rho^\Lambda)_s(x_1,\dots,x_s;x_1',\dots,x_s')=(\mathrm{e}^{-iH_s^\Lambda t}\rho^\Lambda\,\mathrm{e}^{iH_s^\Lambda t})_s(x_1,\dots,x_s;x_1',\dots,x_s'),$$

$$\begin{aligned}&\big(\Omega(\Lambda)\rho^\Lambda\big)_s(x_1,\dots,x_s;x_1',\dots,x_s')=\frac{N}{V}\left(1-\frac{s}{N}\right)\times\\&\times\int\limits_\Lambda\sum_i\rho_{s+1}^\Lambda(x_1,\dots,x_s,x_{s+1};x_1',\dots,x_s',x_{s+1})g_i^1(x_{s+1})\tilde{g}_i^1(x_{s+1})dx_{s+1},\qquad(3)\\&U^\Lambda(t)\rho_s^\Lambda(x_1,\dots,x_s;x_1',\dots,x_s')=\\&\qquad=(\mathrm{e}^{\Omega(\Lambda)}\,\mathrm{e}^{-iH^\Lambda t}\,\mathrm{e}^{-\Omega(\Lambda)}\rho^\Lambda\,\mathrm{e}^{iH^\Lambda t})_s(x_1,\dots,x_s;x_1',\dots,x_s').\end{aligned}$$

In (3), $g_i^1(x_{s+1})$ is a complete orthonormal system of vectors in the one-particle space $L_2(\Lambda)$.

Let

$$\begin{aligned}&(\mathcal{H}^\Lambda\rho^\Lambda)_s(x_1,\dots,x_s;x_1',\dots,x_s')=\\&\quad=[H_s^\Lambda,\rho_s^\Lambda](x_1,\dots,x_s;x_1',\dots,x_s')+\frac{N}{V}\left(1-\frac{s}{N}\right)\mathrm{Tr}_{x_{s+1}}\times\\&\quad\times\sum_{1\leqslant i\leqslant s}(\phi_{i,s+1}(|x_i-x_{s+1}|)-\phi_{i,s+1}(|x_i'-x_{s+1}|))\times\\&\qquad\times\rho_{s+1}^\Lambda(x_1,\dots,x_{s+1};x_1',\dots,x_{s+1}).\end{aligned}$$

Theorem. *The operator $U^\Lambda(t)$ generates a strongly continuous group of bounded operators on B^Λ, whose generators coincide with the operator $-i\mathcal{H}^\Lambda$ on $\tilde{B}^\Lambda$ everywhere dense in B^Λ.*

Proof. The proof is summarized in the following four steps:

Step 1. Let us show that the operator $U^\Lambda(t)$ is bounded on B^Λ. We begin by evaluating the operator Ω^Λ:

$$|\Omega(\Lambda)|_1=\max{}_s\left|\frac{N}{V}\left(1-\frac{s}{N}\right)\right|=\frac{1}{v(\Lambda)}=\mathrm{const.}\qquad(4)$$

From the boundedness of the operator $\Omega(\Lambda)$ (4), it follows that $\mathrm{e}^{\Omega(\Lambda)}$ is bounded $|\mathrm{e}^{\pm\Omega(\Lambda)}|_1 \leqslant \mathrm{e}^{1/v(\Lambda)}$. The operator $U^\Lambda(t)$, as a product of the bounded operators of $\mathrm{e}^{\pm\Omega(\Lambda)}$ and the unitary operators $\mathrm{e}^{\mp iH_s^\Lambda t}$, is bounded and satisfies the estimate $U^\Lambda(t) \leqslant \mathrm{e}^{2/v(\Lambda)}$ on B^Λ.

Step 2. Strong continuity of the operator $U^\Lambda(t)$ on B^Λ follows from boundedness of the operator $\mathrm{e}^{\pm\Omega(\Lambda)}$ and the strong continuity of the operator $\omega^\Lambda(t)$ on B^Λ [6, 7].

Step 3. The operator $U^\Lambda(t)$ satisfies the group property on B^Λ:

$$U^\Lambda(t_1)U^\Lambda(t_2)\rho^\Lambda = U^\Lambda(t_1+t_2)\rho^\Lambda \quad\text{and}\quad U^\Lambda(t_2)U^\Lambda(t_1)\rho^\Lambda = U^\Lambda(t_2+t_1)\rho^\Lambda.$$

Step 4. The generator of the operator $U^\Lambda(t)$ defined on B^Λ consides with $-\,i\mathcal{H}^\Lambda$ on $\tilde{B}^\Lambda$.

$$\begin{aligned}\lim_{t\to 0}\Bigg|\left(\frac{U^\Lambda(t)\rho^\Lambda-\rho^\Lambda}{t}\right)_s &(x_1,\ldots,x_s;x_1',\ldots,x_s')-\\ &-(-i([H_s^\Lambda,\rho_s^\Lambda](x_1,\ldots,x_s;x_1',\ldots,x_s')+\\ +\frac{N}{V}\left(1-\frac{s}{N}\right)\mathrm{Tr}_{x_{s+1}}&\sum_{1\leqslant i\leqslant s}(\phi_{i,s+1}(|x_i-x_{s+1}|)-\phi_{i,s+1}(|x_i'-x_{s+1}|))\times\\ &\times\rho_{s+1}^\Lambda(x_1,\ldots,x_s,x_{s+1};x_1',\ldots,x_s',x_{s+1}))\Bigg|_1=0.\end{aligned}$$

This implies that multiplication of the infinitesimal operator of the group $U^\Lambda(t)$ on B_s^Λ by i concides with the operator

$$\begin{aligned}&[H_s^\Lambda,]+\frac{N}{V}\left(1-\frac{s}{N}\right)\times\\ &\qquad\times\mathrm{Tr}_{x_{s+1}}\sum_{1\leqslant i\leqslant s}(\phi_{i,s+1}(|x_i-x_{s+1}|)-\phi_{i,s+1}(|x_i'-x_{s+1}|))\end{aligned}\tag{5}$$

on the right-hand side of the BBGKY hierarchy of quantum kinetic equations on $\tilde{B}_s^\Lambda$.

According to [4] and Theorem 2.2 of Ch. XIX of [8], since $U^\Lambda(t)$ is a strongly continuous semigroup on B^Λ with generator on $\tilde{B}_s^\Lambda$ which is dense in B_s^Λ, the abstract Cauchy problem (1), (2) associated with operator (5) has the unique solution

$$\begin{aligned}\rho_s^\Lambda(t,x_1,\ldots,x_s;x_1',\ldots,x_s')) &= \left(U^\Lambda(t)\rho^\Lambda\right)_s(x_1,\ldots,x_s;x_1',\ldots,x_s')=\\ &=\left(\mathrm{e}^{\Omega(\Lambda)}\,\mathrm{e}^{-iH^\Lambda t}\,\mathrm{e}^{-\Omega(\Lambda)}\rho^\Lambda\,\mathrm{e}^{iH^\Lambda t}\right)_s(x_1,\ldots,x_s;x_1',\ldots,x_s')\end{aligned}\tag{6}$$

for each $\rho_s^\Lambda(x_1,\ldots,x_s;x_1',\ldots,x_s') \subset \tilde{B}_s^\Lambda$. For the initial data ρ_s^Λ belonging to a certain subset of B_s^Λ (to the domain of definition of $D(-\,i\mathcal{H}^\Lambda)$), which is everewhere dense in B_s^Λ, (6) is a strong solution of Cauchy problem (1), (2).

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