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INSTANTONS AND CHERN–SIMONS FLOWS IN 6, 7 AND 8 DIMENSIONS

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The existence of K-instantons on a cylinder $M^7 = \mathbb{R}_\tau \times K/H$ over a homogeneous nearly Kähler 6-manifold K/H requires a conformally parallel or a cocalibrated G_2-structure on M^7. The generalized anti-self-duality on M^7 implies a Chern–Simons flow on K/H which runs between instantons on the coset. For K-equivariant connections, the torsionful Yang–Mills equation reduces to a particular quartic dynamics for a Newtonian particle on $\mathbb{C}$. When the torsion corresponds to one of the G_2-structures, this dynamics follows from a gradient or Hamiltonian flow equation, respectively. We present the analytic (kink-type) solutions and plot numerical non-BPS solutions for general torsion values interpolating between the instantonic ones.

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INTRODUCTION

Yang–Mills instantons exist in dimensions d larger than four only when there is additional geometric structure on the manifold M^d (besides the Riemannian one). In order to formulate generalized first-order anti-self-duality conditions which imply the second-order Yang–Mills equations (possibly with torsion), M^d must be equipped with a so-called G-structure, which is a globally defined but not necessarily closed $(d-4)$-form Σ, so that the weak holonomy group of M^d gets reduced.

Instanton solutions in higher dimensions are rare in the literature. In the mid-eighties, Fairlie and Nuyts, and also Fubini and Nicolai discovered the $\mathrm{Spin}(7)$-instanton on $\mathbb{R}^8$. Eight years later, a similar G_2-instanton on $\mathbb{R}^7$ was found by Ivanova and Popov, and also by Günaydin and Nicolai. Our recent work shows that these so-called octonionic instantons are not isolated but embedded into a whole family living on a class of conical non-compact manifolds [1].

The string vacua in heterotic flux compactifications contain non-Abelian gauge fields which in the supergravity limit are subject to Yang–Mills equations with torsion $\mathcal{H}$ determined by the three-form flux. Prominent cases admitting instantons are $\mathrm{AdS}_{10-d} \times M^d$, where M^d is equipped with a G-structure, with G being $SU(3)$, G_2 or $\mathrm{Spin}(7)$ for $d = 6$, 7 or 8, respectively. Homogeneous nearly

Kähler 6-manifolds K/H and (iterated) cylinders and (sine-)cones over them provide simple examples, for which all K-equivariant Yang–Mills connections can be constructed [2, 3]. Natural choices for the gauge group are K or G.

Clearly, the Yang–Mills instantons discussed here serve to construct heterotic string solitons, as was first done in 1990 by Strominger for the gauge five-brane. It is therefore of interest to extend our new instantons to solutions of (string-corrected) heterotic supergravity and obtain novel string/brane vacua [4–6].

In this talk, I present the construction for the simplest case of a cylinder over a compact homogeneous nearly Kähler coset K/H, which allows for a conformally parallel or a cocalibrated G_2-structure. I display a family of non-BPS Yang–Mills connections, which contain two instantons at distinguished parameter values corresponding to those G_2-structures. In these two cases, anti-self-duality implies a Chern–Simons flow on K/H.

Finally, I must apologize for the omission — due to page limitation — of all relevant literature besides my own papers on which this talk is based. The reader can find all references therein.

1. SELF-DUALITY IN HIGHER DIMENSIONS

The familiar four-dimensional anti-self-duality condition for Yang–Mills fields F may be generalized to suitable d-dimensional Riemannian manifolds M,

$$*F = -\Sigma \wedge F \quad \text{for} \quad F = dA + A \wedge A \quad \text{and} \quad \Sigma \in \Lambda^{d-4}(M), \tag{1}$$

if there exists a geometrically natural $(d-4)$-form Σ on M. Applying the gauge-covariant derivative $D = d + [A, \cdot]$, it follows that

$$D{*}F + d\Sigma \wedge F = 0 \Longleftrightarrow \text{Yang–Mills with torsion} \quad \mathcal{H} = *d\Sigma \in \Lambda^3(M). \tag{2}$$

This torsionful Yang–Mills equation extremizes the action

$$S_{\text{YM}} + S_{\text{CS}} = \int_M \text{tr}\big\{F \wedge *F + (-)^{d-3}\Sigma \wedge F \wedge F\big\} = \\ = \int_M \text{tr}\left\{F \wedge *F + \frac{1}{2} d\Sigma \wedge \left(A\, dA + \frac{2}{3}A^3\right)\right\}. \tag{3}$$

