ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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ON THE CONNECTION BETWEEN QUANTUM AND CLASSICAL DESCRIPTIONS

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The review paper presents generalization of d'Alembert's variational principle: the dynamics of a quantum system for an external observer is defined by the exact equilibrium of all acting in the system forces, including the random quantum force $\hbar j$, $\forall\hbar$. Special attention is payed to the systems with (hidden) symmetries. It is shown how the symmetry reduces the number of quantum degrees of freedom down to the independent ones. The sine-Gordon model is considered as an example of such a field theory with symmetry. It is shown why the particles S matrix is trivial in this model.

Предлагается обобщение вариационного принципа Даламбера: динамика квантовой системы для внешнего наблюдателя определяется точным равновесием всех действующих в системе сил, включая случайную квантовую силу $\hbar j$, $\forall\hbar$. Специально рассмотрены системы с (скрытой) симметрией. Показано, как симметрия редуцирует число квантовых степеней свободы до независимых. Рассмотрена модель синус-Гордона как пример теории поля с симметрией. Показано, почему S-матрица частиц тривиальна в этой модели.

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Dedicated to
Alexei Sissakian,
co-author and friend

Preface

Present paper is the review of the work which was performed after my first paper [1]. I returned from time to time to the idea [1] that it seems interesting to embed the total probability conservation condition into the quantum field theory formalism and discuss it with Alexei Sissakian during our team-work. It seems unnecessary to note the suggestion that the S matrix is the unitary operator and it is not evident why this attempt can give something new. But it turns out that there exists the correspondence between the quantum theory and the classic one which is independent of the value of quantum corrections. Besides this, new quantum field

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theory is free from divergences and the value of quantum corrections ingenuously depends on the topology of classical field. All that is new from the point of view of ordinary theory and at last Alexei Sissakian proposed to write on paper all results in detail. Present introductory paper is devoted to the simplest examples, and more interesting field theory models will be published later.

1. INTRODUCTION

The basis of the method of calculations is the following [1]. The S-matrix unitarity condition, $S^\dagger S = 1$, in terms of amplitudes, $S = 1 + iA$, looks as follows:

$$iA^\dagger A = (A - A^\dagger). \tag{1.1}$$

The nonlinearity of this equality points on existence of the cancellations mechanism (of the real part of amplitude) which reduces quadratic form down the linear one. Our purpose is to show how this reduction removes the «unwanted» contributions*.

One may consider the simplest vacuum-into-vacuum transition probability, $|Z|^2$, as the main quantity, where Z is the functional integral over fields,

$$Z = \int D\varphi \; \mathrm{e}^{iS(\varphi)}, \;\; D\varphi = \prod_x d\varphi(x). \tag{1.2}$$

One may include into the action, S, also the linear over field φ term,

$$\int dx \, J(x)\varphi(x) \tag{1.3}$$

to describe production of particles. We will assume on the early stages that $J = 0$. Then the vacuum-into-vacuum transition probability is

$$|Z|^2 = \int D\varphi_+ D\varphi_-^* \;\; \mathrm{e}^{iS(\varphi_+) - iS^*(\varphi_-)}, \tag{1.4}$$

where φ_+ and φ_- are completely independent fields.

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*This means that the theory must be formulated directly in terms of probability. But notice that it is the frequently used method of particle physics. For example, one must integrate over unobserved final state in the inclusive approach to the multiple production phenomena. Another example: describing the very high multiplicity (VHM) processes, the number of produced particles n must be considered as the dynamical parameter. In the frame of S-matrix thermodynamics, where the «rough» description of final state is used, one must also integrate over final particles momenta. In all cases one must consider quantities $\sim |A|^2$ directly, where A is the corresponding amplitude.

It can be shown that Eq. (1.1) means that a reduced form must also exist [1]:

$$|Z|^2 = \lim_{j=e=0} \mathrm{e}^{i\hat{\mathbb{K}}(j,e)} \int DM \mathrm{e}^{iU(\varphi,e)}, \tag{1.5}$$

where $\hat{\mathbb{K}} = \hat{\mathbb{K}}(j,e)$ is a definite differential operator over $j(x)$ and $e(x)$, see the examples (2.42), (6.7). The expansion of $\exp\{i\hat{\mathbb{K}}\}$ generates perturbation series. The functional $U(\varphi, e)$ introduces interaction among quantum degrees of freedom, and the integral measure is δ-functional:

$$DM = \prod_x d\varphi(x) \delta\left(\frac{\delta S(\varphi)}{\delta\varphi(x)} + \hbar j(x)\right). \tag{1.6}$$

Sometimes the δ-like measure [2] is called in mathematical literature as the «Dirac measure». It follows from (1.6) that

— *the quantum system for an external observer looks like classical which is excited by the external random force* $\hbar j$, $\forall\hbar$.

The established generalized correspondence principle* is the main consequence of Eq. (1.1). Therefore the complete set of acceptable field states for external observer** is known having (1.6).

It is important that the restricted problem is considered. We will calculate the imaginary part of amplitude believing that it will be sufficient for us. In this case, the unmeasurable phase of amplitude stays undefined***. A main mathematical problem in the searching for representation (1.5) is to find the way how to find the imaginary part from the modulo square of amplitude. To be more precise, we will find the imaginary part as a result of cancellation of «unwanted» contribution in the modulo square of amplitude.

The δ-function (1.6) solves the problem of definition of contributions into the path integral but cannot solve the problem completely since the action of operator $\hat{\mathbb{K}}$ remains unknown. It must be noted that $\exp\{i\hat{\mathbb{K}}\}$ generates the asymptotic series ordinary in quantum theories [3], and it seems that δ-like measure gives nothing new****. But this is not entirely so. I would like to draw attention to the appearance of source of quantum excitations $\hbar j$ in the r.h.s. of classical Lagrange equation, i.e., the changes of l.h.s. in equation of motion lead to the change of j. It is crucially important that (1.6) is rightful independently of the value of $\hbar$.

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*This formulation of the principle was offered by A. Sissakian.

**Since the «probability» is considered

***Therewith why must the calculations of unnecessary, i.e., unmeasurable, phase be performed? Just in this sense the unitarity condition (1.1) is a necessary one. It says that the real part is the «unwanted» part of the amplitude.

****Looking at the approach from the point of view of the stationary phase methods. In other words, one can think that the present approach gives nothing new to Bohr's correspondence principle.

The theory defined on the Dirac measure (1.6) for this reason has quite unexpected properties, e.g., allows one to perform transformation of the path integral variables. So, it will be shown that in theories with symmetry the reduced form of representation (1.5) exists:

$$|Z|^2 = \lim_{j=e=0} \mathrm{e}^{i\hat{\mathbb{K}}(j,e)} \int DM(j)\, \mathrm{e}^{iU(\varphi_c,e)}, \tag{1.7}$$

where $\hat{\mathbb{K}}$ is again the perturbations generating operator, and U introduces interactions. Note that $\hat{\mathbb{K}}$ and U in (1.7) depend on the sets $\{j_{\xi_k}, j_{\eta_k}\}$, $\{e_{\xi_k}, e_{\xi_k}\}$ of new variables. One must take this auxiliary variables equal to zero at the very end of calculations. At the same time, the transformed measure DM is again δ-like:

$$DM = \prod_k \prod_t d\xi_k(t) d\eta_k(t) \times \\ \times \delta\left(\dot{\xi}_k(t) - \frac{\delta h}{\delta \eta_k(t)} - j_{\xi_k}(t)\right) \delta\left(\dot{\eta}_k(t) + \frac{\delta h}{\delta \xi_k(t)} + j_{\eta_k}(t)\right), \tag{1.8}$$

where t is the time variable, and $h = h(\eta)$ is the transformed Hamiltonian:

$$h(\eta) = H(\varphi_c), \tag{1.9}$$

where $\varphi_c = \varphi_c(\mathbf{x}; \xi, \eta)$ is the given solution of Lagrange equation at $j = 0$.

Formula (1.8) is the main result. Therefore, as follows from this formula, the problem of the quantum field theory with symmetry is reduced down to quantum mechanical one, with potential defined by φ_c.

(A) The Dirac measure (1.6) prescribes that $|Z|^2$ is defined by the sum of $strict$ solutions of equation of motion:

$$\frac{\delta S(\varphi)}{\delta \varphi(x)} = \hbar j(x), \tag{1.10}$$

in vicinity of $j = 0$, i.e., by definition Eq. (1.10) must be solved expanding the solution over j^*. Following the ordinary rule, we obviously leave the contribution which ensures the minimal vacuum energy. On the other hand, having theory on Dirac measure, which calls for the complete set of contributions, we have offered another selection rule in our dynamic theory of S matrix. Namely, we simply propose** that

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*It should be noted that it may be that the limit $j = 0$ is absent. For example, it may happen if the system is unstable against symmetry breaking. This important possibility will not be considered in the present paper.

**This selection rule is used widely in classical mechanics, see, e.g., formulation of Kolmogorov–Arnold–Mozer (KAM) theorem [4].

— *the largest terms in the sum over solutions of (1.10) are significant from the physics point of view.*

To be more precise, this selection rule means that if G is the symmetry of action and TG^* is the symmetry of the extremum of the action, then in the situation of a general position only the trajectories with the highest dimension factor group, (G/TG^*), are sufficient.

It will be seen that this kind of definition of the «ground state» extracts the maximally «feeling» symmetry contributions since other ones will be realized on a zero measure, or, more precisely, only maximal symmetry breaking field configurations, φ_c, are most probable. We will call such a solution of the problem *the field theory with symmetry*. It is the main formal distinction of the present approach.

It is important here that the zero width of δ-function excludes the interference among contributions from various trajectories. Therefore, the formalism naturally takes into account the orthogonality of Hilbert spaces built on various trajectories. This is achieved through the special boundary conditions in the frame of which the total action of the product

$$Z \cdot Z^* = \langle \mathrm{in}|\mathrm{out}\rangle \, \langle \mathrm{out}|\mathrm{in}\rangle$$

always describes a closed path, i.e., the necessary for d'Alembert variational principle time reversible motion. It points to the necessity to be careful with boundary conditions in a considered formalism*.

(B) The Dowker theorem [5] insists that the semiclassical approximation to be exact for path integrals on the simple Lie group manifolds. For this reason one can expect that the quantum-mechanical problems, as well as the field-theoretical ones, may be at least transparent to the symmetry manifolds.

However, we know how to construct correctly the path integral formalism only in the restricted case of canonical variables [6]. At the first sight, the path integrals in terms of generalized coordinates can be defined through the corresponding transformation. But there is an opinion that it is impossible to perform the transformation of path-integral variables: the naive transformation of coordinates gives wrong results because of their stochastic nature in quantum theories**. That is why such a general principle as the conservation of total probability (1.1) should play an important role. Indeed, it is evident that δ-like

*The necessity to count all possible boundary conditions of a given problem was mentioned to the author by L. Lipatov.

**One can find corresponding examples in [6,7]. The most popular method of transformation of the path-integral variables is a «time-sliced» method [8], which induces corrections to the interaction Lagrangian proportional at least to $\hbar^2$ [9], i.e., the problem of transformation is of a quantum nature. For this reason the usage of the «time-sliced» method in general case is cumbersome, see also [10].

Dirac measure (1.6) allows one to perform arbitrary transformation [1] just as in the classical mechanics.

Therefore, the theory on Dirac measure straight away leads to the new for quantum field theory selection rule and latter one gives the theory with symmetry. All this is attained by transition to the appropriate variables, $(\xi, \eta) \in W$ in our notations. The last circumstance means that we go away from ordinary spectral analysis of quantum fluctuations to the description of the classical trajectories topology conserving deformations, since $\varphi_c = \varphi_c(\mathbf{x}; \xi, \eta)$ is given, of symmetry manifold, W^*. It must be underlined that our method of transformation is rightful for arbitrary case, i.e., not only for simple Lie group manifolds, where the semiclassical approximation is exact.

Next, the dimensions of initial phase space of field and of the transformed space of independent degrees of freedom, i.e., of the symmetry manifold, will not coincide. That means that the mapping to the independent degrees of freedom, (ξ, η), will be singular. For this reason the transformation

$$\varphi_c : \varphi \to (\xi, \eta)$$

will be irreversible and the notion of particle should be considered as the wrong idea of quantum field theory with symmetry**.

(C) It will be shown that the result of action of the operator $\exp\{i\hat{\mathbb{K}}\}$ for transformed theories may be expressed as the sum of contributions on all boundaries ∂W:

$$|Z|^2 = |Z|^2_{\rm sc} + \sum_k \int d\xi_k(0) \frac{\partial}{\partial \xi_k(0)} C_\xi + \sum_k \int d\eta_k(0) \frac{\partial}{\partial \eta_k(0)} C_\eta, \qquad (1.11)$$

where the first term presents a semiclassical contribution and C_ξ, C_η contain quantum corrections. This result shows that the quantum corrections greatly depend on the topology of classical trajectory.

This important observation solves a number of problems. For instance, it is known that the Coulomb trajectory is closed because of Bargman–Fock symmetry, independent of the initial conditions. For this reason the corrections on ∂W of the Coulomb problem are canceled and the H-atom problem is pure semiclassical. We will find the same for the sine-Gordon model [11] as the consequence of mapping on the Arnold hypertorus [12].

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*It will be seen from our selection rule that the measure on which particles mechanics is realized is equal to zero in the field theories with symmetry.

**Considering gluon production in the frame of Yang–Mills field theory with symmetry the conclusion that gluons cannot be created should be confirmed by direct calculations, taking into account also the quark fields. That was mentioned to the author by P. Culish and will be shown in later publications. It is noticeable that the mapping in quantum mechanics is not singular and for this reason both representations before and after transformation have the equal status.

It is extremely important to keep in mind that the symmetry constraints cannot be taken into account perturbatively over the interaction constant, g. Indeed, we will see below that the expansion in $polynomial$ theories with symmetry is performed in terms of the inverse interaction constant, $1/g$. It points to the absence of the weak-coupling limit in such theories.

In the end our present aim is

— *to find representation (1.5);*

— *to investigate the main properties of theory defined on the Dirac measure (1.6);*

— *to investigate the structure of perturbation theory generated by operator* $\hat{\mathbb{K}}$ *on the measure (1.8);*

— *to find particles production probabilities for theories with symmetry.*

I understand that the perturbations scheme in terms of new variables, especially in theories with symmetry, is outside the habitual one (see [13–15]) and for this reason the approach will be described in more detail, giving step-by-step the properties of a new quantization scheme by appropriate examples. I think that such a nonformal scheme of the description is much more transparent, although the text may contain reiterations with the used method of description far from completeness.

2. SIMPLEST EXAMPLES

2.1. Introduction. As has been mentioned above, a technical aspect of our idea is the suggestion to calculate the probability without the intermediate step of calculations of the amplitudes. In the present section we restrict ourselves to the simplest problem — to the motion of a particle in a potential $V(x)$.

Let the amplitude $A(x_2, T; x_1, 0)$ describes the motion of the particle from the point x_1 to the point x_2 during the time T. Using the spectral representation:

$$A(x_2, T; x_1, 0) = \sum_n \psi_n(x_2)\psi_n^*(x_1)\, \mathrm{e}^{iE_nT}, \tag{2.1}$$

we have for probability:

$$W(x_2, T; x_1, 0) = \sum_{n_1,n_2} \psi_{n_1}(x_2)\psi_{n_1}^*(x_1)\psi_{n_2}^*(x_2)\psi_{n_2}(x_1)\mathrm{e}^{i(E_{n_1}-E_{n_2})T}. \tag{2.2}$$

Taking into account the ortho-normalizability condition:

$$\int dx\, \psi_n(x)\psi_m^*(x) = \delta_{n,m}, \tag{2.3}$$

the total probability:

$$\int dx_2\, dx_1 W\,(x_2, T; x_1, 0) = \sum_n \delta_{n,n} = \Omega \tag{2.4}$$

is the time-independent quantity which coincides with the number of existing physics states. Therefore, the amplitude (2.1) is time-dependent, but the total probability (2.4) is not. This means that the time is the unwanted parameter from the point of view of experiment described by the probability (2.4). Notice also the role of boundary condition (2.3).

The quantity (2.4) is of no interest to experiment. Much more interesting is the probability $\rho(E)$, where E is the energy experimentally measured. The Fourier transform of $A(x_2, T; x_1, 0)$ with respect to T

$$a(x_2, x_1; E) = \sum_n \frac{\psi_n(x_2)\psi_n^*(x_1)}{E - (E_n + i\varepsilon)} \tag{2.5}$$

leads to the probability:

$$\omega(x_2, x_1; E) = |a(x_2, x_1; E)|^2 = \sum_{n_1, n_2} \frac{\psi_{n_1}(x_2)\psi_{n_1}^*(x_1)}{E - (E_{n_1} + i\varepsilon)} \frac{\psi_{n_2}^*(x_2)\psi_{n_2}(x_1)}{E - (E_{n_2} - i\varepsilon)} \tag{2.6}$$

and the total probability:

$$\rho(E) = \int dx_1\, dx_2\, \omega(x_2, x_1; E) = \sum_n \left| \frac{1}{E - E_n - i\varepsilon} \right|^2 =$$
$$= \frac{1}{\varepsilon} \sum_n \mathrm{Im}\, \frac{1}{E - E_n - i\varepsilon} = \frac{\pi}{\varepsilon} \sum_n \delta(E - E_n). \tag{2.7}$$

The total probability $\rho(E)$ again coincides with the number of existing states but for all that it is seen that the unphysical, i.e., needless, states from the point of view of measurement with $E \neq E_n$ were canceled*.

Let us use now the proper-time representation:

$$a(x_1, x_2; E) = \sum_n \Psi_n(x_1)\, \Psi_n^*(x_2) i \int_0^\infty dT\, \mathrm{e}^{i(E - E_n + i\varepsilon)T} \tag{2.8}$$

to see the integral form of cancellation of unwanted contributions and insert it into definition of total probability ($\varepsilon \to +0$):

$$\rho(E) = \int dx_1\, dx_2 |a(x_1, x_2; E)|^2 =$$
$$= \sum_n \int_0^\infty dT_+ dT_-\, \mathrm{e}^{-(T_+ + T_-)\varepsilon}\, \mathrm{e}^{i(E - E_n)(T_+ - T_-)}. \tag{2.9}$$

*Such contributions enter into the real part of $a(x_2, x_1; E)$.

We will introduce new time variables instead of $T_{\pm}$:

$$T_{\pm} = T \pm \tau, \tag{2.10}$$

where, as it follows from Jacobian of transformation, $|\tau| \leqslant T,\ 0 \leqslant T \leqslant \infty$. But we can put $|\tau| \leqslant \infty$ since $T \sim 1/\varepsilon \to \infty$ is essential in integral over T. As a result,

$$\rho(E) = 4\pi \sum_n \int_0^{\infty} dT\, \mathrm{e}^{-2\varepsilon T} \int_{-\infty}^{+\infty} \frac{d\tau}{\pi}\, \mathrm{e}^{2i(E-E_n)\tau} = \frac{\pi}{\varepsilon} \sum_n \delta(E - E_n). \tag{2.11}$$

In the last integral all contributions with $E \neq E_n$ have been canceled and only the acceptable from physics point of view contributions with $E = E_n$ have survived. This peculiarity of considered interference phenomena which is the consequence of unitarity condition, i.e., its ability to extract only physics states, would have the significant applications.

Note also that the product of amplitudes $a \cdot a^*$ was «linearized» after introduction of «virtual» time $\tau = (T_+ - T_-)/2$, i.e., after transformation (2.10) we start calculation of the imaginary part. The meaning of such variables will be discussed also in Subsec. 2.2.

2.2. The Generalized Stationary-Phase Method.

2.2.1. 0-Dimensional Model. Let us practise considering the «0-dimensional» integral:

$$A = \int_{-\infty}^{+\infty} \frac{dx}{(2\pi)^{1/2}} \exp\left[i\left(\frac{1}{2}ax^2 + \frac{1}{3}bx^3\right)\right], \tag{2.12}$$

with $\operatorname{Im} a \to +0$ and $b > 0$. This example is useful since it allows one to illustrate practically all technical tricks of the approach.