Related to this generalized anti-self-duality is the gradient Chern–Simons flow on M,

$$\frac{dA}{d\tau} = \frac{\delta}{\delta A} S_{\text{CS}} = *\big(d\Sigma \wedge F\big) \sim *d\Sigma \lrcorner F. \tag{4}$$

In fact, this equation follows from generalized anti-self-duality on the cylinder $\widetilde{M} = \mathbb{R}_\tau \times M$ over M (in the $A_\tau = 0$ gauge).

The question is therefore: Which manifolds admit a global $(d-4)$-form? And the answer is: G-structure manifolds, i.e., manifolds with a weak special holonomy. The key cases we shall encounter in this talk are given in Table 1.

Table 1. **Examples of G-structure manifolds in** $d = 6, 7, 8$

d	G	Σ	Cases	Example	Structure
6	$SU(3)$	ω	Kähler	$\mathbb{C}P^3$	$d\omega = 0$
6	$SU(3)$	ω	Nearly Kähler	$S^6 = \dfrac{G_2}{SU(3)}$	$d\omega \sim \mathrm{Im}\,\Omega,\ d\,\mathrm{Re}\,\Omega \sim \omega^2$
7	G_2	ψ	Conf. parallel G_2	$\mathbb{R}_\tau\times$ nearly Kähler	$d\psi \sim \psi\wedge d\tau,\ d{*}\psi \sim -{*}\psi\wedge d\tau$
7	G_2	ψ	Nearly parallel G_2	$X_{k,\ell} = \dfrac{SU(3)}{U(1)_{k,\ell}}$	$d\psi \sim {*}\psi \Rightarrow d{*}\psi = 0$
7	G_2	ψ	Parallel G_2	Cone (nearly Kähler)	$d\psi = 0 = d{*}\psi$
8	Spin(7)	Σ	Parallel Spin(7)	$\mathbb{R}_\tau\times$parallel G_2	$d\Sigma = 0,\ {*}\Sigma = \Sigma$

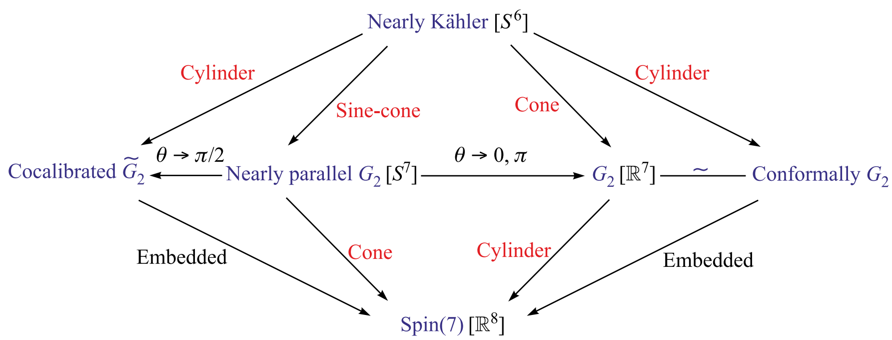


Fig. 1. Iterated cylinders, cones and sine-cones over nearly Kähler 6-manifolds

Some of those cases are related via the scheme shown in Fig. 1, with examples in square brackets.

For this talk I shall consider (reductive nonsymmetric) coset spaces $M = K/H$ in $d = 6$ as well as cylinders and cones over them. In all these cases, the gauge group is chosen to be K.

2. SIX DIMENSIONS: NEARLY KÄHLER COSET SPACES

All known compact nearly Kähler 6-manifolds M^6 are nonsymmetric coset spaces K/H:

$$S^6 = \frac{G_2}{SU(3)}, \quad \frac{Sp(2)}{Sp(1)\times U(1)}, \quad \frac{SU(3)}{U(1)\times U(1)},$$
$$S^3\times S^3 = \frac{SU(2)\times SU(2)\times SU(2)}{SU(2)}. \tag{5}$$

The coset structure $H \lhd K$ implies the decomposition

$$\mathrm{Lie}(K) \equiv \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{with} \quad \mathfrak{h} \equiv \mathrm{Lie}(H) \quad \text{and} \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}. \tag{6}$$