We want to compute the «probability»

$$R = |A|^2 = \int_{-\infty}^{+\infty} \frac{dx_+\, dx_-}{2\pi} \exp\left[i\left(\frac{1}{2}ax_+^2 + \frac{1}{3}bx_+^3\right) - i\left(\frac{1}{2}a^* x_-^2 + \frac{1}{3}bx_-^3\right)\right]. \tag{2.13}$$

New variables:

$$x_{\pm} = x \pm e \tag{2.14}$$

will be introduced to find out the cancellation phenomenon. As a result,

$$R = \int_{-\infty}^{+\infty} \frac{dx\, de}{\pi} \exp\left[-2(x^2 + e^2) \operatorname{Im} a\right] \exp\left[2i(\operatorname{Re} a\ x + 2bx^2)e\right] \exp\left(2i\frac{b}{3}e^3\right), \tag{2.15}$$

where the prescription that $\mathrm{Im}\, a \to +0$ has been used. Note that integrations are performed along the real axis.

We will compute the integral over e perturbatively. For this purpose the transformation:

$$F(e) = \lim_{j=e'=0} \exp\left(\frac{1}{2i}\hat{j}\hat{e}'\right) \mathrm{e}^{2ije} F(e'), \tag{2.16}$$

which is valid for any differentiable function, will be used. In (2.16), two auxiliary variables j and e' have been introduced and the «hat» symbol means the differential over corresponding quantity:

$$\hat{j} = \frac{\partial}{\partial j}, \quad \hat{e}' = \frac{\partial}{\partial e'}. \tag{2.17}$$

The auxiliary variables must be taken equal to zero at the very end of calculations.

Choosing

$$\ln F(e) = -2e^2 \mathrm{Im}\, a + 2i\frac{b}{3}e^3 \tag{2.18}$$

we will find:

$$R = \lim_{j=e=0} \exp\left(\frac{1}{2i}\hat{j}\hat{e}\right) \int\limits_{-\infty}^{+\infty} dx\ \mathrm{e}^{-2(x^2+e^2)\mathrm{Im}\, a} \exp\left(2i\frac{b}{3}e^3\right) \delta(\mathrm{Re}\, a\ x + bx^2 + j). \tag{2.19}$$

Therefore, the destructive interference among two exponents in the product $a \cdot a^*$ unambiguously determines both integrals, over x and over e. The integral over difference $e = (x_+ - x_-)/2$ gives δ-function, and then this δ-function defines the contributions in the last integral over $x = (x_+ + x_-)/2$. Following the definition of δ-function only a strict solution of equation

$$\mathrm{Re}\, a\ x + bx^2 + j = 0 \tag{2.20}$$

gives the contribution into R.

But one can note that this is not the complete solution of the problem: the expansion of operator exponent $\exp\{\frac{1}{2i}\hat{j}\hat{e}\}$ generates the asymptotic series. Note also that it is impossible to remove the source, j, dependence (only harmonic case, $b = 0$, is free from j).

Equation (2.20) at $j = 0$ has the solutions, at $x_1 = 0$ and at $x_2 = -a/b$. Performing trivial transformation $e \to ie$, $\hat{e} \to -i\hat{e}$ of auxiliary variable we find at the limit $\mathrm{Im}\, a = 0$ that the contribution from x_1 extremum (minimum) has the expression*:

$$R = \frac{1}{a} \exp\left(-\frac{1}{2}\hat{j}\hat{e}\right)(1 - 4bj/a^2)^{-1/2} \exp\left(2\frac{b}{3}e^3\right), \tag{2.21}$$

*The contribution of x_2 leads to divergent series.

and the expansion of an operator exponent gives the asymptotic series:

$$R=\frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\frac{(6n-1)!!}{n!}\left(\frac{2b^4}{3a^6}\right)^n, \quad (-1)!!=0!!=1. \tag{2.22}$$

This series is convergent in Borel's sense. Therefore the described destructive interference has not an action upon the value of perturbation series convergence radii.

Let us calculate now R using stationary phase method. The contribution from the minimum x_1 gives ($\mathrm{Im}\, a=0$):

$$A=\mathrm{e}^{-i\hat{j}\hat{x}}\exp\left(-\frac{i}{2a}j^2\right)\exp\left(i\frac{b}{3}x^3\right)(i/a)^{1/2}. \tag{2.23}$$

The corresponding «probability» is

$$R=\frac{1}{a}\,\mathrm{e}^{-i(\hat{j}_+\hat{x}_+-\hat{j}_-\hat{x}_-)}\exp\left[-\frac{i}{2a}(j_+^2-j_-^2)\right]\exp\left[i\frac{b}{3}(x_+^3-x_-^3)\right]. \tag{2.24}$$

Introducing new auxiliary variables:

$$j_\pm=j\pm j_1, \quad x_\pm=x\pm e \tag{2.25}$$

and, correspondingly,

$$\hat{j}_\pm=(\hat{j}\pm\hat{j}_1)/2, \quad \hat{x}_\pm=(\hat{x}\pm\hat{e})/2, \tag{2.26}$$

we find from (2.24):

$$R=\frac{1}{a}\exp\left(-\frac{1}{2}\hat{j}\hat{e}\right)\exp\left(2\frac{b}{3}e^3\right)\exp\left(\frac{2b}{a^2}ej^2\right). \tag{2.27}$$

This expression does not coincide with (2.21), but it leads to the same asymptotic series (2.22). We may conclude that both considered methods of calculation of R are equivalent since the Borel regularization scheme of asymptotic series gives a unique result.

The difference between these two methods of calculation is in different organization of perturbations. So, if $F(e)$, instead of (2.18), is chosen in the form:

$$\ln F(e)=-2e^2\mathrm{Im}\, a+2i\frac{b}{3}e^3+2ibx^2e, \tag{2.28}$$

we may find (2.27) straightforwardly.

Therefore, our method has the freedom in choice of (quantum) source j^*. Indeed, the transition from perturbation theory with Eq. (2.18) to the theory with

*This freedom was mentioned first by A. Ushveridze.

Eq. (2.28) formally looks like the following transformation of the argument of δ-function:

$$\delta(ax + bx^2 + j) = \lim_{e'=j'=0} \mathrm{e}^{-i\hat{j}'\hat{e}'} \, \mathrm{e}^{i(bx^2+j)e'} \delta(ax + j'). \tag{2.29}$$

Here the transformation (2.16) of the Fourier image of δ-function was used. Inserting Eq. (2.29) into (2.19) we easily find (2.27).

During analytic calculations it will be useful to have a corresponding quantum sources of the new dynamical variables. Formally this will be done using transformation (2.29). Note that this transformation will not lead to changing of the Borel regularization procedure.

2.2.2. 1-Dimensional Model. Let us calculate now the probability using the path-integral definition of amplitudes [1]. Calculating the quantity

$$|A|^2 = \langle \mathrm{in}|\mathrm{out}\rangle\langle \mathrm{in}|\mathrm{out}\rangle^* = \langle \mathrm{in}|\mathrm{out}\rangle\langle \mathrm{out}|\mathrm{in}\rangle, \tag{2.30}$$

the converging and diverging waves in the product $A \cdot A^*$ interfere in such a way that the continuum of contributions cancels each other. Indeed, the amplitude

$$A(x_2, T; x_1, 0) = \int_{x(0)=x_1}^{x(T)=x_2} \frac{Dx}{C_T} \, \mathrm{e}^{-iS_T(x)}, \;\; Dx = \prod_{t=0}^{T} \frac{dx(t)}{(2\pi)^{1/2}}, \tag{2.31}$$

where the action S_T is given by the expression:

$$S_T(x) = \int_0^T dt \left(\frac{1}{2} \, \dot{x}^2 - v(x) \right), \tag{2.32}$$

and C_T is the standard normalization coefficient:

$$C_T = \int_{x(0)=x_1}^{x(T)=x_2} Dx \exp\left(\frac{i}{2} \int_0^T dt \; \dot{x}^2 \right). \tag{2.33}$$

Let us calculate the quantity

$$R(x_2, T; x_1, 0) = \int_{x_\pm(0)=x_1}^{x_\pm(T)=x_2} \frac{Dx_+}{C_T} \frac{Dx_-}{C_T^*} \, \mathrm{e}^{-iS_T(x_+)+iS_T(x_-)}. \tag{2.34}$$

We assume for simplicity that the integration in (2.31) is performed over real trajectories. Later a general case of complex trajectories will be considered.

The convergence of functional integral at that is not important. One may restrict the range of integration for better confidence, or introduce into the Lagrangian $i\varepsilon$ term, and later remove the restriction in expression (2.40). It is interesting that the interference phenomena naturally regularize divergent integrals of (2.31) type, accumulating divergence into δ-function.

In order to take into account explicitly the interference between contributions of the trajectories $x_+(t)$ and $x_-(t)$ we shall go over from the integration over two independent trajectories x_+ and x_- to the pair (x, e):

$$x_\pm(t) = x(t) \pm e(t). \tag{2.35}$$

It must be stressed that the transformation (2.35) is linear and for this reason may be done in the path integral. After substituting (2.35) into (2.34), the argument of the exponent takes the form

$$S_T(x+e) - S_T(x-e) = 2\int_0^T dt\, e(\ddot{x} + v'(x)) - U_T(x, e), \tag{2.36}$$

where $U_T(x, e)$ is the remainder of the expansion in powers of $e(t)$ ($U_T = O(e^3)$). Note that in (2.36) we have discarded the «surface» term

$$\int_0^T dt\, \partial_t(e\dot{x}) = e(T)\,\dot{x}(T) - e(0)\,\dot{x}(0) = 0, \tag{2.37}$$

since the boundary points of the trajectories $x_+(0) = x_-(0) = x_1$ and $x_+(T) = x_-(T) = x_2$ are not varied, i.e.,

$$e(0) = e(T) = 0. \tag{2.38}$$

Next,

$$Dx_+\, Dx_- = JDxDe = 2\pi J \prod_{t=0}^{T} dx(t) \prod_{t\neq 0,T} \frac{de(t)}{2\pi}, \tag{2.39}$$

where J is an unimportant Jacobian of the transformation.

As a result of the replacement (2.35), we have

$$R(x_2, T; x_1, 0) = \\ = 2\pi J \int_{x(0)=x_1}^{x(T)=x_2} \frac{Dx}{|C_T|^2} \int_{e(0)=0}^{e(T)=0} De \exp\left[2i\int_0^T dt\, e(\ddot{x} + v'(x)) + U_T(x, e)\right]. \tag{2.40}$$

One can make use of the formula

$$\mathrm{e}^{iU_T(x,e)} = \mathrm{e}^{\hat{\mathbb{K}}(e',j)}\,\mathrm{e}^{iU_T(x,e')}\exp\left[-2i\int\limits_0^T e(t)j(t)dt\right], \tag{2.41}$$

where we have introduced the operator

$$\hat{\mathbb{K}}(e,j) = \lim_{e=j=0}\exp\left\{-\frac{1}{2i}\int\limits_0^T \frac{\delta}{\delta j(t)}\frac{\delta}{\delta e(t)}\right\}, \tag{2.42}$$

after which from (2.40) we have found that

$$\begin{aligned} R(x_2,T;x_1,0) = 2\pi J\,\mathrm{e}^{\hat{\mathbb{K}}(e',j)} \int\limits_{x(0)=x_1}^{x(T)=x_2} \frac{Dx}{|C_T|^2}\,\mathrm{e}^{iU_T(x,e')}\times \\ \times \int\limits_{e(0)=0}^{e(T)=0} De\;\exp\left\{2i\int\limits_0^T dt(\ddot{x}+v'(x)-j)e\right\} = \\ = 2\pi J\,\mathrm{e}^{\hat{\mathbb{K}}(e,j)} \int\limits_{x(0)=x_1}^{x(T)=x_2} \frac{Dx}{|C_T|^2}\,\mathrm{e}^{iU_T(x,e)} \prod_{t\neq 0,T}\delta(\ddot{x}+v'(x)-j), \end{aligned} \tag{2.43}$$

where the functional δ-function

$$\prod_{t\neq 0,T}\delta(\ddot{x}+v'(x)-j) = \int\limits_{e(0)=0}^{e(T)=0} De\;\exp\left\{2i\int\limits_0^T dt(\ddot{x}+v'(x)-j)e\right\} \tag{2.44}$$

has arisen as a result of total reduction of unnecessary contributions from the point of view of equation of motion

$$\ddot{x}(t) + V'(x) = j(t). \tag{2.45}$$

The operator (2.42) is Gaussian so that the system is perturbed by the random force $j(t)$.

If $x(t)$ is the «true» trajectory and the virtual deviation is $e(t)$, then the quantity $e(\ddot{x}+v'(x)-j)$ coincides with the virtual work. It must be equal to zero in classical mechanics since only the time reversible motion is considered. As a result, we came to equation of motion since e is arbitrary in classics.

The difference $S_T(x_+) - S_T(x_-)$ in (2.34) with boundary conditions (2.38) coincides with the action of reversible motion. Upon the substitution (2.35), we have identified the mean trajectory, $x(t)$, and the deviation from it, $e(t)$. One must integrate over $e(t)$ in quantum case, in contrast to classical one. As a result, the measure of the remaining path integral over mean trajectory $x(t)$ takes the Dirac δ-function form which unambiguously chooses the «true» trajectory.

In other words, the proposed definition of the measure of the path integral is generalization of classical d'Alambert's principle on the quantum case. The theory in the frame of this principle can take into account any external perturbations, $j(t)$ in our case, if the time reversibility of motion is conserved. In quantum case, the reversibility is established through the boundary conditions (2.38). Next, one may generalize the approach adding also the probe force which can lead to dynamical symmetry breaking [16]*.

In the semiclassical approximation $\hat{\mathbb{K}}(e,j) = 1$ and taking the limit $e = j = 0$ we find that

$$R(x_2, T; x_1, 0) = 2\pi J \int_{x(0)=x_1}^{x(T)=x_2} \frac{Dx}{|C_T|^2} \prod_{t \neq 0,T} \delta(\ddot{x} + v'(x)). \tag{2.46}$$

Let the solution of the homogeneous equation

$$\ddot{x} + v'(x) = 0 \tag{2.47}$$

be $x_c(t)$, with $x_c(0) = x_1$ and $x_c(T) = x_2$. Then

$$R(x_2, T; x_1, 0) = 2\pi J \int_{x(0)=x_1}^{x(T)=x_2} \frac{Dx}{|C_T|^2} \prod_{t \neq 0,T} \delta(\ddot{x} + v''(x_c)x). \tag{2.48}$$

The remaining integral is calculated by the standard methods**. As a result, we find

$$R(x_2, T; x_1, 0) = \frac{1}{2\pi} \left| \frac{\partial^2 S_T(x_c)}{\partial x_c(0)\, \partial x_c(T)} \right|_{x_c(0)=x_1, x_c(T)=x_2}. \tag{2.49}$$

Next, let us recall that the full derivative of the classical action is

$$dS = p_2 dx_2 - p_1 dx_1, \tag{2.50}$$

---

*It is important that if the expectation value of the probe force is not equal to zero, then the symmetry is broken. This important possibility will not be considered in the present work.

**Here it is more convenient to represent (2.48) as a production of two Gaussian integrals; later on more effective method of calculation of the functional determinant will be offered.

where p_2 and p_1 are, respectively, the final and initial momenta. Noting this definition,

$$\left|\frac{\partial^2 S_T}{\partial x_1\, \partial x_2}\right| dx_2 = dp_1, \tag{2.51}$$

we find that

$$\int dx_1\, dx_2 R(x_2, T; x_1, 0) = \int \frac{dx_1\, dp_1}{2\pi} = \Omega^2, \tag{2.52}$$

which coincides with (2.4), i.e., it agrees with conservation of total probability since (2.52) again coincides with the total number of physical states.

Deriving (2.52), we somewhat simplify the problem considering a unique solution of Eq. (2.47). A more complicate and important examples will be considered in the next sections.

2.3. Complex Trajectories. Let us consider the one-dimensional motion with fixed energy E on the complex trajectory*. The corresponding amplitude has the form:

$$A(x_1, x_2; E) = i \int_0^\infty dT\, \mathrm{e}^{iET} \int_{x_1 = x(0)}^{x_2 = x(T)} D_{C_+} x\, \mathrm{e}^{iS_{C_+}(x)}, \tag{2.53}$$

where the action

$$S_{C_+}(x) = \int_{C_+} dt \left(\frac{1}{2}\dot{x}^2 - v(x)\right) \tag{2.54}$$

and the measure

$$D_{C_+} x = \prod_{t \in C_+} \frac{dx(t)}{(2\pi)^{1/2}} \tag{2.55}$$

are defined on the shifted in the upper half-time plane Mills' contour $C_+ = C_+(T)$ [17]:

$$t \to t + i\varepsilon, \quad \varepsilon \to +0, \quad 0 \leqslant t \leqslant T. \tag{2.56}$$

Therefore, we will consider integration over real functions of complex variables:

$$x^*(t) = x(t^*). \tag{2.57}$$

It must be underlined also that the boundary conditions in (2.53) have the classical meaning, i.e., they do not vary, and x_1, x_2 are the real quantities.

*The necessity to extend the formalism on the case of complex trajectories was mentioned to the author by A. Slavnov.

The probability looks as follows:

$$R(E) = \int_0^\infty e^{iE(T_+ - T_-)} \int_{x_\pm(0)=x_1}^{x_\pm(T_\pm)=x_2} D_{C_+}x_+ D_{C_-}x_- \times$$
$$\times\ e^{iS_{C_+(T_+)}(x_+) - iS_{C_-(T_-)}(x_-)}, \quad (2.58)$$

where $C_-(T) = C_+^*(T)$ is the time contour in the lower half of the complex time plane.

New time variables

$$T_\pm = T \pm \tau \quad (2.59)$$

will be used. Considering $\operatorname{Im} E \to +0$ we can consider T and τ as the independent variables:

$$0 \leqslant T \leqslant \infty, \quad -\infty \leqslant \tau \leqslant \infty. \quad (2.60)$$

We will apply the boundary conditions, see (2.58):

$$x_1 = x_+(0) = x_-(0), \quad x_2 = x_+(T_+) = x_-(T_-). \quad (2.61)$$

Inserting (2.59) one can find in zero order over τ from (2.61) that

$$x_+(0) = x_-(0), \quad x_+(T) = x_-(T). \quad (2.62)$$

Now we will introduce also the mean trajectory $x(t) = (x_+(t) + x_-(t))/2$ and the deviation $e(t)$ from $x(t)$:

$$x_\pm(t) = x(t) \pm e(t). \quad (2.63)$$

We have considered $e(t)$ and τ as the virtual quantities. The integrals over e and τ will be calculated perturbatively. In zero order over e and τ, i.e., in the semiclassical approximation, x is the classical path, and T is the total time of classical motion. Note that one can do surely the linear transformations (2.63) in the path integrals.

The higher terms over τ put unphysical constraints on the trajectory $x(t)$:

$$\frac{d^{(2n+1)}x(T)}{dT^{(2n+1)}} = 0, \; n = 0, 1, 2, \ldots,$$

since $e(t)$ must be arbitrary. Therefore, to avoid this constraints and since the boundaries have classical unvaried meaning, we will use the minimal boundary conditions:

$$e(0) = e(T) = 0, \quad (2.64)$$

which ensures the time reversibility. Note that it is sufficient to have (2.64) if the integrals over $e(t)$ are calculated perturbatively. At the same time,

$$x(0) = x_1, \ x(T) = x_2. \tag{2.65}$$

Let us extract now the linear over e and τ terms from the closed-path action:

$$S_{C_+(T_+)}(x_+) - S_{C_-(T_-)}(x_-) = \\ = -2\tau H_T(x) - \int\limits_{C^{(+)}(T)} dt\, e(\ddot{x} + v'(x)) - \tilde{H}_T(x;\tau) - U_T(x,e), \tag{2.66}$$

where

$$C^{(+)}(T) = C_+(T) + C_-(T) \tag{2.67}$$

is the total-time path, H_T is the Hamiltonian:

$$2H_T(x) = -\frac{\partial}{\partial T}(S_{C_+(T)}(x) + S_{C_-(T)}(x)), \tag{2.68}$$

and

$$-\tilde{H}_T(x;\tau) = S_{C_+(T+\tau)}(x) - S_{C_-(T-\tau)}(x) + 2\tau H_T(x), \tag{2.69}$$

$$-U_T(x,e) = S_{C_+(T)}(x+e) - S_{C_-(T)}(x-e) + \int\limits_{C^{(+)}} dt\, e(\ddot{x} + v'(x)) \tag{2.70}$$

are the remainder terms, where $v'(x) = \partial v(x)/\partial x$. Deriving the decomposition (2.66), the definition

$$C_-(T) = C_+^*(T) \tag{2.71}$$

and the boundary conditions (2.64) were used.

One can find the compact form of expansion of

$$\mathrm{e}^{-i\tilde{H}_T(x;\tau) - iU_T(x,e)}$$

over τ and e using formulae (2.16):

$$\exp\{-i\tilde{H}_T(x;\tau) - iU_T(x,e)\} = \exp\left\{\frac{1}{2i}\hat{\omega}\hat{\tau}' - i\int\limits_{C^{(+)}(T)} dt\, \hat{j}(t)\hat{e}'(t)\right\} \times \\ \times \exp\left\{2i\omega\tau + i\int\limits_{C^{(+)}(T)} dt\, j(t)e(t)\right\} \exp\{-i\tilde{H}_T(x;\tau') - iU_T(x,e')\}. \tag{2.72}$$

At the end of calculations the auxiliary variables (ω, τ', j, e') should be taken equal to zero.

Using (2.66) and (2.72) we find from (2.58) that

$$R(E) = 2\pi \int_0^\infty dT \exp\left\{ \frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt\, \hat{j}(t)\hat{e}(t) \right\} \times$$
$$\times \int Dx \exp\{-i\tilde{H}_T(x;\tau) - iU_T(x,e)\}\delta(E+\omega-H_T(x)) \times$$
$$\times \prod_{C^{(+)}} \delta(\ddot{x} + v'(x) - j). \quad (2.73)$$

The expansion over the differential operators:

$$\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt\, \hat{j}(t)\hat{e}(t) = \frac{1}{2i}\left(\frac{\partial}{\partial\omega}\frac{\partial}{\partial\tau} + \mathrm{Re} \int_{C+} dt \frac{\delta}{\delta j(t)}\frac{\delta}{\delta e(t)} \right) \quad (2.74)$$

will generate the perturbation series. We propose that it is summable in Borel sense.

The first δ-function in (5.33) fixes the conservation of energy:

$$E + \omega = H_T(x), \quad (2.75)$$

where E is the observed energy; $H_T(x)$ is the energy at the mean trajectory at the time moment T, and ω is the energy of quantum fluctuations. The second δ-function*

$$\prod_{t\in C^{(+)}} \delta(\ddot{x}+v'(x)-j) = (2\pi)^2 \int \prod_{t\in C^{(+)}} \frac{de(t)}{\pi}\delta(e(0))\delta(e(T)) \times$$
$$\times \exp\left[-2i\mathrm{Re}\int_{C_+} dt\, e(\ddot{x}+v'(x)-j)\right] =$$
$$= \prod_{t\in C_+(T)} \delta(\mathrm{Re}\,(\ddot{x}+v'(x)-j))\delta(\mathrm{Im}\,(\ddot{x}+v'(x)-j)) \quad (2.76)$$

---

*Following shorthand entry of δ-function of the complex argument:

$$\prod_{C^{(+)}} \delta(f(t)) = \prod_{C_+}\delta(f(t))\prod_{C_-}\delta(f(t)) =$$
$$= \prod_{C_+}\delta(\mathrm{Re}\ f(t) + i\mathrm{Im}\ f(t))\delta(\mathrm{Re}\ f(t) - i\mathrm{Im}\ f(t)) = \prod_{C_+}\delta(\mathrm{Re}\ f(t))\,\delta(\mathrm{Im}\ f(t))$$

will be useful during calculations. The condition (2.57) is important here. The inessential constant can be canceled by normalization. So, as a result of analytical continuation of $C_\pm$ on the real axis the product of two δ-functions reduces to single one since $\delta^2(\mathrm{Re}\, f(x)) = \delta(0)\delta(\mathrm{Re}\, f(x)) = \delta(0)\delta(f(x))$, and $\delta(0)$ must be canceled by normalization. Offered abbreviated notation will allow one to consider δ-function on the complex time contour as the ordinary one.

fixes the function $x(t)$ of complex argument on $C^{(+)}$ completely by the equation

$$\ddot{x} + v'(x) = j. \tag{2.77}$$

The physics meaning of δ-function (2.76) was discussed in Subsec. 2.3 noting that the unitarity condition of quantum theories plays the same role as d'Alambert's variational principle in classical mechanics.

In (2.77), $j(t)$ describes the external quantum force. The solution $x_j(t)$ of this equation will be found expanding it over $j(t)$:

$$x_j(t) = x_c(t) + \int dt_1 \, G(t, t_1) j(t_1) + \dots \tag{2.78}$$

This is sufficient since $j(t)$ is the auxiliary variable*. In this decomposition $x_c(t)$ is the strict solution of unperturbed equation:

$$\ddot{x} + v'(x) = 0. \tag{2.79}$$

Note that the functional δ-function in (2.76) does not contain the end-point values of $x(t)$, at $t = 0$ and $t = T$. This means that if we integrate over x_1 and x_2, then the initial conditions to Eq. (2.79) are not fixed and the integration over them must be performed.

Inserting (2.78) into (2.77) we find the equation for the Green function:

$$(\partial^2 + v''(x_c))_t G(t, t'; x_c) = \delta(t - t'). \tag{2.80}$$

It is too hard to find the exact solution of this equation if $x_c(t)$ is the nontrivial function of t. We will see that the canonical transformation to the (action–angle)-type variables can help to avoid this problem, see the following section.

2.4. Conclusions.

1. The path integral must be defined on the Mills time contour. This condition will be important in the field theories with high space-time symmetries (such as the Yang–Mills type theory) since it seems that for such theories with symmetry one cannot perform surely the analytic continuation over time variable**.

2. The quantization can be performed without transition to the canonical formalism, using only the Lagrange one which is more natural for relativistic field theories.

3. Only the exact solutions of the equation of motion must be taken into account defining the contributions into the functional integral.

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*See also footnote on page 1024.

**The fact that a theory must satisfy certain conditions upon analytic continuation over time variable is clear from [18].

3. PATH INTEGRALS ON DIRAC MEASURE

3.1. Introduction. In the present section we will offer two methods which may simplify calculation of path integrals on Dirac measure. They are based on the possibility to perform transformation of the path-integral variables.

We will consider two examples. In the first example the transformation to the (action–angle)-type variables will be considered. This example shows how much the calculations of path integrals may be simplified.

In the second part of the present section the coordinate transformation will be described. For the sake of definiteness the transformation to cylindrical coordinates will be considered.

3.2. Canonical Transformation. Let us introduce the first-order formalism. We will insert in (2.73)

$$1 = \int Dp \prod_t \delta(p - \dot{x}). \tag{3.1}$$

As a result,

$$R(E) = 2\pi \int_0^\infty dT \exp\left[\frac{1}{2i}(\hat{\omega}\hat{\tau} + \mathrm{Re}\int_{C_+(T)} dt\, \hat{j}(t)\hat{e}(t))\right] \times \\ \times \int DxDp\, \mathrm{e}^{-i\tilde{H}_T(x;\tau) - iU_T(x,e)} \times \\ \times \delta(E + \omega - H_T(x)) \prod_t \delta\left(\dot{x} - \frac{\partial H_j}{\partial p}\right) \delta\left(\dot{p} + \frac{\partial H_j}{\partial x}\right), \tag{3.2}$$

where

$$H_j = \frac{1}{2}p^2 + v(x) - jx \tag{3.3}$$

may be considered as the total Hamiltonian which is time-dependent through $j(t)$. Notice that in the present simplest case x and p are independent parameters and therefore (3.3) defines the Hamiltonian.

Instead of pair $(x(t), p(t))$ we introduce new pair $(\theta(t), h(t))$ inserting in (3.2)

$$1 = \int \prod_t d\theta\, dh\, \delta\left(h - \frac{1}{2}p^2 - v(x)\right) \delta\left(\theta - \int^x dx\, (2(h - v(x)))^{-1/2}\right). \tag{3.4}$$

Note that the integral measures in (3.2) and (3.4) are both δ-like, i.e., have the equal power. It allows one to change the order of integration and first integrate over (x, p). We find that

$$R(E) = 2\pi \int_0^\infty dT \exp\left[\frac{1}{2i}(\hat{\omega}\hat{\tau} + \mathrm{Re}\int_{C_+(T)} dt\, \hat{j}(t)\hat{e}(t))\right] \times$$
$$\times \int D\theta Dh\, \mathrm{e}^{-i\tilde{H}_T(x_c;\tau) - iU_T(x_c,e)} \times$$
$$\times\, \delta(E + \omega - h(T)) \prod_t \delta\left(\dot{\theta} - \frac{\partial H_c}{\partial h}\right) \delta\left(\dot{h} + \frac{\partial H_c}{\partial \theta}\right), \quad (3.5)$$

where

$$H_c = h - jx_c(h, \theta) \quad (3.6)$$

is the transformed Hamiltonian and $x_c(\theta, h)$ is the given solution of algebraic equation:

$$\theta = \int^x dx(2(h - v(x)))^{-1/2}, \quad (3.7)$$

i.e., x_c is the classical trajectory parameterized in terms of $h(t)$ and $\theta(t)$.

As follows from (3.5), new variables, $h(t)$ and $\theta(t)$, are subjected to the action of quantum force $j(t)$ and the topology of classical trajectory x_c remains unchanged.

So, instead of Eq. (2.77) we must solve the equations:

$$\dot{h} = j\frac{\partial x_c}{\partial \theta}, \qquad \dot{\theta} = 1 - j\frac{\partial x_c}{\partial h}, \quad (3.8)$$

which have a simpler structure. Expanding the solutions over j, we will find the infinite set of recursive equations. This is the important peculiarity of used quantization scheme.

Note now that $j\partial x_c/\partial\theta$ and $j\partial x_c/\partial h$ in the r.h.s. can be considered as the new sources. We will use this property of Eqs. (3.8) and introduce in the perturbation theory new «renormalized» sources:

$$j_h = j\frac{\partial x_c}{\partial \theta}, \qquad j_\theta = j\frac{\partial x_c}{\partial h}, \quad (3.9)$$

i.e., j_ξ and j_η are the forces on the cotangent bundle. We will use transformations (2.29):

$$\prod_t \delta(\dot{h} - j\frac{\partial x_c}{\partial \theta}) =$$
$$= \exp\left[\frac{1}{2i}\mathrm{Re}\int_{C_+} dt\, \hat{j}_h(t)\hat{e}_h(t)\right] \exp\left(2i\mathrm{Re}\int_{C_+} e_h j\frac{\partial x_c}{\partial \theta}\right) \prod_t \delta(\dot{h} - j_h) \quad (3.10)$$

and

$$\prod_t \delta\left(\dot\theta - 1 + j\frac{\partial x_c}{\partial h}\right) = \\ = \exp\left[\frac{1}{2i}\mathrm{Re}\int_{C_+} dt\,\hat{j}_\theta(t)\hat{e}_\theta(t)\right]\exp\left(2i\mathrm{Re}\int_{C_+} e_\theta j\frac{\partial x_c}{\partial h}\right)\prod_t \delta(\dot\theta - 1 - j_\theta) \quad (3.11)$$

to introduce them. The rescaling of source j leads to the rescaling of auxiliary field e. In the new perturbation theory we will have two sources j_h, j_θ and two auxiliary fields e_h, e_θ. Notice that the momentum p never arose.

Inserting (3.10), (3.11) into (3.5) we find:

$$R(E) = 2\pi\int_0^\infty dT\exp\left[\frac{1}{2i}(\hat\omega\hat\tau - i\int_{C^{(+)}} dt\,(\hat{j}_h(t)\,\hat{e}_h(t) + \hat{j}_\theta(t)\,\hat{e}_\theta(t)))\right]\times \\ \times\int DhD\theta\,\mathrm{e}^{-i\tilde{H}_T(x_c;\tau) - iU_T(x_c,e_c)}\times \\ \times\,\delta(E + \omega - h(T))\prod_t \delta(\dot\theta - 1 - j_\theta)\delta(\dot h - j_h), \quad (3.12)$$

where

$$e_c = e_h\frac{\partial x_c}{\partial\theta} - e_\theta\frac{\partial x_c}{\partial h} \quad (3.13)$$

carry the simplectic structure of Hamilton equations of motion, and the «hat» symbol means differential operator over corresponding quantity. At the very end one should take all auxiliary variables, $(e_h, j_h, e_\theta, j_\theta)$, equal to zero.

Hiding the $x_c(t)$ dependence into e_c we solve the problem of the functional determinants, see (3.12), and simplify the Hamilton equations of motion as much as possible:

$$\dot h(t) = j_h(t), \qquad \dot\theta(t) = 1 + j_\theta(t). \quad (3.14)$$

We will use the boundary conditions

$$h(0) = h_0, \quad \theta(0) = \theta_0 \quad (3.15)$$

as the extension of boundary conditions in (2.58). This leads to the following Green function of transformed perturbation theory:

$$g(t - t') = \Theta(t - t'), \quad (3.16)$$

with the properties of projection operator:

$$\int dt\,dt' g^2(t - t') = \int dt\,dt' g(t - t'), \\ \int dt\,dt' g(t - t')g(t' - t) = 0, \quad (3.17)$$

and, at the same time, we will assume that

$$g(0) = 1. \tag{3.18}$$

It is important to note that $\operatorname{Im} g(t)$ is regular on the real time axis. This is the very simplification of the perturbation theory since it eliminates the doubling of degrees of freedom. One may use here the analytical continuation to the real time axis.

As a result, shifting C_+ and C_- contours on the real time axis we find:

$$R(E) = \\ = 2\pi \int_0^\infty dT \exp\left[\frac{1}{2i}(\hat{\omega}\hat{\tau} + \int_0^\infty dt_1\, dt_2 \Theta(t_1 - t_2)(\hat{e}_h(t_1)\, \hat{h}(t_2) + \hat{e}_\theta(t_1)\, \hat{\theta}(t_2)))\right] \times \\ \times \int dh_0\, d\theta_0\, \mathrm{e}^{-i\tilde{H}_T(x_c;\tau) - iU_T(x_c, e_c)}\, \delta(E + \omega - h_0 + h(T)), \tag{3.19}$$

where the solutions of Eqs. (3.14) were used. In this expression $x_c(t) = x_c(h_0 - h(t), t + \theta_0 - \theta(t))$ and $(h(t), e_h(t), \theta(t), e_\theta(t))$ are the auxiliary fields. At the very end one must take them equal to zero.

3.3. Selection Rule. Let us consider the theory with Lagrangian

$$L(x) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \frac{g}{4}x^4. \tag{3.20}$$

The Dirac measure gives the equation (of motion):

$$\ddot{x} + \omega^2 x + g x^3 = j. \tag{3.21}$$

It has two solutions:

$$x_1(t) = x_c(t) + O(j), \quad x_2(t) = O(j). \tag{3.22}$$

For this reason,

$$R(E) = R_1(E; x_1) + R_2(E; x_2) \tag{3.23}$$

and which one defines $R(E)$ is a question. Following to our selection rule it is just R_1. This will be shown.

Let us return now to the example with Lagrangian (3.20). In the semiclassical approximation

$$R_1(E; x_1) = \int_0^\infty dT \int_0^\infty dh_0 \int_{-\infty}^{+\infty} d\theta_0\, \mathrm{e}^{-iU_T(x_c, 0)} \delta(E - h_0). \tag{3.24}$$

Therefore,

$$R_1(E;x_1) \sim \int_{-\infty}^{+\infty} d\theta_0 \equiv \Omega, \tag{3.25}$$

i.e., it is proportional to the volume of group of time translations.

At the same time

$$R_2(E;x_2) = O(1) \tag{3.26}$$

in the semiclassical approximation. Therefore,

$$R = R_1(1 + O(1/\Omega)). \tag{3.27}$$

This result explains the source of chosen selection rule.

3.4. Coordinate Transformation. In this section the coordinate transformation of two-dimensional quantum mechanical model with potential

$$v = v((x_1^2 + x_2^2)^{1/2}) \tag{3.28}$$

will be considered. Repeating calculations of the previous sections,

$$R(E) = 2\pi \int_0^\infty dT \exp\left[\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt\, \hat{\mathbf{j}}(t)\hat{\mathbf{e}}(t)\right] \times \\ \times \int D^{(2)}M(x)\, \mathrm{e}^{-i\tilde{H}_T(x;\tau) - iU_T(x,e)}, \tag{3.29}$$

where the δ-like Dirac measure is

$$D^{(2)}M(x) = \delta(E + \omega - H_T(x)) \prod_t d^2x(t)\delta^{(2)}(\ddot{x} + v'(x) - j). \tag{3.30}$$

In the classical mechanics the problem with potential (3.28) is solved in the cylindrical coordinates:

$$x_1 = r\cos\phi, \qquad x_2 = r\sin\phi. \tag{3.31}$$

We insert into (3.29)

$$1 = \int DrD\phi \prod_t \delta(r - (x_1^2 + x_2^2)^{1/2})\delta\left(\phi - \arctan\frac{x_2}{x_1}\right) \tag{3.32}$$

to perform the transformation. Note that the transformation (3.31) is not canonical. As a result we will find a new measure:

$$D^{(2)}M(r,\phi) = \delta(E + \omega - H_T(x)) \prod_t dr\, d\phi\, J(r,\phi), \tag{3.33}$$

where the Jacobian of transformation

$$J(r,\phi)=\int\prod d^2x\delta^{(2)}(\ddot{x}+v'(x)-j)\delta\left(\phi-\arctan\frac{x_2}{x_1}\right)\delta(r-(x_1^2+x_2^2)^{1/2}) \tag{3.34}$$

is the product of two δ-functions:

$$J(r,\phi)=\prod_t r^2(t)\delta(\ddot{r}-\dot{\phi}^2r+v'(r)-j_r)\,\delta(\partial_t(\dot{\phi}r^2)-rj_\phi), \tag{3.35}$$

where $v'(r)=\partial v(r)/\partial r$, and

$$j_r=j_1\cos\phi+j_2\sin\phi,\quad j_\phi=-j_1\sin\phi+j_2\cos\phi \tag{3.36}$$

are the components of $\mathbf{j}$ in the cylindrical coordinates.

It is useful to organize the perturbation theory in terms of j_r and j_ϕ. For this purpose the following transformation of arguments of δ-functions will be used:

$$\prod_t\delta(\ddot{r}-\dot{\phi}^2r+v'(r)-j_r)=$$
$$=\exp\left(-i\int_{C^{(+)}}dt\,\hat{j}_r'\hat{e}_r\right)\exp\left(i\int_{C^{(+)}}dt\,j_re_r\right)\prod_t\delta(\ddot{r}-\dot{\phi}^2r+v'(r)-j_r') \tag{3.37}$$

and

$$\prod_t\delta(\partial_t(\dot{\phi}r^2)-rj_\phi)=$$
$$=\exp\left(-i\int_{C^{(+)}}dt\,\hat{j}_\phi'\hat{e}_\phi\right)\exp\left(i\int_{C^{(+)}}dt\,j_\phi re_\phi\right)\prod_t r(t)\,\delta(\partial_t(\dot{\phi}r^2)-j_\phi'). \tag{3.38}$$

Here j_r and j_ϕ were defined in (3.36). Therefore, we get to the path integral formalism written in terms of cylindrical coordinates. This is a very simplification which will help to solve a lot of mechanical problems. One can note that as a result of mapping our problem reduced to the description of quantum fluctuations of the surface of cylinder:

$$R(E)=2\pi\int_0^\infty dT\exp\left[\frac{1}{2i}\hat{\omega}\hat{\tau}-i\int_{C^{(+)}(T)}dt\,(\hat{j}_r(t)\,\hat{e}_r(t)+\hat{j}_\phi(t)\,\hat{e}_\phi(t))\right]\times$$
$$\times\int D^{(2)}M(r,\phi)\,\mathrm{e}^{-i\tilde{H}_T(x;\tau)-iU_T(x,e_C)}, \tag{3.39}$$

where

$$D^{(2)}M(r,\phi)=\delta(E+\omega-H_T(r,\phi))\prod_t r^2(t)\,dr(t)\,d\phi(t)\times$$
$$\times\,\delta(\ddot{r}-\dot{\phi}^2r+v'(r)-j_r)\delta(\partial_t(\dot{\phi}r^2)-j_\phi) \tag{3.40}$$

and

$$e_{C,1} = e_r \cos\phi - re_\phi \sin\phi, \quad e_{C,2} = e_r \sin\phi + re_\phi \cos\phi. \tag{3.41}$$

This is the final result. The transformation looks quite classically but (3.39) cannot be deduced from naive coordinate transformation of initial path integral for amplitude.