Interestingly, the reflection automorphism of symmetric spaces gets generalized to a so-called tri-symmetry automorphism $S : K \to K$ with $S^3 = id$ implying

$$s : \mathfrak{k} \to \mathfrak{k} \quad \text{with} \quad s|_{\mathfrak{h}} = \mathbb{1} \quad \text{and} \quad s|_{\mathfrak{m}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\, J = \exp\left\{\frac{2\pi}{3} J\right\}, \tag{7}$$

effecting a $2\pi/3$ rotation on TM^6. I pick a Lie-algebra basis

$$\{I_{a=1,\ldots,6},\, I_{i=7,\ldots,\dim G}\} \quad \text{with} \quad [I_a, I_b] = f^i_{ab} I_i + f^c_{ab} I_c, \tag{8}$$

involving the structure constants $f^{\bullet}_{ab}$. The Cartan–Killing form then reads

$$\langle \cdot, \cdot \rangle_{\mathfrak{k}} = -\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}(\cdot) \circ \mathrm{ad}(\cdot)) = 3\, \langle \cdot, \cdot \rangle_{\mathfrak{h}} = 3\, \langle \cdot, \cdot \rangle_{\mathfrak{m}} = \mathbb{1}. \tag{9}$$

Expanding all structures in a basis of canonical one-forms e^a framing $T^*(G/H)$,

$$g = \delta_{ab}\, e^a e^b, \quad \omega = \frac{1}{2} J_{ab}\, e^a \wedge e^b, \quad \Omega = -\frac{1}{\sqrt{3}} (f + iJf)_{abc}\, e^a \wedge e^b \wedge e^c, \tag{10}$$

we see that the almost complex structure (J_{ab}) and the structure constants f_{abc} rule everything.

Nearly Kähler 6-manifolds are special in that the torsion term in (2) *vanishes by itself*! What is more, this property is actually *equivalent* to the generalized anti-self-duality condition (1):

$$*F = -\omega \wedge F \iff 0 = d\omega \wedge F \sim \mathrm{Im}\,\Omega \wedge F \iff \text{DUY equations}, \tag{11}$$

where the Donaldson–Uhlenbeck–Yau (DUY) equations* state that

$$F^{2,0} = F^{0,2} = 0 \quad \text{and} \quad \omega \lrcorner F = 0. \tag{12}$$

Another interpretation of this anti-self-duality condition is that it projects F to the 8-dimensional eigenspace of the endomorphism $*(\omega \wedge \cdot)$ with eigenvalue -1, which contains the part of $F^{1,1}$ orthogonal to ω. Equations (11) imply also $\mathrm{Re}\,\Omega \wedge F = 0$ and the (torsion-free) Yang–Mills equations $D{*}F = 0$. Clearly, they separately extremize both S_{YM} and S_{CS} in (3), but of course yield only BPS-type classical solutions. In components the above relations take the form

$$\frac{1}{2}\epsilon_{abcdef} F_{ef} = -J_{[ab} F_{cd]} \iff 0 = f_{abc} F_{bc}, \tag{13}$$

$$\implies \omega_{ab} F_{ab} = 0, \quad (Jf)_{abc} F_{bc} = 0, \quad D_a F_{ab} = 0. \tag{14}$$

I notice that each Chern–Simons flow $\dot{A}_a \sim f_{abc} F_{bc}$ on M^6 ends in an instanton.

*Also known as «Hermitian Yang–Mills equations».

Let me look for K-equivariant connections A on M^6. If I restrict their value to $\mathfrak{h}$, the answer is unique: the only «H-instanton» is the so-called canonical connection

$$A^{\text{can}} = e^i\, I_i \longrightarrow F^{\text{can}} = -\frac{1}{2} f^i_{ab}\, e^a \wedge e^b\, I_i, \tag{15}$$

where $e^i = e^i_a e^a$. Generalizing to «K-instantons», I extend to

$$A = e^i\, I_i + e^a\, \Phi_{ab}\, I_b \quad \text{with ansatz} \quad (\Phi_{ab}) =: \Phi = \phi_1\, \mathbb{1} + \phi_2 J, \tag{16}$$

which is in fact general for G_2 invariance on S^6. Its curvature is readily computed to

$$F_{ab} = F^{1,1}_{ab} + F^{2,0\oplus 0,2}_{ab} = (|\Phi|^2 - 1)\, f^i_{ab}\, I_i + [(\bar{\Phi}^2 - \Phi)\, f]_{abc}\, I_c \tag{17}$$

and displays the tri-symmetry invariance under $\Phi \to \exp(-2\pi/3J)\Phi$. The solutions to the BPS conditions (11) are

$$\bar{\Phi}^2 = \Phi \Longrightarrow \Phi = 0 \quad \text{or} \quad \Phi = \exp\left\{\frac{2\pi k}{3} J\right\} \quad \text{for} \quad k = 0, 1, 2, \tag{18}$$

which yields three flat K-instanton connections besides the canonical curved one,

$$A^{(k)} = e^i\, I_i + e^a\, (s^k I)_a \quad \text{and} \quad A^{\text{can}} = e^i\, I_i. \tag{19}$$

3. SEVEN DIMENSIONS: CYLINDER OVER NEARLY KÄHLER COSETS

Let me step up one dimension and consider 7-manifolds M^7 with weak G_2 holonomy associated with a G_2-structure three-form ψ. Here, the 7 generalized anti-self-duality equations project F onto the -1 eigenspace of $*(\psi \wedge \cdot)$, which is 14-dimensional and isomorphic to the Lie algebra of G_2,

$$*F = -\psi \wedge F \Longleftrightarrow *\psi \wedge F = 0 \Longleftrightarrow \psi \lrcorner F = 0, \tag{20}$$

providing an alternative form of the condition. In components, it reads

$$\frac{1}{2}\epsilon_{abcdefg} F_{fg} = -\psi_{[abc} F_{de]} \Longleftrightarrow 0 = \psi_{abc} F_{bc}. \tag{21}$$

For the parallel and nearly parallel G_2 cases, the previous accident (11) recurs,

$$d\psi \sim *\psi \Longrightarrow d\psi \wedge F = 0 \Longrightarrow D * F = 0, \tag{22}$$

and the torsion decouples. Note that on a general weak G_2-manifold there are two different flows,

$$\frac{dA(\sigma)}{d\sigma} = *d\psi \lrcorner F(\sigma) \quad \text{and} \quad \frac{dA(\sigma)}{d\sigma} = \psi \lrcorner F(\sigma) \quad \text{for} \quad \sigma \in \mathbb{R}, \tag{23}$$

which coincide in the nearly parallel case. The second flow ends in an instanton on M^7.

In this talk I focus on cylinders $M^7 = \mathbb{R}_\tau \times K/H$ over nearly Kähler cosets, with a metric $g = (d\tau)^2 + \delta_{ab}\, e^a e^b$, on which I study the Yang–Mills equation with a torsion given by

$$*\mathcal{H} = \frac{1}{3}\kappa\, d\omega \wedge d\tau \Longleftrightarrow T_{abc} = \kappa f_{abc} \tag{24}$$

with a real parameter κ. We shall see that for special values of κ my torsionful Yang–Mills equation

$$D{*}F + \frac{1}{3}\kappa\, d\omega \wedge d\tau \wedge F = 0 \tag{25}$$

descends from an anti-self-duality condition (20).

Taking the $A_0 = 0$ gauge and borrowing the ansatz (16) from the nearly Kähler base, I write

$$\begin{aligned} A_a &= e^i_a I_i + [\Phi(\tau)\, I]_a \Rightarrow F_{0a} = [\dot\Phi\, I]_a, \\ F_{ab} &= (|\Phi|^2{-}1)\, f^i_{ab}\, I_i + [(\bar\Phi^2{-}\Phi)\, f]_{abc}\, I_c \end{aligned} \tag{26}$$

which depends on a complex function $\Phi(\tau)$ (values in the $(\mathbb{1}, J)$ plane). Sticking this into (25) and computing for a while, one arrives at

$$\ddot\Phi = (\kappa - 1)\Phi - (\kappa{+}3)\bar\Phi^2 + 4\bar\Phi\Phi^2 =: \frac{1}{3}\frac{\partial V}{\partial\bar\Phi}. \tag{27}$$

Nice enough, I have obtained a ϕ^4 model with an action

$$\begin{aligned} S[\Phi] &\sim \int_{\mathbb{R}} d\tau\, \left\{3|\dot\Phi|^2 + V(\Phi)\right\} \quad \text{for} \\ V(\Phi) &= (3{-}\kappa) + 3(\kappa{-}1)|\Phi|^2 - (3{+}\kappa)(\Phi^3{+}\bar\Phi^3) + 6|\Phi|^4 \end{aligned} \tag{28}$$

devoid of rotational symmetry (for $\kappa{\neq}{-3}$) but enjoying tri-symmetry in the complex plane. It leads me to a mechanical analog problem of a Newtonian particle on $\mathbb{C}$ in a potential $-V$. I obtain the same action by plugging (26) directly into (3) with $d\Sigma = *\mathcal{H}$ from (24).