Inserting

$$1 = \int DpDl \prod_t \delta(p - \dot{r})\delta(l - \dot{\phi}r^2) \tag{3.42}$$

into (3.39) we can introduce the motion in the phase space with Hamiltonian

$$H_j = \frac{1}{2}p^2 + \frac{l^2}{2r^2} + v(r) - j_r r - j_\phi \phi. \tag{3.43}$$

The Dirac measure becomes four-dimensional:

$$D^{(4)}M(r,\phi,p,l) = \delta(E + \omega - H_T(r,\phi,p,l)) \prod_t dr(t)\, d\phi(t)\, dp(t)\, dl(t) \times \\ \times \delta\left(\dot{r} - \frac{\partial H_j}{\partial p}\right) \delta\left(\dot{\phi} - \frac{\partial H_j}{\partial l}\right) \delta\left(\dot{p} + \frac{\partial H_j}{\partial r}\right) \delta\left(\dot{l} + \frac{\partial H_j}{\partial \phi}\right). \tag{3.44}$$

Note absence of the coefficient r^2 in this expression. This is the result of special choice of transformation (3.38).

Since the Hamilton group manifolds are more rich than Lagrange ones, the measure (3.44) can be considered as the starting point of farther transformations. One must note that the (action–angle) variables are most useful [12]. Note also that to avoid the technical problems with equations of motion and with functional determinants it is useful to linearize the argument of δ-functions in (3.44) hiding nonlinear terms into the corresponding auxiliary variables e_c.

3.5. Conclusions

1. Our perturbation theory describes the quantum fluctuations of the parameters (h, θ) of classical trajectory x_c. It is more complicated than canonical one, over an interaction constant [19], since demands investigation of analytic properties of $4N$-dimensional integrals, where $2N$ is the phase–space dimension. Indeed, in the considered case with $N = 1$ the perturbations-generating operator, $\hat{\mathbb{K}}$, see (3.12), contains derivatives over four auxiliary parameters, $(j_h, e_h, j_\theta, e_\theta)$.

Our transformed theory describes the «direct» deformations of classical trajectory $x_c = x_c(h, \theta)$, i.e., just h and θ are the objects of quantization in the considered example. In other words, the quantum deformation of the invariant hypersurface, (h, θ), is described in the new quantum theory. This possibility is the consequence of δ-likeness of measure, i.e., it is based on the conservation of total probability.

The Dirac measure allows one to perform classical transformations of the measure and to use high resources of classical mechanics. For example, the interesting possibility may arise in connection with Kolmogorov–Arnold–Mozer (KAM) theorem [4]: the system which is not strictly integrable can show the stable motion peculiar to integrable systems. This is the argument in favor of the idea that there may be another, non-topological, mechanism of suppression of the quantum excitations.

2. One can note that the transformed perturbation theory describes only the retarded quantum fluctuations, see definition of Green function (3.16). This feature of the theory can lead to the imaginary time irreversibility of quantum processes and it must be explained.

The starting expression (2.58) describes the reversible in time motion since total action $S_{C_+(T_+)}(x_+) - S_{C_-(T_-)}(x_-)$ is time reversible. But the unitarity condition forced us to consider the interference picture between expanding and converging waves. This is fixed by the boundary conditions $e(0) = e(T) = 0$. The quantum theory remains time reversible up to canonical transformation to the invariant hypersurface of the constant energy. The causal Green function $G(t, t')$, see (2.80), is able to describe both advanced and retarded perturbations and the theory contains the doubling of degrees of freedom. It means that the theory «keeps in mind» the time reversibility. But after the canonical transformation, using the above-mentioned boundary conditions, and continuing the theory to the real time, the quantum perturbations were transferred on the inner degrees of freedom of classical trajectory. As a result, the memory of doubling of the degrees of freedom disappears and the theory becomes «time irreversible».

The key step in this calculations was an extraction of the classical trajectory x_c which cannot be defined without definition of boundary conditions. Just x_c introduces the direction of motion and the order of quantum perturbations of trajectories inner degrees of freedom playes no role, i.e., the mechanical motion is time reversible while the corrections to energy of trajectory, h, and to the phase, θ, cannot be time reversible. Therefore, the considered irreversibility of the quantum mechanics in terms of (h, θ) seems to be imaginary.

4. REDUCTION OF QUANTUM DEGREES OF FREEDOM

4.1. Introduction. It will be shown in this section that the quantum fluctuations of angular variables may be removed if the classical motion is periodic. This cancellation mechanism can be used for path-integral explanation of integrability of the quantum-mechanical problems, for example, of H-atom problem where the classical trajectories are closed independently of the initial conditions*. The main

*The approach may be extended to the case of rigid rotator problem [20]. Last one is isomorphic to the Pöschl–Teller problem [21].

result of the present section is based on the statement that the topology properties of classical trajectory take special significance*.

Our technical problem consists in necessity to extract the quantum angular degrees of freedom. For this purpose we will define path integral in the phase space of action–angle variables. For simplicity we will demonstrate the effect of cancellations on the one-dimensional λx^4 model. In the following subsection the brief description of unitary definition of the path-integral measure will be given. The perturbation theory in terms of action–angle variables will be contracted in Subsec. 4.3 (the scheme of transformed perturbation theory was given firstly in [1]). In Subsec. 4.4 the cancellation mechanism will be demonstrated.

4.2. Unitary Definition of the Path-Integral Measure. We will calculate the probability

$$R(E) = \int dx_1\, dx_2 |A(x_1, x_2; E)|^2 \tag{4.1}$$

to introduce the unitary definition of path-integral measure [1]. Here

$$A(x_1, x_2; E) = i \int_0^\infty dT\, \mathrm{e}^{iET} \int_{x(0)=x_1}^{x(T)=x_2} Dx\, \mathrm{e}^{iS_{C_+(T)}(x)} \tag{4.2}$$

is the amplitude of the particle with energy E moving from x_1 to x_2. The action

$$S_{C_+(T)}(x) = \int_{C_+(T)} dt \left(\frac{1}{2}\dot{x}^2 - \frac{\omega_0^2}{2} x^2 - \frac{\lambda}{4} x^4 \right) \tag{4.3}$$

is defined on the Mills contour [17]:

$$C_\pm(T) : t \to t \pm i\epsilon, \quad \epsilon \to +0, \quad 0 \leqslant t \leqslant T. \tag{4.4}$$

So, we will omit the calculation of the amplitude.

Inserting (4.2) into (4.1) we find, see the previous section, that

$$R(E) = 2\pi \int_0^\infty dT \exp\left[\frac{1}{2i} \hat{\omega} \hat{\tau} - i \int_{C^{(+)}(T)} dt\, \hat{j}(t)\, \hat{e}(t) \right] \int Dx\, \mathrm{e}^{-i\tilde{H}(x;\tau) - iU_T(x,e)} \times \\ \times \delta(E + \omega - H_T(x)) \prod_t \delta(\ddot{x} + \omega_0^2 x + \lambda x^3 - j). \tag{4.5}$$

*Since the action of perturbations-generating operator of transformed theory, $\hat{\mathbb{K}}$, maps quantum corrections on the boundaries of cotangent foliation, ∂W, see (4.41).

The «hat» symbol means differentiation over corresponding auxiliary quantity. For instance,

$$\hat{\omega} \equiv \frac{\partial}{\partial \omega}, \quad \hat{j}(t) = \frac{\delta}{\delta j(t)}. \tag{4.6}$$

It will be assumed that

$$\begin{aligned} &\hat{j}(t \in C_{\pm}) j(t' \in C_{\pm}) = \delta(t - t'), \\ &\hat{j}(t \in C_{\pm}) j(t' \in C_{\mp}) = 0. \end{aligned} \tag{4.7}$$

The time integral over contour $C^{(\pm)}(T)$ means that

$$\int\limits_{C^{(\pm)}(T)} = \int\limits_{C_{+}(T)} \pm \int\limits_{C_{-}(T)} . \tag{4.8}$$

At the end of calculations the limit $(\omega, \tau, j, e) = 0$ must be calculated. The explicit form of $\tilde{H}(x;\tau)\ U_T(x,e)$ will be given later; $H_T(x)$ is the Hamiltonian at the time moment $t = T$.

The functional δ-function unambiguously determines the contributions in the path integral. For this purpose we must find the strict solution $x_j(t)$ of the equation of motion:

$$\ddot{x} + \omega_0^2 x + \lambda x^3 - j = 0, \tag{4.9}$$

expanding it over j. In zero order over j we have the classical trajectory x_c which is defined by the equation of motion:

$$\ddot{x} + \omega_0^2 x + \lambda x^3 = 0. \tag{4.10}$$

This equation is equivalent to the following one:

$$t + \theta_0 = \int^{x} dx \{2(h_0 - \omega_0^2 x^2 - \lambda x^4)\}^{-1/2}. \tag{4.11}$$

The solution of this equation is the periodic elliptic function.

Here (h_0, θ_0) are the constants of integration of Eq. (4.10), i.e., (h_0, θ_0) are the coordinates of point on the surface defined by elliptic function. The integration over (h_0, θ_0) is assumed since the integration over all trajectories in (4.2) must be performed, i.e., (h_0, θ_0) takes on all values available by elliptic function. Let W be the corresponding manyfold. One can say therefore that classical trajectory belongs to W completely.

The mapping of our problem on the action–angle phase space will be performed using representation (4.5) [22]. Using the obvious definition of the action:

$$I = \frac{1}{2\pi} \oint \{2(h - \omega_0^2 x^2 - \lambda x^4)\}^{1/2} \tag{4.12}$$

and of the angle variable [12]

$$\phi = \frac{\partial h}{\partial I}\int^{x_c}\{2(h - \omega_0^2 x^2 - \lambda x^4)\}^{-1/2}, \tag{4.13}$$

we easily find from (4.5) that

$$R(E) = 2\pi\int_0^\infty dT\exp\left[\frac{1}{2i}\hat{\omega}\hat{\tau} - i\int_{C^{(+)}(T)} dt\,\hat{j}(t)\,\hat{e}(t)\right]\times$$
$$\times\int DID\phi\,\mathrm{e}^{-i\tilde{H}(x_c;\tau)-iU_T(x_c,e)}\times$$
$$\times\,\delta(E+\omega-h_T(I))\prod_t\delta\left(\dot{I} - j\frac{\partial x_c}{\partial\phi}\right)\delta\left(\dot{\phi} - \Omega(I) + j\frac{\partial x_c}{\partial I}\right), \tag{4.14}$$

where $x_c = x_c(I,\phi)$ is the solution of Eq. (4.13) with $h = h(I)$ as the solution of Eq. (4.12) and the frequency

$$\Omega(I) = \frac{\partial h}{\partial I}. \tag{4.15}$$

Representation (4.14) is not the full solution of our problem: the action and angle variables are still interdependent since they both are excited by the same source $j(t)$. This reflects the Lagrange nature of the path-integral description of phase-space motion. The true Hamilton's description must contain independent quantum sources of action and angle variables.

4.3. Perturbation Theory on the Cotangent Manifold. The structure of source terms, $j\partial x_c/\partial\phi$ and $j\partial x_c/\partial I$, shows that the source of quantum fluctuations is the classical trajectories perturbation and j is the auxiliary variable. It allows one to regroup the perturbation series in the following manner. Let us consider the action of the perturbation-generating operators on δ-functions:

$$\exp\left[-i\int_{C^{(+)}(T)} dt\,\hat{j}(t)\,\hat{e}(t)\right]\mathrm{e}^{-iU_T(x,e)}\prod_t\delta\left(\dot{I} + j\frac{\partial x_c}{\partial\phi}\right)\delta\left(\dot{\phi} - \Omega(I) - j\frac{\partial x_c}{\partial I}\right) =$$
$$= \int D_{C^{(+)}}e_I D_{C^{(+)}}e_\phi\exp\left[i\int_{C^{(+)}} dt(e_I\dot{I} + e_\phi(\dot{\phi} - \Omega(I)))\right]\mathrm{e}^{-iU_T(x,e_c)}, \tag{4.16}$$

where

$$e_c(e_I,e_\phi) = e_I\frac{\partial x_c}{\partial\phi} - e_\phi\frac{\partial x_c}{\partial I}. \tag{4.17}$$

The integrals over (e_I, e_ϕ) will be calculated perturbatively:

$$\mathrm{e}^{-iU_T(x,e_c)} = \sum_{n_I,n_\phi=0}^{\infty} \frac{1}{n_I! n_\phi!} \int \prod_{k=1}^{n_I} (dt_k e_I(t_k)) \prod_{k=1}^{n_\phi} (dt'_k e_\phi(t'_k)) \times \\ \times P_{n_I,n_\phi}(x_c, t_1, \ldots, t_{n_I}, t'_1, \ldots, t_{n_\phi}), \quad (4.18)$$

where

$$P_{n_I,n_\phi}(x_c, t_1, \ldots, t_{n_I}, t'_1, \ldots, t_{n_\phi}) = \prod_{k=1}^{n_I} \hat{e}'_I(t_k) \prod_{k=1}^{n_\phi} \hat{e}'_\phi(t'_k) \, \mathrm{e}^{-iU_T(x,e'_c)}, \quad (4.19)$$

where $e'_c \equiv e_c(e'_I, e'_\phi)$ and the derivatives in (4.19) are calculated at $e'_I = 0$, $e'_\phi = 0$. At the same time,

$$\prod_{k=1}^{n_I} e_I(t_k) \prod_{k=1}^{n_\phi} e_\phi(t'_k) = \\ = \prod_{k=1}^{n_I} (i\hat{j}_I(t_k)) \prod_{k=1}^{n_\phi} (i\hat{j}_\phi(t'_k)) \exp\Big[-i \int_{C^{(+)}} dt (j_I(t) e_I(t) + j_\phi(t) e_\phi(t))\Big]. \quad (4.20)$$

The limit $(j_I, j_\phi) = 0$ is assumed. Inserting (4.19), (4.20) into (4.16) we will find new representation for $R(E)$:

$$R(E) = 2\pi \int_0^\infty dT \exp\Big[\frac{1}{2i} \hat{\omega} \hat{\tau} - i \int_{C^{(+)}(T)} dt \, (\hat{j}_I(t) \, \hat{e}_I(t) + \hat{j}_\phi(t) \, \hat{e}_\phi(t))\Big] \times \\ \times \int DID\phi \, \mathrm{e}^{-i\tilde{H}(x_c;\tau) - iU_T(x_c,e_c)} \times \\ \times \delta(E + \omega - h_T(I)) \prod_t \delta(\dot{I} - j_I) \delta(\dot{\phi} - \Omega(I) - j_\phi), \quad (4.21)$$

in which the action and the angle are the decoupled degrees of freedom.

Solving the canonical equations of motion

$$\dot{I} = j_I, \quad \dot{\phi} = \Omega(I) + j_\phi, \quad (4.22)$$

the boundary conditions

$$I_j(0) = I_0, \quad \phi_j(0) = \phi_0 \quad (4.23)$$

will be used. This will lead to the following Green function:

$$g(t - t') = \Theta(t - t'), \quad (4.24)$$

with boundary condition: $\Theta(0) = 1$. The solutions of Eqs. (4.22) have the form:

$$\begin{aligned} I_j(t) &= I_0 + \int dt' g(t-t') j_I(t') \equiv I_0 + I'(t), \\ \phi_j(t) &= \phi_0 + \tilde{\Omega}(I_j)t + \int dt' g(t-t') j_\phi(t') \equiv \phi_0 + \tilde{\Omega}(I_0 + I')t + \phi'(t), \end{aligned} \tag{4.25}$$

where

$$\tilde{\Omega}(I_j) = \frac{1}{t} \int dt' g(t-t') \Omega(I_0 + I'(t')). \tag{4.26}$$

Inserting (4.25) into (4.21) we find

$$R(E) = 2\pi \int_0^\infty dT \exp\left[\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt \, (\hat{j}_I(t)\, \hat{e}_I(t) + \hat{j}_\phi(t)\, \hat{e}_\phi(t))\right] \times \\ \times \int_0^\infty dI_0 \int_0^{2\pi} d\phi_0 \, \mathrm{e}^{-i\tilde{H}(x_c;\tau) - iU_T(x_c, e_c)} \delta(E + \omega - h_T(I_j)), \tag{4.27}$$

where

$$x_c = x_c(I_j, \phi_j) = x_c(I_0 + I(t), \phi_0 + \tilde{\Omega}(I_0 + I)t + \phi(t)), \tag{4.28}$$

and e_c was defined in (4.17). Note that the measure of the integrals over (I_0, ϕ_0) was defined without the Faddeev–Popov ansatz and there are not any «hosts» since the Jacobian of transformation is equal to one.

We can extract the Green function into the perturbation-generating operator using the equalities:

$$\hat{j}_I(t) = \int dt' g(t-t') \hat{I}(t), \hat{j}_\phi = \int dt' g(t-t') \hat{\phi}(t), \tag{4.29}$$

which evidently follow from (4.25). Hence,

$$R(E) = 2\pi \int_0^\infty dT \exp\left\{\frac{1}{2i}\hat{\omega}\hat{\tau} - i \int_{C^{(+)}(T)} dt\, dt' g(t'-t)(\hat{I}(t)\, \hat{e}_I(t') + \hat{\phi}(t)\, \hat{e}_\phi(t'))\right\} \times \\ \times \int_0^\infty dI_0 \int_0^{2\pi} d\phi_0 \, \mathrm{e}^{-i\tilde{H}(x_c;\tau) - iU_T(x_c, e_c)} \delta(E + \omega - h_T(I_0 + I)), \tag{4.30}$$

where x_c was defined in (4.28).

We can define the formalism without doubling the degrees of freedom. One can use the fact that the action of perturbation-generating operators and the analytical continuation to the real times are commuting operations. This can be easily seen using the definition (4.7). As a result, the expression

$$R(E) = 2\pi \int_0^\infty dT \exp\left\{ \frac{1}{2i}\hat{\omega}\hat{\tau} - i\int_0^T dt\, dt' \Theta(t'-t)(\hat{I}(t)\,\hat{e}_I(t') + \hat{\phi}(t)\,\hat{e}_\phi(t'))\right\}\times$$
$$\times \int_0^\infty dI_0 \int_0^{2\pi} d\phi_0\, \mathrm{e}^{-i\tilde{H}(x_c;\tau) - iU_T(x_c,e_c)}\, \delta(E+\omega-h_T(I_0+I(T))), \quad (4.31)$$

where

$$\tilde{H}_T(x_c;\tau) = 2\sum_{n=1}^\infty \frac{\tau^{2n+1}}{(2n+1)!}\frac{d^{2n}}{dT^{2n}} h(I_0+I(T)), \quad (4.32)$$

and

$$-U_T(x_c,e_c) = S(x_c+e_c) - S(x_c-e_c) - 2\int_0^T dt\, e_c \frac{\delta S(x_c)}{\delta x_c} \quad (4.33)$$

defines quantum theory on the cotangent manifold W.