For the case of $K/H = S^6 = G_2/SU(3)$, equation (27) produces in fact *all* G-equivariant Yang–Mills connections on $\mathbb{R}_\tau \times K/H$. On $Sp(2)/(Sp(1)\times U(1))$ and $SU(3)/(U(1){\times}U(1))$, however, the most general G-equivariant connections involve two respective three complex functions of τ. The corresponding Newtonian dynamics on $\mathbb{C}^2$ respective $\mathbb{C}^3$ is of similar type but constrained by the conservation of Noether charges related to relative phase rotations of the complex functions.

4. SEVEN DIMENSIONS: SOLUTIONS

Finite-action solutions require Newtonian trajectories between zero-potential critical points $\hat{\Phi}$. With two exotic exceptions, $dV(\hat{\Phi}) = 0 = V(\hat{\Phi})$ yields precisely the BPS configurations on K/H:

- $\hat{\Phi} = \mathrm{e}^{2\pi ik/3}$ with $V(\hat{\Phi}) = 0$ for all values of κ and $k = 0, 1, 2$;
- $\hat{\Phi} = 0$ with $V(\hat{\Phi}) = 3-\kappa$ vanishing only at $\kappa = 3$.

Kink solutions will interpolate between two different critical points, while bounces will return to the critical starting point. Thus for generic κ values one may have kinks of «transversal» type, connecting two third roots of unity, as well as bounces. For $\kappa = 3$ «radial» kinks, reaching such a root from the origin, may occur as well. Numerical analysis reveals the domains of existence in κ (see Table 2).

Table 2. **Existence domains of kink and bounce solutions**

κ interval	$(-\infty, -3]$	$(-3, +3)$	$+3$	$(+3, +5)$	$[+5, +\infty)$
Types of trajectory	Radial bounce	Transversal kink	Radial kink	Radial bounce	—

In Fig. 2 I display contour plots of the potential and finite-action trajectories for eight choices of κ. They reveal three special values of κ: At $\kappa = -3$ rotational symmetry emerges; this is a degenerate situation. At $\kappa = -1$ and at $\kappa = +3$, the trajectories are straight, indicating integrability. Indeed, behind each of these two cases lurks a first-order flow equation, which originates from anti-self-duality and hence a particular G_2-structure ψ.

Let me first discuss $\kappa = +3$. For this value I find that

$$3\ddot{\Phi} = \frac{\partial V}{\partial \bar{\Phi}} \Longleftarrow \sqrt{2}\dot{\Phi} = \pm\frac{\partial W}{\partial \bar{\Phi}} \quad \text{with} \quad W = \frac{1}{3}(\Phi^3+\bar{\Phi}^3) - |\Phi|^2, \tag{29}$$

which is a gradient flow with a real superpotential W, as

$$V = 6\left|\frac{\partial W}{\partial \bar{\Phi}}\right|^2 \quad \text{for} \quad \kappa = +3. \tag{30}$$

It admits the obvious analytic radial kink solution,

$$\Phi(\tau) = \exp\left(\frac{2\pi \mathrm{i} k}{3}\right)\left(\frac{1}{2} \pm \frac{1}{2}\,\tanh\,\frac{\tau}{2\sqrt{3}}\right). \tag{31}$$

What is the interpretation of this gradient flow in terms of the original Yang–Mills theory? Demanding that the torsion in (24) comes from a G_2-structure, $*\mathcal{H} = d\psi$, I am led to

$$\psi = \frac{1}{3}\kappa\,\omega \wedge d\tau + \alpha\,\mathrm{Im}\,\Omega \Longrightarrow d\psi \sim \kappa\,\mathrm{Im}\,\Omega \wedge d\tau \sim \psi \wedge d\tau, \tag{32}$$