Now we can use the last δ-function:

$$R(E) = 2\pi \int_0^\infty dT \exp\left\{ \frac{1}{2i}(\hat{\omega}\hat{\tau} + \int_0^T dt\, dt' \Theta(t'-t)(\hat{I}(t)\,\hat{e}_I(t') + \hat{\phi}(t)\,\hat{e}_\phi(t'))\right\}\times$$
$$\times \int_0^\infty dI_0 \int_0^{2\pi} \frac{d\phi_0}{\Omega(E+\omega)}\, \mathrm{e}^{-i\tilde{H}(x_c;\tau) - iU_T(x_c,e_c)}. \quad (4.34)$$

Here

$$x_c(t) = x_c(I_0(E+\omega) + I(t) - I(T), \phi_0 + \tilde{\Omega}t + \phi(t)). \quad (4.35)$$

Equation (4.34) contains unnecessary contributions: the action of the operator

$$\int_0^T dt\, dt' \Theta(t-t')\, \hat{e}_I(t)\, \hat{I}(t') \quad (4.36)$$

on $\tilde{H}_T$, defined in (4.32), leads to the time integrals with zero integration range:

$$\int_0^T dt\, \Theta(T-t)\Theta(t-T) = 0.$$

Using this fact,

$$R(E)=2\pi\int\limits_0^\infty dT\exp\left[\frac{1}{2i}\int\limits_0^T dt\,dt'\Theta(t'-t)(\hat{I}(t)\,\hat{e}_I(t')+\hat{\phi}(t)\,\hat{e}_\phi(t'))\right]\times$$
$$\times\int\limits_0^\infty dI_0\int\limits_0^{2\pi}\frac{d\phi_0}{\Omega(E)}\,\mathrm{e}^{-iU_T(x_c,e_c)},\quad (4.37)$$

where
$$x_c(t)=x_c(I_0(E)+I(t)-I(T),\phi_0+\tilde{\Omega}t+\phi(t))\qquad (4.38)$$

is the periodic function:

$$x_c(I_0(E)+I(t)-I(T),(\phi_0+2\pi)+\tilde{\Omega}t+\phi(t))=$$
$$=x_c(I_0(E)+I(t)-I(T),\phi_0+\tilde{\Omega}t+\phi(t)).\quad (4.39)$$

Now we can consider the cancellation of angular perturbations.

4.4. Cancellation of Angular Perturbations

4.4.1. Simplest Example. Introducing the perturbation-generating operator into the integral over ϕ_0:

$$R(E)=2\pi\int\limits_0^\infty dT\exp\left[\frac{1}{2i}\int\limits_0^T dt\,dt'\,\Theta(t'-t)\,\hat{I}(t)\,\hat{e}_I(t')\right]\times$$
$$\times\int\limits_0^\infty dI_0\int\limits_0^{2\pi}\frac{d\phi_0}{\Omega(E)}\exp\left[\frac{1}{2i}\int\limits_0^T dt\,dt'\,\Theta(t'-t)\,\hat{\phi}(t)\,\hat{e}_\phi(t')\right]\mathrm{e}^{-iU_T(x_c,e_c)},\quad (4.40)$$

the mechanism of cancellations of the angular perturbations becomes evident. One can formulate the statement:

(i) if

$$\exp\left[\frac{1}{2i}\int\limits_0^T dt\,dt'\Theta(t'-t)\hat{\phi}(t)\,\hat{e}_\phi(t')\right]\mathrm{e}^{-iU_T(x_c,e_c)}=$$
$$=\mathrm{e}^{-iU_T(x_c,e_c)}|_{e_\phi=\phi=0}+\frac{dF(\phi_0)}{d\phi_0},\quad (4.41)$$

and

(ii) if
$$F(\phi_0+2\pi)=F(\phi_0),\qquad (4.42)$$

then:

$$R(E) = 2\pi \int_0^{2\pi} \frac{d\phi_0}{\Omega(E)} \int_0^{\infty} dT\, dI_0 \exp\left[\frac{1}{2i}\int_0^T dt\, dt'\, \Theta(t'-t)\,(\hat{I}(t)\,\hat{e}_I(t')\right] \times \\ \times \exp\left[S\left(x_c + e\frac{\partial x_c}{\partial \phi_0}\right) - S\left(x_c - e\frac{\partial x_c}{\partial \phi_0}\right)\right], \quad (4.43)$$

i.e., we find the expression in which the angular corrections were canceled. In this case the problem becomes semiclassical over the angular degrees of freedom.

For the $(\lambda x^4)_1$-model

$$S\left(x_c + e\frac{\partial x_c}{\partial \phi_0}\right) - S\left(x_c - e\frac{\partial x_c}{\partial \phi_0}\right) = S_0(x_c) - 2\lambda \int_0^T dt\, x_c(t) \left\{ e\frac{\partial x_c}{\partial \phi_0}\right\}^3, \quad (4.44)$$

where [1]

$$S_0(x_c) = \oint_T dt\, \left(\frac{1}{2}\dot{x}_c^2 - \frac{\omega_0^2}{2}x_c^2 - \frac{\lambda}{4}x_c^4\right) \quad (4.45)$$

is the closed time-path action and

$$x_c(t) = x_c(I_0(E) + I(t) - I(T), \phi_0 + \tilde{\Omega}t). \quad (4.46)$$

Here $I(t)$ and $I(T)$ are the auxiliary variables.

The condition (4.42) requires that the classical trajectory x_c with all derivatives over I_0, ϕ_0 is the periodic function. In the considered case of $(\lambda x^4)_1$-model, x_c is periodic function with period $1/\Omega$, see (4.39). Therefore, we can concentrate the attention on the condition (4.41) only.

Expanding $F(\phi_0)$ over λ:

$$F(\phi_0) = \lambda F_1(\phi_0) + \lambda^2 F_2(\phi_0) + \ldots \quad (4.47)$$

we find that

$$\frac{d}{d\phi_0}F_1(\phi_0) = \\ = \int_0^T \prod_{k=1}^3 dt'_k \hat{\phi}(t'_k) \left(\left(-\frac{6}{(2i)^3}\right)\int_0^T dt \prod_{k=1}^3 \Theta(t-t'_k) x_c(t) \left(\frac{\partial x_c}{\partial I_0}\right)^3 \mathrm{e}_k^{iS_0(x_c)}\right) = \\ = \int_0^T dt' \hat{\phi}(t') B_1(\phi), \quad (4.48)$$

where

$$B_1(\phi) = \left\{ -\frac{6}{(2i)^3} \int_0^T dt\, \Theta(t-t') \times \right.$$
$$\left. \times \prod_{k=1}^{2} (\Theta(t-t'_k)\hat{\phi}(t'_k)) x_c(t) \left(\frac{\partial x_c}{\partial I_0}\right)^3 \mathrm{e}^{iS_0(x_c)} \right\}. \quad (4.49)$$

This example shows that the sum over all powers of λ can be written in the form:

$$\frac{d}{d\phi_0} F(\phi_0) = \int_0^T dt' \hat{\phi}(t') B(\phi), \quad (4.50)$$

where, using the definition (4.35),

$$B(\phi) = \int_0^T dt\, \tilde{B}(\phi_0 + \phi(t)). \quad (4.51)$$

Therefore,

$$\hat{\phi}(t') B(\phi) = \frac{d}{d\phi_0} \int_0^T dt\, \delta(t-t') \tilde{B}(\phi_0 + \phi(t)) \quad (4.52)$$

coincides with the total derivative over initial phase ϕ_0, and

$$F(\phi_0) = \tilde{B}(\phi_0 + \phi(t))|_{\phi=0}. \quad (4.53)$$

This result ends the proof of (4.41).

4.4.2. General Case. Now we will offer the following important statement:

— each order of perturbation theory in the invariant subspace can be represented as the sum of total derivative over the subspace coordinate.

This statement directly follows from structure of perturbations-generating operator $\hat{\mathbb{K}}$ and the assumption (3.18). It explains the statement, offered in Preface.

Let us remind that integration with last δ-function gives the result of action of operator $\hat{\mathbb{K}}$ written in the form:

$$R(E) = 2\pi \int_0^\infty dT \int_0^{2\pi} \frac{d\varphi_0}{\Omega(E)} : \mathrm{e}^{-iU(x_c, \hat{e}/2i)} :, \quad (4.54)$$

where the colons mean normal product,

$$\hat{e} = \hat{j}_\varphi \frac{\partial x_c}{\partial I} - \hat{j}_I \frac{\partial x_c}{\partial \varphi}, \quad (4.55)$$

and by definition U_T is the odd over $\hat{e}_c$ functional:

$$U_T(x_c, e_c) = 2 \int_0^T \sum_{n=1} (\hat{e}_c(t)/2i)^{2n+1} u_n(x_c), \tag{4.56}$$

where u_n is the function of only x_c at the time t. Inserting (4.55), one can write:

$$: \mathrm{e}^{-iU_(x_c, \hat{e}/2i)} := \prod_{n=1}^{\infty} \prod_{k=0}^{2n+1} : \mathrm{e}^{-iU_{k,n}(j, x_c)} :, \tag{4.57}$$

where

$$U_{k,n}(j, x_c) = \int_0^T dt (\hat{j}_\varphi(t))^{2n-k+1} (\hat{j}_I(t))^k \, b_{k,n}(x_c(t)), \tag{4.58}$$

and the explicit form of $b_{k,n}(x_c)$ is not important.

Using the evident definition:

$$\hat{j}_X = \int_0^T dt' \Theta(t - t') \hat{X}(t'), \quad X = \varphi, I,$$

it is easy to find that

$$j_X(t_1) b_{k,n}(x_c(t_2)) = \Theta(t_1 - t_2) \frac{\partial b_{k,n}(x_c(t_2))}{\partial X_0},$$

since $x_c = x_c(X + X_0)$, or shortly:

$$j_1 b_2 = \Theta_{12} \partial_{X_0} b_2 = \partial_{X_0} (\Theta_{12} b_2), \tag{4.59}$$

since the indexes (k, n) are not important.

Let us start consideration from the first term with $k = 0$. In this case we describe only the angular fluctuations. Noting that ∂_{X_0} and $\hat{j}$ commute we can consider the lowest order over $\hat{j}$. The typical term looks as follows (omitting the index X_0):

$$\hat{j}_1 \hat{j}_2 \cdots \hat{j}_m b_1 b_2 \cdots b_m.$$

It is sufficient to show that this expression is the total derivative over X_0.

Case $m = 1$. In this approximation we have, see (4.59):

$$\hat{j}_1 b_1 = \Theta_{11} \partial_0 b_1 \neq 0. \tag{4.60}$$

Here (3.18) was used.

Case $m = 2$. This order is less trivial:

$$\hat{j}_1\hat{j}_2 b_1 b_2 = \Theta_{21} b_1^2 b_2 + b_1^1 b_2^1 + \Theta_{12} b_1 b_2^2, \tag{4.61}$$

where

$$b_i^n \equiv \partial^n b_i. \tag{4.62}$$

At first glance (4.61) is not the total derivative. But inserting

$$1 = \Theta_{12} + \Theta_{21}$$

we can symmetrize it:

$$\hat{j}_1\hat{j}_2 b_1 b_2 = \Theta_{21}(b_1^2 b_2 + b_1^1 b_2^1) + \Theta_{12}(b_1 b_2^2 + b_1^1 b_2^1) = \\ = \partial_0(\Theta_{21} b_1^1 b_2 + \Theta_{12} b_1 b_2^1) \equiv \partial_0(b_1^1 \to b_2 + b_2^1 \to b_1), \tag{4.63}$$

since the explicit form of the function is not important. Therefore, the second-order term can be also reduced to the total derivative. Notice that (4.63) shows time reversibility.

Case $m = 3$. In this order one can find that

$$\hat{j}_1\hat{j}_2\hat{j}_3 b_1 b_2 b_3 = \partial_0 \left\{ \sum_{i \neq j \neq k = 1}^{3} (i^2 \to j \to k + i^1 \to j^1 \to k) \right\}. \tag{4.64}$$

The mth order contribution is also total derivative:

$$\hat{j}_1\hat{j}_2 \cdots \hat{j}_m b_1 b_2 \cdots b_m = \partial_0 \Bigg\{ \sum_{i_1 \neq i_2 \neq i_3 \neq \ldots \neq i_m = 1}^{m} (i_1^m \to i_2 \to i_3 \to \ldots \to i_m + \\ + i_1^{m-1} \to i_2^1 \to i_3 \to \ldots \to i_m + i_1^{m-2} \to i_2^1 \to i_3^1 \to \ldots \to i_m + \ldots \\ \ldots + i_1^1 \to i_2^1 \to i_3^1 \to \ldots \to i_{m-1}^1 \to i_m) \Bigg\}. \tag{4.65}$$

Let us consider now the case with $k \neq 0$. The typical term looks as follows:

$$\hat{j}_1^1 \hat{j}_2^1 \cdots \hat{j}_l^1 \hat{j}_{l+1}^2 \hat{j}_{l+2}^2 \cdots \hat{j}_m^2 b_1 b_2 \cdots b_m, \qquad 0 < l < m, \tag{4.66}$$

where, for instance,

$$\hat{j}_k^1 \equiv \hat{j}_I(t_k), \quad \hat{j}_k^2 \equiv \hat{j}_\varphi(t_k) \tag{4.67}$$

and

$$\hat{j}_1^i b_2 = \Theta_{12} \partial_0^i b_2. \tag{4.68}$$

Case $m = 2, l = 1$. In this case:

$$\hat{j}_1^1 \hat{j}_2^2 b_1 b_2 = \\ = \Theta_{21}(b_2 \partial_0^1 \partial_0^2 b_1 + (\partial_0^2 b_2)(\partial_0^1 \partial_0^2 b_1)) + \Theta_{12}(b_1 \partial_0^1 \partial_0^2 b_2 + (\partial_0^2 b_2)(\partial_0^1 \partial_0^2 b_1)) = \\ = \partial_0^1(\Theta_{21} b_2 \partial_0^2 b_1 + \Theta_{12} b_1 \partial_0^2 b_2) + \partial_0^2(\Theta_{21} b_2 \partial_0^1 b_1 + \Theta_{12} b_1 \partial_0^1 b_2). \quad (4.69)$$

Therefore we have the total-derivative structure yet. This property is conserved in arbitrary order over m and l since the time-ordered structure does not depend on upper index of $\hat{j}$, see (4.68).

One can conclude that the contribution is defined by topology properties of classical trajectory x_c. We will see that this important property of perturbation theory remains unchanged also for field theories with symmetry.

4.5. Conclusions

1. It was shown that the real-time quantum problem can be semiclassical over the part of the degrees of freedom and quantum over other ones. Following the result of this section, one may introduce the (probably naive) interpretation of the quantum systems integrability (we suppose that the classical system is integrable and can be mapped on the compact hypersurface in the phase space [12]): the quantum system is strictly integrable due to cancellation of all quantum degrees of freedom. The mechanism of cancellation of the quantum corrections is varied from case to case.

For some problems (as the rigid rotator, or the Pöschl–Teller), the cancellation of angular degrees of freedom is enough since they carry only the angular ones. In another case (as in the Coulomb problem, or in the one-dimensional models), the problem may be partly integrable since the quantum fluctuations of action degrees of freedom just survive. Theirs absence in the Coulomb problem needs special discussion (one must take into account the dynamical (hidden) symmetry of the Coulomb problem [23]).

The transformation to the action–angle variables maps the N-dimensional Lagrange problem on the $2N$-dimensional phase–space torus. If the winding number on this hypertorus is a constant (i.e., the topological charge is conserved) one can expect the same cancellations. This is important for the field-theoretical problems (for instance, for sine-Gordon model [24]).

2. In the classical mechanics the following approximated method of calculations is used [12]. The canonical equations of motion:

$$\dot{I} = a(I, \phi), \quad \dot{\phi} = b(I, \phi) \quad (4.70)$$

are changed on the averaged equations:

$$\dot{J} = \frac{1}{2\pi} \int_0^{2\pi} d\phi a(J, \phi), \quad \dot{\phi} = b(J, \phi). \quad (4.71)$$

It is possible if the oscillations can be extracted from the systematic evolution of the degrees of freedom.

In our case,

$$a(I,\phi) = j\frac{\partial x_c}{\partial \phi}, \quad b(I,\phi) = \Omega(I) - j\frac{\partial x_c}{\partial I}. \tag{4.72}$$

Inserting this definitions into (4.71) we find evidently wrong result since in this approximation the problem looks like pure semiclassical for the case of periodic motion:

$$\dot{J} = 0, \quad \dot{\phi} = \Omega(J). \tag{4.73}$$

The result of this section was used here. This shows that the procedure of extraction of the oscillations from the systematic evolution is not trivial and this method should be used carefully in the quantum theories. (This approximation of dynamics is «good» on the time intervals $\sim 1/|a|$ [12].)

5. EXAMPLE: H-ATOM

5.1. Introduction. The mapping

$$J : T \to W, \tag{5.1}$$

where T is the $2N$-dimensional phase space and W is a linear space, solves the mechanical problem iff

$$J = \otimes_1^N J_i, \tag{5.2}$$

where J_i are the first integrals in involution, see, e.g., [12]*. The aim of this section is to adopt this procedure for H-atom.

The mapping (5.1) introduces integral $manifold$ $J_\omega = J^{-1}(\omega)$ in such a way that the $classical$ phase space flow belongs to J_ω $completely$. We wish to quantize the J_ω manifold instead of flow in T noting that the quantum trajectory also should belong to J_ω completely. This important conclusion was demonstrated in the previous section by transformation of the path-integral measure to the canonical variables (ξ, η). New perturbation theory is extremely simple since W is the linear space.

The «direct» mapping (5.1) used in [26] assumes that J is known. But it seems inconvenient having in mind the general problem of nonlinear waves quantization, when the number of degrees of freedom $N = \infty$, or if the transformation is not canonical. We will consider by this reason the «inverse» approach

*The formalism of reduction (5.1) in classical mechanics is described also in [25].

assuming that just the classical flow is known. Then, since the flow belongs to J_ω completely [26], we would be able to find the quantum motion in W. It is the main technical result illustrated in this section.

The manifold J_ω is invariant relatively to some subgroup G_ω [27] in accordance with topological class of classical flow. This introduces the J_ω classification, and summation over all (homotopic) classes should be performed. Note, the classes are separated by the boundary bifurcation lines in W [27]. If the quantum perturbations are switched on adiabatically, then the homotopic group should stay unbroken. It is the ordinary statement for quantum mechanics, but, generally speaking, this is not true for field theories.

We will calculate the bound state energies in the Coulomb potential*. This popular problem was considered by many authors, using various methods, see, e.g., [23]. The path-integral solution of this problem was offered firstly in [28].

The classical flow of this problem can be parameterized by the angular momentum l, corresponding angle φ and by the normalized on total Hamiltonian Runge–Lenz vector length n. So, we will consider the mapping (p is the conjugate to r radial momentum in the cylindrical coordinates):

$$J_{l,n} : (p, l, r, \varphi) \to (l, n, \varphi) \tag{5.3}$$

to construct the perturbation theory in the $W = (l, n, \varphi)$ space. That is, W is not considered as the cotangent foliation on T.

The mapping (5.3) assumes additional reduction of the four-dimensional incident phase space up to three-dimensional linear subspace**. Just this reduction phenomena leads to corresponding stability of n concerning quantum perturbations and will allow one to solve our H-atom problem completely***.

In Subsec. 5.2 we will show how the mapping (5.3) can be performed for path-integral differential measure. In Subsec. 5.3 the consequence of reduction will be derived and in Subsec. 5.4 the perturbation theory in the W space will be analyzed. The calculations are based on the formalism offered in the previous sections.

5.2. Mapping. We will calculate the integral [26]:

$$\rho(E) = \int_0^\infty dT \, \mathrm{e}^{-i\hat{\mathbb{K}}(j,e)} \int DM(p, l, r, \varphi) \, \mathrm{e}^{-iU(r,e)}, \tag{5.4}$$

---

*We will restrict ourselves by the plane problem. Corresponding phase space $T = (p, l, r, \varphi)$ is 4-dimensional.

**W would not have the simplectic structure. Actually in the considered case $W = R + TW$, where R is the zero-modes space and TW is the simplectic subspace.

***In other words, we would demonstrate that the hidden Bargmann–Fock [23] $O(4)$ symmetry stayes unbroken concerning quantum perturbations.

where $\rho(E)$ is the $probability$ to find a particle with energy E, i.e., we should find [22] that normalized on the zero-modes volume

$$\rho(E) = \pi \sum_n \delta(E - E_n), \tag{5.5}$$

where E_n are the bound states energies. For H-atom problem $E_n \leqslant 0$. This condition will define considered homotopy class.

Expansion over operator

$$\hat{\mathbb{K}}(j, e) = \frac{1}{2} \int_0^T dt (\hat{j}_r \hat{e}_r + \hat{j}_\varphi \hat{e}_\varphi), \quad \hat{X}(t) \equiv \frac{\delta}{\delta X(t)} \tag{5.6}$$

generates the perturbation series. It will be seen that in our case we may omit the question of perturbation theories convergence.

The differential measure

$$DM(p, l, r, \varphi) = \delta(E - H_0) \prod_t dr(t)\, dp(t)\, dl(t)\, d\varphi(t) \times \\ \times \delta\left(\dot{r} - \frac{\partial H_j}{\partial p}\right) \delta\left(\dot{p} + \frac{\partial H_j}{\partial r}\right) \delta\left(\dot{\varphi} - \frac{\partial H_j}{\partial l}\right) \delta\left(\dot{l} + \frac{\partial H_j}{\partial \varphi}\right), \tag{5.7}$$

with total Hamiltonian ($H_0 = H_j|_{j=0}$)

$$H_j = \frac{1}{2} p^2 - \frac{l^2}{2r^2} - \frac{1}{r} - j_r r - j_\varphi \varphi \tag{5.8}$$

allows one to perform arbitrary transformation of variables because of its δ-likeness. Notice that H_j contains only the «Lagrange forces» j_r and j_φ.

The functional

$$U(r, e) = -s_0(r) + \\ + \int_0^T dt \left[\frac{1}{((r + e_r)^2 + r^2 e_\varphi^2)^{1/2}} - \frac{1}{((r - e_r)^2 + r^2 e_\varphi^2)^{1/2}} + 2\frac{e_r}{r} \right] \tag{5.9}$$

describes the interaction between various quantum modes, and $s_0(r)$ defines the nonintegrable phase factor [22]. The quantization of this factor determines the bound state energy. Such a factor will appear if the phase of amplitude cannot be fixed*. Note that the Hamiltonian (5.8) contains the energy of radial $j_r r$ and angular $j_\varphi \varphi$ excitation, independently.

*As, for instance, in the Aharonov–Bohm case.

Let us introduce the functional

$$\Delta = \int \prod_t d^2\xi d^2\eta\, \delta(r(t) - r_c(\xi,\eta)) \times \\ \times\, \delta(p(t) - p_c(\xi,\eta))\, \delta(l(t) - l_c(\xi,\eta))\, \delta(\varphi(t) - \varphi_c(\xi,\eta)), \quad (5.10)$$

which is defined by given functions $(r_c, p_c, \varphi_c, l_c)(\xi,\eta)$. If given functions (ξ,η) zeroes argument of δ-functions in (5.10), then it is assumed that the functional determinant

$$\Delta_c = \int \prod_t d^2\bar{\xi} d^2\bar{\eta} \delta\left(\frac{\partial r_c}{\partial \xi}\cdot\bar{\xi} + \frac{\partial r_c}{\partial \eta}\cdot\bar{\eta}\right) \delta\left(\frac{\partial p_c}{\partial \xi}\cdot\bar{\xi} + \frac{\partial p_c}{\partial \eta}\cdot\bar{\eta}\right) \times \\ \times\, \delta\left(\frac{\partial \varphi_c}{\partial \xi}\cdot\bar{\xi} + \frac{\partial \varphi_c}{\partial \eta}\cdot\bar{\eta}\right) \delta\left(\frac{\partial l_c}{\partial \xi}\cdot\bar{\xi} + \frac{\partial l_c}{\partial \eta}\cdot\bar{\eta}\right) \neq 0. \quad (5.11)$$

Note that this is the condition only for $(r_c, p_c, \varphi_c, l_c)(\xi,\eta)$.

To perform the mapping we will insert

$$1 = \Delta/\Delta_c \quad (5.12)$$

into (5.4) and integrate over $r(t)$, $p(t)$, $\varphi(t)$, and $l(t)$. As a result, we find the measure:

$$DM(\xi,\eta) = \frac{1}{\Delta_c}\delta(E - H_0) \prod_t d^2\xi d^2\eta \delta\left(\dot{r_c} - \frac{\partial H_j}{\partial p_c}\right) \times \\ \times\, \delta\left(\dot{p_c} + \frac{\partial H_j}{\partial r_c}\right) \delta\left(\dot{\varphi}_c - \frac{\partial H_j}{\partial l_c}\right) \delta\left(\dot{l_c} + \frac{\partial H_j}{\partial \varphi_c}\right). \quad (5.13)$$

Note that the functions $(r_c, p_c, \varphi_c, l_c)(\xi,\eta)$ must obey only one condition (5.11).

A simple algebra gives:

$$DM(\xi,\eta) = \frac{\delta(E - H_0)}{\Delta_c} \prod_t d^2\xi d^2\eta \int \prod_t d^2\bar{\xi} d^2\bar{\eta} \times \\ \times\, \delta^2\left(\bar{\xi} - \left(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\right)\right) \delta^2\left(\bar{\eta} - \left(\dot{\eta} + \frac{\partial h_j}{\partial \xi}\right)\right) \times \\ \times\, \delta\left(\frac{\partial r_c}{\partial \xi}\cdot\bar{\xi} + \frac{\partial r_c}{\partial \eta}\cdot\bar{\eta} + \{r_c, h_j\} - \frac{\partial H_j}{\partial p_c}\right) \times$$

$$\times\,\delta\left(\frac{\partial p_c}{\partial\xi}\cdot\bar\xi+\frac{\partial p_c}{\partial\eta}\cdot\bar\eta+\{p_c,h_j\}+\frac{\partial H_j}{\partial r_c}\right)\times$$
$$\times\,\delta\left(\frac{\partial\varphi_c}{\partial\xi}\cdot\bar\xi+\frac{\partial\varphi_c}{\partial\eta}\cdot\bar\eta+\{\varphi_c,h_j\}-\frac{\partial H_j}{\partial l_c}\right)\times$$
$$\times\,\delta\left(\frac{\partial l_c}{\partial\xi}\cdot\bar\xi+\frac{\partial l_c}{\partial\eta}\cdot\bar\eta+\{l_c,h_j\}+\frac{\partial H_j}{\partial\varphi_c}\right). \quad (5.14)$$

The Poisson notation

$$\{X,h_j\}=\frac{\partial X}{\partial\xi}\frac{\partial h_j}{\partial\eta}-\frac{\partial X}{\partial\eta}\frac{\partial h_j}{\partial\xi}$$

was introduced in (5.14).

Next, the «auxiliary» quantity h_j has been introduced by the following equalities:

$$\{r_c,h_j\}-\frac{\partial H_j}{\partial p_c}=0, \qquad \{p_c,h_j\}+\frac{\partial H_j}{\partial r_c}=0,$$
$$\{\varphi_c,h_j\}-\frac{\partial H_j}{\partial l_c}=0, \qquad \{l_c,h_j\}+\frac{\partial H_j}{\partial\varphi_c}=0. \quad (5.15)$$

Then the functional determinant Δ_c is canceled and

$$DM(\xi,\eta)=\delta(E-H_0)\prod_t d^2\xi d^2\eta\,\delta^2\left(\dot\xi-\frac{\partial h_j}{\partial\eta}\right)\delta^2\left(\dot\eta+\frac{\partial h_j}{\partial\xi}\right). \quad (5.16)$$

It is the desired result of transformation of the measure for given generating functions $(r_c,p_c,\varphi_c,l_c)(\xi,\eta)$. In this case the «Hamiltonian» $h_j(\xi,\eta)$ is defined by four equations (5.15).

But there is another possibility. Let us assume that

$$h_j(\xi,\eta)=H_j(r_c,p_c,\varphi_c,l_c), \quad (5.17)$$

and the functions $(r_c,p_c,\varphi_c,l_c)(\xi,\eta)$ are unknown. Then Eqs. (5.15) are the equations for these functions. It is not hard to see that Eqs. (5.15) simultaneously with equations fixed by δ-functions in (5.16) are equivalent to the incident equations if the equality (5.17) is hold. Indeed, for example,

$$\dot r_c=\frac{\partial r_c}{\partial\xi}\cdot\dot\xi+\frac{\partial r_c}{\partial\eta}\cdot\dot\eta=\{r_c,h_j\}=\frac{\partial H_j}{\partial p_c}, \quad (5.18)$$

where (5.16) and (5.15) were used successfully.

So, incident dynamical problem was divided into two parts. First, one defines the trajectory in the W space through Eqs. (5.15). Second, one defines the dynamics, i.e., the time dependence, through the equations fixed by δ-functions in the measure (5.16).

Therefore, we should consider $r_c,\ p_c,\ \varphi_c,\ l_c$ as the solutions in the $\xi,\ \eta$ parameterization. The desired parameterization of classical orbits has the form (one can find it in arbitrary textbook of classical mechanics):

$$r_c = \frac{\eta_1^2(\eta_1^2+\eta_2^2)^{1/2}}{(\eta_1^2+\eta_2^2)^{1/2}+\eta_2\cos\xi_1},\ p_c = \frac{\eta_2\sin\xi_1}{\eta_1(\eta_1^2+\eta_2^2)^{1/2}},\ \varphi_c=\xi_1,\ l_c=\eta_1, \quad (5.19)$$

i.e., r_c and p_c are ξ_2-independent. At the same time,

$$h_j = \frac{1}{2(\eta_1^2+\eta_2^2)^{1/2}} - j_r r_c - j_\varphi \xi_1 \equiv h(\eta) - j_r r_c - j_\varphi \xi_1. \quad (5.20)$$

Noting that the derivatives of h_j over ξ_2 are equal to zero*, we find that

$$DM(\xi,\eta) = \delta(E-h(T))\prod_t d^2\xi d^2\eta\,\delta\left(\dot{\xi}_1-\omega_1+j_r\frac{r_c}{\partial\eta_1}\right)\times$$
$$\times\,\delta\left(\dot{\xi}_2-\omega_2+j_r\frac{r_c}{\partial\eta_2}\right)\delta\left(\dot{\eta}_1-j_r\frac{\partial r_c}{\partial\xi_1}-j_\varphi\right)\delta(\dot{\eta}_2), \quad (5.21)$$

where

$$\omega_i = \frac{\partial h}{\partial\eta_i} \quad (5.22)$$

are the conserved in classical limit $j_r = j_\varphi = 0$ «velocities» in the W space.

5.3. Reduction. We see from (5.21) that the length of Runge–Lenz vector is not perturbated by the quantum forces j_r and j_φ. To investigate the consequence of this fact it is useful to project these forces on the axis of W space. This means splitting of $j_r,\ j_\varphi$ on $j_\xi,\ j_\eta$. The equality

$$\prod_t \delta\left(\dot{\xi}_1-\omega_1+j_r\frac{r_c}{\partial\eta_1}\right) =$$
$$= \exp\left(\frac{1}{2i}\int_0^T dt\hat{j}_{\xi_1}\hat{e}_{\xi_1}\right)\exp\left(2i\int_0^T dt j_r e_{\xi_1}\frac{\partial r_c}{\partial\eta_1}\right)\prod_t\delta(\dot{\xi}_1-\omega_1+j_{\xi_1})$$

becomes evident if the Fourier representation of δ-function is used (see also [26]). The same transformation of arguments of other δ-functions in (5.21) can be

*To have the condition (5.11) we should assume that $\partial r_c/\partial\xi_2 \sim \epsilon \neq 0$. We put $\epsilon = 0$ completing the transformation.

applied. Then, noting that the last δ-function in (5.21) is source-free, we find the same representation as (5.4) with

$$\hat{\mathbb{K}}(j,e) = \int_0^T dt(\hat{j}_{\xi_1}\hat{e}_{\xi_1} + \hat{j}_{\xi_2}\hat{e}_{\xi_2} + \hat{j}_{\eta_1}\hat{e}_{\eta_1}), \tag{5.23}$$

where the operators $\hat{j}$ are defined by the equality:

$$\hat{j}_X(t) = \int_0^T dt'\Theta(t-t')\hat{X}(t') \tag{5.24}$$

and $\Theta(t-t')$ is the Green function of our perturbation theory [26].

We should change also

$$e_r \to e_c = e_{\eta_1}\frac{\partial r_c}{\partial \xi_1} - e_{\xi_1}\frac{\partial r_c}{\partial \eta_1} - e_{\xi_2}\frac{\partial r_c}{\partial \eta_2}, \quad e_\varphi \to e_{\xi_1} \tag{5.25}$$

in Eq. (5.9). The differential measure takes the simplest form:

$$DM(\xi,\eta) = \delta(E - h(T))\prod_t d^2\xi d^2\eta \times$$
$$\times\, \delta(\dot{\xi}_1 - \omega_1 - j_{\xi_1})\,\delta(\dot{\xi}_2 - \omega_2 - j_{\xi_2})\,\delta(\dot{\eta}_1 - j_{\eta_1})\delta(\dot{\eta}_2). \tag{5.26}$$

Note now that the ξ, η variables are contained in r_c only:

$$r_c = r_c(\xi_1, \eta_1, \eta_2).$$

This means that the action of the operator $\hat{j}_{\xi_2}$ gives identical to zero contributions into perturbation theory series. And, since $\hat{e}_{\xi_2}$ and $\hat{j}_{\xi_2}$ are conjugate operators, see (5.23), we can put

$$j_{\xi_2} = e_{\xi_2} = 0.$$

This conclusion ends the reduction:

$$\hat{\mathbb{K}}(j,e) = \int_0^T dt(\hat{j}_{\xi_1}\hat{e}_{\xi_1} + \hat{j}_{\eta_1}\hat{e}_{\eta_1}), \tag{5.27}$$

$$e_c = e_{\eta_1}\frac{\partial r_c}{\partial \xi_1} - e_{\xi_1}\frac{\partial r_c}{\partial \eta_1}. \tag{5.28}$$

The measure has the form:

$$DM(\xi,\eta) =$$
$$= \delta(E - h(T))\, d\xi_2(0)\, d\eta_2(0)\prod_t d\xi_1 d\eta_1 \delta(\dot{\xi}_1 - \omega_1 - j_{\xi_1})\delta(\dot{\eta}_1 - j_{\eta_1}) \tag{5.29}$$

since $V = V(r_c, e_c, \xi_1)$ is ξ_2-independent and

$$\int \prod_t dX(t)\delta(\dot{X}) = \int dX(0).$$

5.4. Perturbations. One can see from (5.29) that the reduction cannot solve the H-atom problem completely: there are nontrivial corrections to the orbital degrees of freedom ξ_1, η_1. By this reason, we should consider the expansion over $\hat{\mathbb{K}}$.

Using last δ-functions in (5.29) we find, see also [26] (normalizing $\rho(E)$ on the integral over $\xi_2(0)\eta_2(0)$):

$$\rho(E) = \int_0^\infty dT\, \mathrm{e}^{-i\hat{\mathbb{K}}(j,e)} \int dM\, \mathrm{e}^{-iU(r_c,e)}, \tag{5.30}$$

where

$$dM = \frac{d\xi_1 d\eta_1}{\omega_2(E)}. \tag{5.31}$$

The operator $\hat{\mathbb{K}}(j, e)$ was defined in (5.27), and

$$U(r_c, e_c) = -s_0(r) + \\ + \int_0^T dt \left[\frac{1}{((r_c + e_c)^2 + r_c^2 e_{\xi_1}^2)^{1/2}} - \frac{1}{((r_c - e_c)^2 + r_c^2 e_{\xi_1}^2)^{1/2}} + 2\frac{e_c}{r_c} \right] \tag{5.32}$$

with $e_c,\ e_{\xi_1}$ was defined in (5.28), (5.25) and

$$r_c(t) = r_c(\eta_1 + \eta(t), \bar{\eta}_2(E, T), \xi_1 + \omega_1(t) + \xi(t)), \quad E \equiv h(\eta_1 + \eta(T), \bar{\eta}_2), \tag{5.33}$$

where $\bar{\eta}_2(E, T)$ is the solution of equation $E = h$.

The integration range over ξ_1 and η_1 is as follows:

$$0 \leqslant \xi_1 \leqslant 2\pi, \quad -\infty \leqslant \eta_1 \leqslant +\infty. \tag{5.34}$$

The first inequality defines the principal domain of the angular variable φ and the second ones take into account the clockwise and anticlockwise motions of particle on the Kepler orbits.

We can write:

$$\rho(E) = \int_0^\infty dT \int dM :\, \mathrm{e}^{-iV(r_c,\hat{e})} : \tag{5.35}$$

since the operator $\ln\hat{\mathbb{K}}$ is linear over $\hat{e}_{\xi_1}, \hat{e}_{\eta_1}$. The colons mean «normal product» with differential operators staying to the left of functions, and $U(r_c, \hat{e})$ is the functional of operators:

$$2i\hat{e}_c = \hat{j}_{\eta_1}\frac{\partial r_c}{\partial \xi_1} - \hat{j}_{\xi_1}\frac{\partial r_c}{\partial \eta_1}, \quad 2i\hat{e}_{\xi_1} = \hat{j}_{\xi_1}. \tag{5.36}$$

Expanding $U(r_c, \hat{e})$ over $\hat{e}_c$ and $\hat{e}_{\eta_1}$ we find:

$$U(r_c, \hat{e}) = -s_0(r_c) + 2\sum_{n+m\geqslant 1} C_{n,m}\int_0^T dt\, \hat{e}_c^{2n+1}\hat{e}_{\eta_1}^m \frac{1}{r_c^{2n+2}}, \tag{5.37}$$

where $C_{n,m}$ are the numerical constants. We see that the interaction part presents the expansion over $1/r_c$ and, therefore, the expansion over U generates the expansion over $1/r_c$.

As a result, see Subsec. 4.5, we have

$$\rho(E) = \int_0^\infty dT \int dM\{\, \mathrm{e}^{is_0(r_c)} + B_{\xi_1}(\xi_1, \eta_1) + B_{\eta_1}(\xi_1, \eta_1)\}. \tag{5.38}$$

The first term is the pure semiclassical contribution and the last ones are the quantum corrections. The functionals B are the total derivatives:

$$B_{\xi_1}(\xi_1, \eta_1) = \frac{\partial}{\partial \xi_1} b_{\xi_1}(\xi_1, \eta_1), \quad B_{\eta_1}(\xi_1, \eta_1) = \frac{\partial}{\partial \eta_1} b_{\eta_1}(\xi_1, \eta_1). \tag{5.39}$$

This means that the mean value of quantum corrections in the ξ_1 direction is equal to zero:

$$\int_0^{2\pi} d\xi_1 \frac{\partial}{\partial \xi_1} b_{\xi_1}(\xi_1, \eta_1) = 0 \tag{5.40}$$

since r_c is the closed trajectory independently of initial conditions, see (5.19).

In the η_1 direction the motion is classical:

$$\int_{-\infty}^{+\infty} d\eta_1 \frac{\partial}{\partial \eta_1} b_{\eta_1}(\xi_1, \eta_1) = 0 \tag{5.41}$$

since (i) b_{η_1} is the series over $1/r_c^2$ and (ii) $r_c \to \infty$ when $|\eta_1| \to \infty$. Therefore,

$$\rho(E) = \int_0^\infty dT \int dM\, \mathrm{e}^{is_0(r_c)}. \tag{5.42}$$

This is the desired result.

Noting that

$$s_0(r_c) = kS_1(E), \quad k = \pm 1, \pm 2, \ldots,$$

where $S_1(E)$ is the action over one classical period T_1:

$$\frac{\partial S_1(E)}{\partial E} = T_1(E),$$

and using the identity [22]:

$$\sum_{-\infty}^{+\infty} \mathrm{e}^{inS_1(E)} = 2\pi \sum_{-\infty}^{+\infty} \delta(S_1(E) - 2\pi n),$$

we find

$$\rho(E) = \pi\Omega \sum_n \delta(E + 1/2n^2), \tag{5.43}$$

where Ω is the zero-modes volume.

5.5. Conclusions. The demonstrated above mechanism of reduction is universal: one can introduce from the very beginning the arbitrary number of coordinates (ξ, η). But later on the formalism automatically, through dependence of classical trajectory on coordinates of W, will extract the necessary set of variables (ξ, η). At the same time $\dim(\xi, \eta) = \dim W$ and the integrals over other ones will give the volume

$$V_0 = \int \prod d\xi(0)\, d\eta(0),$$

see (5.29), where $\dim V_0 = 2$.

Notice that appearance of the «0-dimensional» integral measure

$$d\xi_2(0)\, d\eta_2(0)$$

in (5.29) reflects the hidden $O(4)$ symmetry of H-atom problem [23]. Therefore, following our selection rule, we must consider in the first place the classical trajectory which leads to the maximal value of $\dim V_0$, i.e., we must consider the contributions with maximal number of zero modes.

6. EXAMPLE: SINE-GORDON MODEL

6.1. Introduction. First of all, we will describe «canonical» transformation in the path-integral formalism. The method of canonical transformations in spite of its expected effectiveness is unpopular in quantum theories since on this way there exists the problem: it is necessary to find the transformation from Lagrangian to

Hamiltonian descriptions. This transition, in general, is very difficult if $\varphi(x)$ and $\dot{\varphi}(x) = p(x)$ are not independent quantities [13]. But we may use the following trick. We start from the simplest verse of the canonical formalism introducing the «first-order» description* and after transformation come to independent canonically conjugate pairs, (ξ, η), i.e., come to Hamiltonian description. It is evident that, in general, the transformation

$$\varphi_c : (\varphi, p) \to (\xi, \eta)$$

will not be canonical. The formalism of the present section is the same as in the H-atom problem but there is some distinction.

We will continue in this section description of influence of the phase-space structure on the result of quantum-mechanical measurements started in the previous sections. Now we will calculate the expectation value of the «order parameter» (mass-shell particles production vertex) $\Gamma(q; u)$ [29]:

$$\rho(q) = \langle |\Gamma(q; u)|^2 \rangle_u,$$

where q is the mass-shell ($q^2 = m^2$) particles momentum and $\langle\rangle_u$ means averaging over the field $u(x, t)$. Just the procedure of averaging would be the object of our interest considering the quantum Hamiltonian system with symmetry G. By definition, ρ is the *probability* to find one mass-shell particle. Certainly, $\rho(q) = 0$ on the sourceless vacuum but, generally speaking, $\rho(q) \neq 0$ in a field with nonzero energy density.

Calculations will be illustrated by the integrable (1+1)-dimensional model with nonpolynomial Lagrangian

$$L = \frac{1}{2}(\partial_\mu u)^2 + \frac{m_h^2}{\lambda^2}[\cos(\lambda u) - 1]. \tag{6.1}$$

We will consider the following formulation of the problem. Formally nothing prevents to linearize partly our problem considering the Lagrangian

$$L = \frac{1}{2}[(\partial_\mu u)^2 - m_h^2 u^2] + \frac{m_h^2}{\lambda^2}\left[\cos(\lambda u) - 1 + \frac{\lambda^2}{2}u^2\right] \equiv L_0(u) - v(u) \tag{6.2}$$

to describe creation (and absorption) of the mass m_h particles. Then the last term in (6.2),

$$v(u) = -\frac{m_h^2}{\lambda^2}\left[\cos(\lambda u) - 1 + \frac{\lambda^2}{2}u^2\right], \tag{6.3}$$

*In other words, we will still stay in the frame of Lagrangian formalism.

describes interactions. The corresponding to this theory order parameter is

$$\Gamma(q;u)=\int dx\,dt\,\mathrm{e}^{iqx}(\partial^2+m_h^2)u(x,t),\quad q^2=m_h^2. \tag{6.4}$$

It will be shown by explicit calculations that

$$\rho(q)=0 \tag{6.5}$$

as the consequence of unbroken $\tilde{sl}(2,C)$ Kac–Moody algebra on which the solitons of theory (6.1) live*, see, e.g., [31] and references cited therein**. The solution (6.5) seems interesting since it can be interpreted as the explicit demonstration of field $u(x,t)$ confinement. The main purpose of this paper is to investigate how the solution (6.5) appears.

We will be able to find exact equality (6.5) since the model (6.1) possesses infinite number of integrals of motion. It is well known that each integral of motion in involution allows one to shrink a number of phase space $\bar{\gamma}$ variables on two units, see, e.g., [12]. Resulting phase space γ is called as the reduced phase space [25]. The summation over all reduced phase–space topological classes [27] is assumed.

By this way the field-theoretical problem will be reduced to the quantum-mechanical one. We would consider η as the «particles» generalized momentum and would introduce ξ as the conjugate to η coordinate of soliton. The $2N$-dimensional phase space (cotangent manifold) γ_N with local coordinates (ξ,η) on it has natural simplectic structure, and $DM(\gamma_N)=D^NM(\xi,\eta)$ in practical calculations (see Subsec. 6.2). The summation over N is assumed.

The quantum corrections to semiclassical approximation of transformed theory are simply calculable since η are conserved in the classical limit. This is the particularity of solitons dynamics (solitons momenta are the conserved quantities). One can consider the developed in this paper formalism as the path-integral version of nonlinear waves (solitons in our case) quantum theory (the canonical quantization of sine-Gordon model in the soliton sector was described also in [14].)

In Subsec. 6.3 we will demonstrate Eq. (6.5). It will be shown that this solution is a consequence of the previously developed proposition (we would justify it

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*Trivialness of soliton S matrix was shown in [30].

**It may be useful at this point to compare our approach with ordinary thermodynamics of ferromagnetic. The external magnetic field is $\sim\langle\mu\rangle$, where the *order parameter* $\langle\mu\rangle$ is the mean value of the spin, and the phase transition means that $\langle\mu\rangle\neq 0$, i.e., $\langle\mu\rangle=0$ means that corresponding symmetry stays unbroken. We will suppose that the mean value of $|\Gamma(q,u)|^2$, which is the function of external fields parameter q, plays the same role for field theories with symmetry, i.e., $\langle|\Gamma(q,u)|^2\rangle_u=0$ means that corresponding symmetry stays unbroken. Therefore in our approach only the «external» display of symmetry can be described.

in Subsec. 6.2) that the semiclassical approximation is exact for sine-Gordon model [11]. The semiclassical approximation in the γ_N phase space will be considered in Subsec. 6.2.

We would not use the complicated algebra to show the reduction procedure explicitly noting that all solutions of model (6.1) are known [24]. Then, using the δ-likeness of measure $DM(\tilde\gamma)$, we will find in Subsec. 6.2 $DM(\gamma_N)$ considering the mapping as an ordinary transformation to useful variables*. Corresponding perturbation theory, see Subsec. 6.3, in the momentum space J was described in [26]. In Subsec. 6.2 the path-integral definition of $\rho(q)$ will be given.

We would conclude (this is the main result) that a theory in the «nonlinear waves» sector may be nontrivial ($\rho \neq 0$) iff the manifold γ is not compact.

6.2. Reduction Procedure. *6.2.1. Introduction into formalism.*

Our aim is to calculate the integral:

$$\rho(q) = \mathrm{e}^{-i\hat{\mathbb{K}}(j,e)} \int DM(u,p) |\Gamma(q;u)|^2 \, \mathrm{e}^{iS_O(u) - iU(u,e)}, \tag{6.6}$$

where $\Gamma(q;u)$ was defined in (6.4). In this expression the expansion over operator

$$\hat{\mathbb{K}}(j,e) = \mathrm{Re} \int_{C_+} dxdt \frac{\delta}{\delta j(x,t)} \frac{\delta}{\delta e(x,t)} \equiv \mathrm{Re} \int_{C_+} dx\, dt\, \hat{j}(x,t) \hat{e}(x,t) \tag{6.7}$$

generates the perturbation theory series. We will assume that this series exists. The functionals $U(u,e)$ and $S_O(u)$ are defined by the equalities:

$$\begin{aligned} V(u+e) - V(u-e) &= U(u,e) + \int dx\, dt\, e(x,t)\, v'(u), \\ S_0(u+e) - S_0(u-e) &= S_O(u) + \int dx\, dt\, e(x,t) (\partial^2 + m_h^2)\, u(x,t). \end{aligned} \tag{6.8}$$

The action $S_0(u)$ corresponds to the free part of Lagrangian (6.1), and $V(u)$ describes interactions. The quantity $S_O(u)$ is not equal to zero since the soliton

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*We will apply inverse reduction procedure. Let G be a group of canonical transformations acting on the simplectic manifold $\tilde\gamma$, and let $\bar{G}$ be the Lie algebra of G with G^* dual of it. Then the momentum [32] mapping $J:\ \tilde\gamma \to G^*$ introduces the integrals of motion which reduce the $\tilde\gamma$ manifold. Noting that the set of levels $J^{-1}(\eta)$ (solution of equations $J(\pi) = \eta$, $\pi \in \tilde\gamma$) is a manifold then $\gamma_\eta = J^{-1}(\eta)/\bar{G}_\eta$ is the reduced phase space, where $\bar{G}_\eta$ is the co-adjoint isotropy subgroup of G. Therefore, the differential measure $dM = dM(\eta, \gamma_\eta)$ for reduced phase space. For integrable mechanical systems (infinite dimensional as well, see, e.g., [24]) γ_η shrinks to the point and in this case $dM = dM(\eta)$ is the measure of momentum manifold. Just this simplest case would be considered working with Lagrangian (6.1) and more general and interesting case with measure $DM = DM(\eta, \gamma_\eta)$, $\gamma_\eta \neq \emptyset$, will be considered later. So, the reduction procedure of our Hamiltonian system with symmetry G looks like canonical transformation [31]. This problem is nontrivial since, generally speaking, $\dim\tilde\gamma$ and $\dim\gamma$ are not the same for model (6.1).

configurations have nontrivial topological charge (see also [1]). All time integrals in this expressions were defined on the Mills time contour [17]:

$$2\mathrm{Re}\int_{C_+} = \int_{C_+} + \int_{C_-}$$

and

$$C_\pm : t \to t \pm i\epsilon, \quad \epsilon \to +0, \quad -\infty \leqslant t \leqslant +\infty$$

to avoid the possible light-cone singularities of the perturbation theory. The variational derivatives in (6.7) are defined by the following way:

$$\frac{\delta u(x, t \in C_i)}{\delta u(x', t' \in C_j)} = \delta_{ij}\delta(x - x')\delta(t - t'), \quad i, j = +, -.$$

The auxiliary variables (j, e) must be taken equal to zero at the very end of calculations.

Considering the first-order formalism with new coordinates (u, p), the measure $DM(u, p)$ has the form:

$$DM(u,p) = \prod_{x,t} du(x,t)\, dp(x,t)\delta\left(\dot{u} - \frac{\delta H_j(u,p)}{\delta p}\right)\delta\left(\dot{p} + \frac{\delta H_j(u,p)}{\delta u}\right) \quad (6.9)$$

with the total «Hamiltonian»

$$H_j(u,p) = \int dx \left\{\frac{1}{2}p^2 + \frac{1}{2}(\partial_x u)^2 - \frac{m_h^2}{\lambda^2}[\cos(\lambda u) - 1] - ju\right\}. \quad (6.10)$$

The problem will be considered assuming that $u(x, t)$ belongs to Schwartz space:

$$u(x,t)|_{|x|=\infty} = 0 \left(\mathrm{mod}\frac{2\pi}{\lambda}\right). \quad (6.11)$$

This means that $u(x, t)$ tends to zero $(\mathrm{mod}\, 2\pi/\lambda)$ at $|x| \to \infty$ faster than any power of $1/|x|$. Note that $\dot{u} = p$, i.e., u and p are not the independent quantities.

The measure (6.9) allows one to perform arbitrary transformations. But, as was explained in Introduction, we will use the analog of canonical transformation which conserves the form of equations of motion. Hence, it is sufficient on this stage of calculations to know only the fact that this transformation exists [24]. One may propose that finally we should find for N-soliton topology:

$$D^N M(\xi,\eta) = \prod_t d^N\xi(t)\, d^N\eta(t)\, \delta^{(N)}\left(\dot{\xi} - \frac{\partial h_j(\xi,\eta)}{\partial \eta(t)}\right)\delta^{(N)}\left(\dot{\eta} + \frac{\partial h_j(\xi,\eta)}{\partial \xi(t)}\right), \quad (6.12)$$

where h_j is the «transformed Hamiltonian»:

$$h_j = h_N(\eta) - \int dx j(x,t)\, u_N(x;\xi,\eta) \tag{6.13}$$

and $u_N(x;\xi,\eta)$ is the N-soliton configuration the time dependence of which is parameterized by (ξ,η). Therefore, the local coordinates (ξ,η) are defined by the equations:

$$\dot{\xi} = \frac{\partial h_j}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial h_j}{\partial \xi}, \tag{6.14}$$

where h_j must obey the Poisson conditions*:

$$\{u_c(x,t), h_j\} = \frac{\delta H_j}{\delta p_c(x,t)}, \quad \{p_c(x,t), h_j\} = -\frac{\delta H_j}{\delta u_c(x,t)}. \tag{6.15}$$

One can see, choosing

$$h_j(\xi,\eta) = H_j(u_c,p_c), \tag{6.16}$$

that the initial equations have been restored:

$$\dot{u}_c = \frac{\partial u_c}{\partial \xi}\dot{\xi} + \frac{\partial u}{\partial \eta}\dot{\eta} = \{u_c, h_j\} = \frac{\delta H_j}{\delta p_c}.$$

The same we will have for $\dot{p}_c$. Therefore (u_c,p_c) are solutions of equations of motion (6.14), if the equality (6.16) is held.

The field theory case in $(1+1)$-dimensional configuration space needs additional explanations. First of all, the analog of (5.10) must be introduced:

$$\Delta(u,p) = \int \prod_t d^N\xi(t)\, d^N\eta(t) \prod_{x,t} \delta(u(x,t) - u_c(x;\xi,\eta))\, \delta(p(x,t) - p_c(x;\xi,\eta)) \tag{6.17}$$

if the N-soliton configuration is considered. Notice that the one-dimensional δ-functions are introduced in (6.17), and u_c, p_c are the functions of sets (ξ,η), $\dim(\xi,\eta) = 2N$. Introducing (6.17) we make an attempt to «hide» the time dependence entirely into the set of *independent* variables (ξ,η).

Comparing (6.9) and (6.12) one can note that x dependence disappeared and the transformed measure depends on the number $N = 1, 2, \ldots$ Therefore, there occurs the reduction of the quantum degrees of freedom since the power of the coordinate set is continuum and the number of solitons N is the countable set. This means that the proposed transformation to coordinates of solitons will be unavoidably singular.

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*See the previous section.

Notice then that the x dependence of $\Delta(u,p)$ remains unimportant since last one always appears under the integrals over all $u(x,t)$ and $p(x,t)$. At the same time, it is important that introduced in the previous section Δ_c disappeared in the final result, if the integral form of Poisson brackets (6.15) is held*.

One can try to propose also the local form of canonical commutators (6.15), if the definition (6.16) is held. Indeed, one can find inserting (6.16) into (6.15) that:

$$\{u_c(x,t), H_j(u_c,p_c)\} = \frac{\delta H_j(u_c,p_c)}{\delta p_c(x,t)}, \quad \{p_c(x,t), H_j(u_c,p_c)\} = -\frac{\delta H_j(u_c,p_c)}{\delta u_c(x,t)}. \tag{6.18}$$

This equalities must hold for arbitrary j. Using the definition:

$$H_j(x_c,p_c) = \int dy \tilde{H}_j(x_c,p_c),$$

where $\tilde{H}_j$ is the Hamiltonian density, one can write from (6.18):

$$\int dy \{u_c(x;\xi,\eta), u_c(y;\xi,\eta)\} \frac{\delta \tilde{H}_j}{\delta u_c(y,t)} + \\ + \int dy (\{u_c(x;\xi,\eta), p_c(y;\xi,\eta)\} - \delta(x-y)) \frac{\delta \tilde{H}_j}{\delta p_c(y,t)} = 0$$

and

$$\int dy \{p_c(x;\xi,\eta), p_c(y;\xi,\eta)\} \frac{\delta \tilde{H}_j}{\delta p_c(y,t)} - \\ - \int dy (\{u_c(x;\xi,\eta), p_c(y;\xi,\eta)\} - \delta(x-y)) \frac{\delta \tilde{H}_j}{\delta u_c(y,t)} = 0.$$

Then one can propose the solutions of these equations:

$$\begin{gathered} \{u_c(x;\xi,\eta), u_c(y;\xi,\eta)\} = \{p_c(x;\xi,\eta), p_c(y;\xi,\eta)\} = 0, \\ \{u_c(x;\xi,\eta), p_c(y;\xi,\eta)\} = \delta(x-y). \end{gathered} \tag{6.19}$$

But it is interesting that the local commutators (6.19) are not satisfied**. One can see this inserting the soliton solution into (6.19). On the other hand, the integral form (6.18) is satisfied. All this means that u_c and p_c are not the completely

*See the transformation (5.12), described in the previous section. For more confidence one can introduce the appropriate cells in the x space [24].

**That circumstance was mentioned firstly by V. Voronyuk.

independent variables. It must be stressed that the local relations (6.19) are not the necessary conditions in our formalism.

In our terms, the quantum force $j(x,t)$ excites the (ξ,η) manifold only, leaving the topology of classical trajectory $(u,p)_c$ unchanged. We can use them immediately since the complete set of canonical coordinates (ξ,η) of sine-Gordon model is known, see, e.g., [24].

6.2.2. Perturbation Theory on the Cotangent Bundle. The classical Hamiltonian h_j is the sum:

$$h_j(\eta) = \int dp\,\sigma(r)\sqrt{r^2+m_h^2} + \sum_{i=1}^{N} h(\eta_i), \tag{6.20}$$

where $\sigma(r)$ is the continuous spectrum and $h(\eta)$ is the soliton energy. Note the absence of interaction energy among solitons.

New degrees of freedom $(\xi,\eta)(t)$ must obey equations (6.14):

$$\begin{gathered}\dot{\xi}_i = \Omega(\eta_i) - \int dx j(x,t)\frac{\partial u_N(x;\xi,\eta)}{\partial \eta_i}, \quad \Omega(\eta) \equiv \frac{\partial h(\eta)}{\partial \eta},\\ \dot{\eta}_i = \int dx j(x,t)\frac{\partial u_N(\xi,\eta)}{\partial \xi_i}.\end{gathered} \tag{6.21}$$

Hence the sources of quantum perturbations are proportional to the time-local fluctuations of soliton configurations

$$\frac{\partial u_N(x;\xi,\eta)}{\partial \eta_i}, \quad \frac{\partial u_N(x;\xi,\eta)}{\partial \xi_i}.$$

One can split the Lagrange source onto «Hamiltonian» ones:

$$j(x,t) \to (j_\xi, j_\eta).$$

This gives weight functional $U(u_N; e_\xi, e_\eta)$ and operator $\hat{\mathbb{K}}(e_\xi, e_\eta; j_\xi, j_\eta)$. As a result,

$$\rho(q) = \sum_N \mathrm{e}^{-i\hat{K}(e_\xi,e_\eta;j_\xi,j_\eta)} \int D^N M(\xi,\eta)\, \mathrm{e}^{iS_O(u_N)}\, \mathrm{e}^{-iU(u_N;e_\xi,e_\eta)} \times \\ \times |\Gamma(q;u_N)|^2, \tag{6.22}$$

where, using vector notations, we get

$$\hat{\mathbb{K}}(e_\xi, e_\eta; j_\xi, j_\eta) = \frac{1}{2}\int dt\, \{\hat{j}_\xi(t)\cdot\hat{e}_\xi(t) + \hat{j}_\eta(t)\cdot\hat{e}_\eta(t)\}. \tag{6.23}$$

The measure takes the form:

$$D^N M(\xi,\eta) = \prod_{i=1}^{N}\prod_t d\xi_i(t)\, d\eta_i(t)\, \delta(\dot{\xi}_i - \Omega(\eta_i) - j_{\xi,i}(t))\, \delta(\dot{\eta}_i - j_{\eta,i}(t)). \quad (6.24)$$

The effective interaction potential

$$U(u_N; e_\xi, e_\eta) = -\frac{2m^2}{\lambda^2}\int dx\, dt\, \sin\, \lambda u_N\, (\sin\, \lambda e - \lambda e) \quad (6.25)$$

with

$$e(x,t) = e_\xi(t)\,\frac{\partial u_N(x;\xi,\eta)}{\partial \eta(t)} - e_\eta(t)\,\frac{\partial u_N(x;\xi,\eta)}{\partial \xi(t)}. \quad (6.26)$$

Performing the shifts:

$$\begin{aligned} \xi_i(t) &\to \xi_i(t) + \int dt' g(t-t') j_{\xi,i}(t') \equiv \xi_i(t) + \xi_i'(t), \\ \eta_i(t) &\to \eta_i(t) + \int dt' g(t-t') j_{\eta,i}(t') \equiv \eta_i(t) + \eta_i'(t), \end{aligned} \quad (6.27)$$

we can move the Green function $g(t-t')$ into the operator:

$$\hat{\mathbb{K}}(e_\xi, e_\eta; \xi', \eta') = \frac{1}{2}\int dt\, dt' g(t-t')\{\hat{\xi}'(t')\cdot \hat{e}_\xi(t) + \hat{\eta}'(t')\cdot \hat{e}_\eta(t)\}. \quad (6.28)$$

Notice that the Green function $g(t-t')$ of Eqs. (6.21) is again the step function:

$$g(t-t') = \Theta(t-t'). \quad (6.29)$$

Its imaginary part is equal to zero for real times and this allows one to shift $C_\pm$ to the real-time axis (see [26]).

Consequently,

$$D^N M(\xi,\eta) = \prod_{i=1}^{N}\prod_t d\xi_i(t)\, d\eta_i(t)\, \delta(\dot{\xi}_i - \Omega(\eta+\eta'))\delta(\dot{\eta}_i) \quad (6.30)$$

with

$$u_N = u_N(x; \xi+\xi', \eta+\eta'). \quad (6.31)$$

The equations

$$\dot{\xi}_i = \Omega(\eta_i + \eta_i') \quad (6.32)$$

are trivially integrable. In quantum case $\eta_i' \neq 0$ this equation describes the motion on nonhomogeneous and anisotropic manifold. So, the expansion over

$(\hat{\xi}',\ \hat{e}_\xi,\ \hat{\eta}',\ \hat{e}_\eta)$ generates the local in time deformations of γ_N manifold, $(\xi,\eta)\in\gamma_N$ completely. The weight of this deformations is defined by $U(u_N;e_\xi,e_\eta)$.

Using the definition

$$\int Dx\delta(\dot{x})=\int dx(0)=\int dx_0,$$

functional integrals are reduced to the ordinary integrals over initial data $(\xi,\eta)_0$. These integrals define the zero modes volume.

6.3. Quantum Corrections. The proof of (6.5) we would divide into two parts. First of all, we would consider the semiclassical approximation (Subsubsec. 6.3.1), and in Subsubsec. 6.3.2. we will show that this approximation is exact.

6.3.1. Introduction and Definitions. The N-soliton solution u_N depends on $2N$ parameters. Half of them, N, can be considered as the position of solitons and other N as the solitons momentum. Generally at $|t|\to\infty$ the u_N solution decomposed on the single solitons u_s and on the double-soliton bound states u_b [24]:

$$u_N(x,t)=\sum_{j=1}^{n_1}u_{s,j}(x,t)+\sum_{k=1}^{n_2}u_{b,k}(x,t)+O(\mathrm{e}^{-|t|}).$$

We will see later that main elements of our formalism are the one-soliton u_s and two-soliton bound state u_b configurations. Its (ξ,η) parameterizations, confirmed to Eqs. (6.15), have the form:

$$u_s(x;\xi,\eta)=-\frac{4}{\lambda}\arctan\{\exp\left(m_h x\cosh\beta\eta-\xi\right)\},\quad \beta=\frac{\lambda^2}{8} \tag{6.33}$$

and

$$u_b(x;\xi,\eta)=-\frac{4}{\lambda}\arctan\left\{\tan\frac{\beta\eta_2}{2}\frac{m_h x\sinh\beta\eta_1/2\,\cos\beta\eta_2/2-\xi_2}{m_h x\cosh\beta\eta_1/2\,\sin\beta\eta_2/2-\xi_1}\right\}. \tag{6.34}$$

The (ξ,η) parameterization of solitons individual energies $h(\eta)$ takes the form:

$$h_s(\eta)=\frac{m_h}{\beta}\cosh\beta\eta,\quad h_b(\eta)=\frac{2m_h}{\beta}\cosh\frac{\beta\eta_1}{2}\sin\frac{\beta\eta_2}{2}\geqslant 0.$$

The bound-states energy h_b depends on η_1 and η_2. First one defines inner motion of two bounded solitons and second one — the bound-states center-of-mass motion. Correspondingly, we will call these parameters as the internal and external ones. Note that the inner motion is periodic, see (6.24).

Performing last integration in (6.22) with measure (6.30) we find:

$$\rho(q)=\sum_N\int\prod_{i=1}^{N}\{d\xi_0\,d\eta_0\}_i\,\mathrm{e}^{-i\hat{\mathbb{K}}}\,\mathrm{e}^{iS_O(u_N)}\,\mathrm{e}^{-iU(u_N;e_\xi,e_\eta)}|\Gamma(q;u_N)|^2, \tag{6.35}$$

where

$$u_N = u_N(\eta_0 + \eta', \xi_0 + \Omega(t) + \xi') \tag{6.36}$$

and

$$\Omega(t) = \int dt' \Theta(t - t') \, \Omega(\eta_0 + \eta'(t')). \tag{6.37}$$

In the semiclassical approximation $\xi' = \eta' = 0$ we have

$$u_N = u_N(x; \eta_0, \xi_0 + \Omega(\eta_0)t). \tag{6.38}$$

Note now that if the surface term

$$\int \partial_\mu (\mathrm{e}^{iqx} \partial^\mu u_N) = 0, \tag{6.39}$$

then

$$\int d^2x \, \mathrm{e}^{iqx} (\partial^2 + m_h^2) \, u_N(x,t) = -(q^2 - m_h^2) \int d^2x \, \mathrm{e}^{iqx} u_N(x,t) = 0 \tag{6.40}$$

since q^2 belongs to mass shell by definition. The condition (6.39) is satisfied since u_N belongs to Schwartz space (the periodic boundary condition for $u(x,t)$ does not alter this conclusion). Therefore, in the semiclassical approximation Eq. (6.5) is held.

Expending the operator exponent in (6.35) we will find the expansion over

$$\begin{aligned} \rho_{n,m}(q) = \frac{(1/2i)^n}{n!} \frac{(1/2i)^m}{m!} \lim_{(\xi',\eta',e_\xi,e_\eta)=0} \sum_N \int d^N \xi_0 \, d^N \eta_0 \times \\ \times \int \prod_{i=1}^n \{ dt_i \, dt'_i \theta(t_i - t'_i) \hat{\xi}'(t'_i) \} \times \\ \times \int \prod_{i=1}^m \{ dt_i \, dt'_i \theta(t_i - t'_i) \, \hat{\eta}'(t'_i) \} \, \mathrm{e}^{iS_O(u_N)} |\Gamma(q; u_N)|^2 \times \\ \times \left\{ \prod_{i=1}^n \hat{e}_\xi(t_i) \prod_{j=1}^m \hat{e}_\eta(t_j) \, \mathrm{e}^{-iU(u_N; e_\xi, e_\eta)} \right\} \Bigg|_{e=0}, \end{aligned} \tag{6.41}$$

where $U(u_N; e_\xi, e_\eta)$ was defined in (6.25), (6.26). Notice that the action of operators $\hat{\xi}'$, $\hat{\eta}'$ creates terms

$$\int d^2x \, \mathrm{e}^{iqx} \theta(t - t') (\partial^2 + m^2) u_N(x,t) \neq 0. \tag{6.42}$$

6.3.2. Quantum Corrections. Now we will show that *the semiclassical approximation is exact in the soliton sector of (6.1), (6.11) theory.*

The structure of the perturbation theory is readily seen in the «normal-product» form:

$$\rho(q) = \sum_N \int \prod_{i=1}^{N} \{d\xi_0\, d\eta_0\}_i :\, \mathrm{e}^{-iU(u_N;\hat{j}/2i)}\, \mathrm{e}^{iS_O(u_N)} |\Gamma(q;u_N)|^2 :, \qquad (6.43)$$

where

$$\hat{j} = \hat{j}_\xi \frac{\partial u_N}{\partial \eta} - \hat{j}_\eta \frac{\partial u_N}{\partial \xi} = \omega \hat{j}_X \frac{\partial u_N}{\partial X} \qquad (6.44)$$

and

$$\hat{j}_X = \int dt' \Theta(t-t')\, \hat{X}(t') \qquad (6.45)$$

with $2N$-dimensional vector $X = (\xi, \eta)$. In Eq. (6.44), ω is the ordinary simplectic matrix.

The colons in (6.43) mean that the operator $\hat{j}$ should stay to the left of all functions. The structure (6.44) shows that each order over $\hat{j}_{X_i}$ is proportional at least to the first-order derivative of u_N over conjugate to X_i variable.

The expansion of (6.43) over $\hat{j}_X$ can be written [26] in the form of total derivatives (omitting the semiclassical approximation):

$$\rho(q) = \sum_N \int \prod_{i=1}^{N} \{d\xi_0\, d\eta_0\}_i \left\{ \sum_{i=1}^{2n} \frac{\partial}{\partial X_{0i}} P_{X_i}(u_N) \right\}, \qquad (6.46)$$

where $P_{X_i}(u_N)$ is the infinite sum of «time-ordered» polynomials (see [26]) over u_N and its derivatives. The explicit form of $P_{X_i}(u_N)$ is complicated since the interaction potential is nonpolynomial. But it is enough to know, see (6.44), that

$$P_{X_i}(u_N) \sim \omega_{ij} \frac{\partial u_N}{\partial X_{0j}}. \qquad (6.47)$$

Therefore,

$$\rho(q) = 0 \qquad (6.48)$$

since (i) each term in (6.46) is the total derivative, (ii) we have (6.47) and (iii) u_N belongs to Schwartz space.

We can conclude that the equality (6.48) is hold since

$$\frac{\partial u_N}{\partial X_0} = 0 \ \text{ at } \ X_0 \in \partial W, \qquad (6.49)$$

where ∂W is the boundary of W.

In our consideration we did not touch the continuous spectrum contributions. In considered approach these contributions are absent since they are realized on zero measure: theirs contributions are $\sim \{\text{volume of } \gamma_N\}^{-1}$.

7. SUMMARY

Let as summarize the general results of the the present and of the previous sections.

1. The m- into n-particles transition (non-normalized) *probability* R_{nm} would have on the Dirac measure the following symmetrical form:

$$\rho_{nm}(p_1,\ldots,p_n,q_1,\ldots,q_m)=\left\langle \prod_{k=1}^{m}|\Gamma(q_k;u)|^2\prod_{k=1}^{n}|\Gamma(p_k;u)|^2\right\rangle_u =$$
$$= \mathrm{e}^{-i\hat{K}(j,e)}\int DM(u)\ \mathrm{e}^{iS_O(u)-iU(u,e)}\prod_{k=1}^{m}|\Gamma(q_k;u)|^2\prod_{k=1}^{n}|\Gamma(p_k;u)|^2 \equiv$$
$$\equiv \hat{\mathcal{O}}(u)\prod_{k=1}^{m}|\Gamma(q_k;u)|^2\prod_{k=1}^{n}|\Gamma(p_k;u)|^2. \quad (7.1)$$

Here $p(q)$ are the in(out)-going particle momenta. It should be underlined that this representation is strict and is valid for arbitrary Lagrange theory of arbitrary dimensions.

2. The operator $\hat{\mathcal{O}}$ contains three elements. The Dirac measure DM, the functionals S_O, $U(x,e)$, and the operator $\hat{\mathbb{K}}(j,e)$.

The expansion over the operator

$$\hat{\mathbb{K}}(j,e)=\frac{1}{2}\mathrm{Re}\int_{C_+} dx\,dt\frac{\delta}{\delta j(x,t)}\frac{\delta}{\delta e(x,t)}\equiv\frac{1}{2}\mathrm{Re}\int_{C_+} dx\,dt\,\hat{j}(x,t)\hat{e}(x,t) \quad (7.2)$$

generates the perturbation series. We will assume that this series exists (at least in Borel sense).

3. The functionals $U(u,e)$ and $S_O(u)$ are defined by the equalities:

$$S_O(u)=(S_0(u+e)-S_0(u-e))+2\mathrm{Re}\int_{C_+} dx\,dt\,e(x,t)(\partial^2+m^2)u(x,t), \quad (7.3)$$

$$U(u,e)=V(u+e)-V(u-e)-2\mathrm{Re}\int_{C_+} dx\,dt\,e(x,t)v'(u), \quad (7.4)$$

where $S_0(u)$ is the free part of the Lagrangian and $V(u)$ describes interactions. The quantity $S_O(u)$ is not equal to zero if u have nontrivial topological charge.

4. The measure $DM(u,p)$ has the Dirac form:

$$DM(u,p)=\prod_{x,t}du(x,t)\,dp(x,t)\delta\left(\dot{u}-\frac{\delta H_j(u,p)}{\delta p}\right)\delta\left(\dot{p}+\frac{\delta H_j(u,p)}{\delta u}\right) \quad (7.5)$$

with the total Hamiltonian

$$H_j(u,p)=\int dx\left\{\frac{1}{2}p^2+\frac{1}{2}(\nabla u)^2+v(u)-ju\right\}. \qquad (7.6)$$

This last one includes the energy ju of quantum fluctuations.

5. Dirac measure contains following information:

a. Only *strict* solutions of equations

$$\dot{u}-\frac{\delta H_j(u,p)}{\delta p}=0,\ \dot{p}+\frac{\delta H_j(u,p)}{\delta u}=0 \qquad (7.7)$$

with $j=0$ should be taken into account. This «rigidness» of the formalism means the absence of pseudosolutions (similar to multiinstanton, or multikink) contribution.

b. ρ_{nm} is described by the *sum* of all solutions of Eq. (7.7), independently of their «nearness» in the functional space.

c. ρ_{nm} did not contain the interference terms from various topologically nonequivalent contributions. This displays the orthogonality of corresponding Hilbert spaces.

d. The measure (7.5) includes $j(x)$ as the external adiabatic source. Its fluctuation disturbs the solutions of Eq. (7.7) and *vice versa* since the measure (7.5) is strict.

e. In the frame of the adiabatical condition, the field disturbed by $j(x)$ belongs to the same manifold (topology class) as the classical field defined by (7.7) [26].

f. The Dirac measure is derived for *real-time* processes only, i.e., (7.5) is not valid for tunneling ones. For this reason, the above conclusions should be taken carefully.

g. It can be shown that theory on the measure (7.5) restores ordinary (canonical) perturbation theory.

6. The parameter $\Gamma(q;u)$ plays the role of particle production vertex. It is connected directly with *external* particle energy, momentum, spin, polarization, charge, etc., and is sensitive to the symmetry properties of the interacting fields system. For the sake of simplicity, $u(x)$ is the real scalar field. The generalization would be evident.

As a consequence of (7.5), $\Gamma(q;u)$ is the function of the external particle momentum q and is a *linear* functional of $u(x)$:

$$\Gamma(q;u)=-\int dx\,\mathrm{e}^{iqx}\frac{\delta S_0(u)}{\delta u(x)}=\int dx\,\mathrm{e}^{iqx}(\partial^2+m^2)u(x),\quad q^2=m^2 \qquad (7.8)$$

for the mass m field. This parameter presents the momentum distribution of the interacting field $u(x)$ on the remote hypersurface σ_∞ if $u(x)$ is the regular function. Notice, the operator (∂^2+m^2) cancels the mass-shell states of $u(x)$.

The construction (7.8) means, because of the Klein–Gordon operator and since the external states being mass-shell by definition [33], that the solution $\rho_{nm} = 0$ is possible for a particular topology (compactness and analytic properties) of $quantum$ field $u(x)$. So, $\Gamma(q; u)$ carries the following remarkable properties:

— it directly defines the observables,

— it is defined by the topology of $u(x)$,

— it is the linear functional of the actions symmetry group element $u(x)$.

If (7.7) have nontrivial solution $u_c(x,t)$, then this «extended objects» quantization problem arises. We solve it introducing convenient dynamical variables [34]. Then the measure (7.5) admits the transformation:

$$u_c: \ (u,p) \to (\xi,\eta) \in W = \frac{G}{G_c}, \tag{7.9}$$

and the transformed measure has the form:

$$DM(u,p) = \prod_{x,tC} d\xi(t)\, d\eta(t) \delta\left(\dot{\xi} - \frac{\delta h_j(\xi,\eta)}{\delta\eta}\right) \delta\left(\dot{\eta} + \frac{\delta h_j(\xi,\eta)}{\delta\xi}\right), \tag{7.10}$$

where $h_j(\xi,\eta) = H_j(u_c,p_c)$ is the transformed Hamiltonian.

It is evident that (ξ,η) are parameters of integration of Eqs. (7.7) and they form the factor space $W = G/G_c$. As a result of mapping of the perturbation generating operator $\hat{\mathbb{K}}$ on the manifold W, the equations of motion became linearized:

$$DM = \prod_t \delta\left(\dot{\xi} - \frac{\delta h(\eta)}{\delta\eta} - j_\xi\right) \delta\left(\dot{\eta} - j_\eta\right). \tag{7.11}$$

If Feynman's $i\epsilon$-prescription is adopted, then the Green function of Eq. (7.11):

$$g(t-t') = \Theta(t-t') \tag{7.12}$$

with boundary property:

$$\Theta(0) = 1.$$

7. Expansion of $\exp\{\hat{\mathbb{K}}(j,e)\}$ gives the «strong coupling» perturbation series. Its analysis shows that the action of the integro-differential operator $\hat{\mathcal{O}}$ leads to the following representation:

$$\rho_{nm}(p,q) = \int_W \left\{ d\xi(0)\, \frac{\partial}{\partial\xi(0)} \rho^{\xi}_{nm}(p,q) + d\eta(0)\, \frac{\partial}{\partial\eta(0)} \rho^{\eta}_{nm}(p,q) \right\}. \tag{7.13}$$

This means that the contributions into $R_{nm}(p,q)$ are accumulated strictly on the boundary, «bifurcation manifold», ∂W, i.e., depend directly on the topology property of W.

8. It was shown that the MP is absent in the frame of Lagrangian (6.1). For this purpose one should modify the sine-Gordon Lagrangian adding, for instance, the term

$$\frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}M^2\Phi^2 - \frac{c}{3}u\Phi^2 \tag{7.14}$$

to describe collision of «external» field Φ on the solitons. This model allows one to introduce the nontrivial probabilities $\rho(q_1, q_2, \ldots)$ considering creation (and absorption) of the field Φ. Note that field $u(x)$ is still «confined» even with this adding.

CONCLUSION

The final goal of the present approach is to construct the workable at arbitrary distances, i.e., for arbitrary momenta of produced hadrons, S-matrix formalism for theories with (hidden) symmetry. But this aim remains unachieved in the present paper. In subsequent papers more realistic field models in 4D Minkowski space–time metric will be described. But one should not consider the demonstrated examples of Yang–Mills S matrix as the definite proves since I am not sure that the used $O(4) \times O(2)$ solution of Yang–Mills equation in the Minkowski in the situation of general position guarantees the largest contribution. Moreover, only the $SU(2)$ theory will be considered. Unfortunately, we cannot find in the frame of 't Hooft ansatz [35] the solution for larger $SU(N)$ group [36].

It will be to show how one or another physical phenomena may be seen in the field theory with symmetry. Namely,

— *no plain-waves production exists in theories with symmetry*,

i.e., for instance, the gluons cannot be seen in a free state since simply the last ones are absent in quantum theory of the symmetry manifolds, or, in other words, since the gluon states and the «states» of the symmetry manifold belong to the orthogonal Hilbert spaces. The quark fields will not be included in this simplest example. But more realistic model with quarks shows that

— *inclusion of matter cannot change the previous conclusion that the gluons cannot be created.*

In the other example we will show how the

— *binding potential may arise among quarks.*

Here the situation of general position selection rule will be extremely important: it will be used that the situation, when $(q\bar{q})$ potential is independent of the scale of Yang–Mills fields, is mostly probable.

The quantum field theory with constraints will obey the following important property:

— *the perturbation theory of quantum systems with symmetry may be free of any divergences,* i.e., it may^* be rightful at arbitrary distances, for VHM case as well. It is the evident consequence of lessening of the number of dynamical degrees of freedom because of symmetry constraints**.

There exists also the intriguing question of asymptotic freedom. The point is that there is no running coupling constants in our strong coupling perturbation theory without divergences. On the other hand, the asymptotic freedom is the experimental fact. We will show how

— *the effect of asymptotic freedom may arise*

in our quantum theory of the symmetry manifolds. The main question here is to find the experimentally observable corrections to the asymptotic freedom law.

In summary, the aim of future publications would be the question: Is the offered approach complete from physical point of view? It is important since offered quantization scheme in the situation of general position on Dirac measure must be true for arbitrary distances, since it is free from arbitrary scale parameters***.

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*One cannot be sure that the approach is universal, can be used, for instance, in quantum gravity case.

**And it is unnecessary to have in that case any new mechanism, such as the supersymmetry, for example, to achieve the field theory without divergences. Possible scenario of such a theory will be discussed later.

***That is why I hope that it may give the predictions acceptable from physical point of view at arbitrary distances.

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