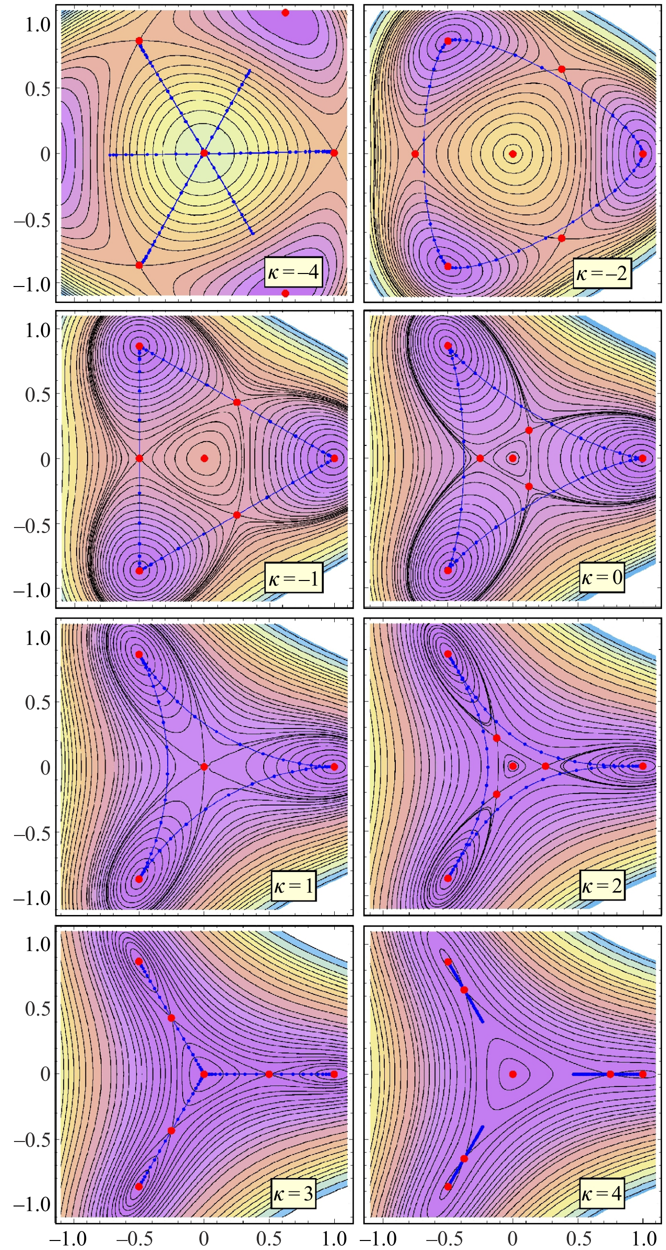


Fig. 2. Contour plots of the potential and finite-action trajectories for various κ values

where α is undetermined. This is a conformally parallel G_2-structure, and (20) quantizes the coefficients to $\alpha = 1$ and $\kappa = 3$, fixing

$$\psi = \omega\wedge d\tau + \operatorname{Im}\Omega = r^{-3}(r^2\omega\wedge dr + r^3 \operatorname{Im}\Omega) = r^{-3}\psi_{\text{cone}} \quad \text{with} \quad \mathrm{e}^{\tau} = r, \tag{33}$$

where I displayed the conformal relation to the parallel G_2-structure on the cone over K/H.

Alternatively, with this G_2-structure the 7 anti-self-duality equations (20) turn into

$$\omega\lrcorner F \sim J_{ab}\, F_{ab} = 0 \qquad \text{and} \qquad \dot{A} \sim d\omega\lrcorner F \sim e^a\, f_{abc}\, F_{bc}. \tag{34}$$

With the ansatz (26), the first relation is automatic, and the second one indeed reduces to (29). As a consistency check, one may verify that

$$\int_{K/H} \operatorname{tr}\{\omega\wedge F\wedge F\} \propto\ W(\Phi) + \frac{1}{3}. \tag{35}$$

I now come to the other instance of straight trajectories, $\kappa = -1$. For this value I find that

$$3\ddot{\Phi} = \frac{\partial V}{\partial\bar{\Phi}} \Longleftarrow \sqrt{2}\dot{\Phi} = \pm i\frac{\partial H}{\partial\bar{\Phi}} \quad \text{with} \quad H = \frac{1}{3}(\Phi^3+\bar{\Phi}^3) - |\Phi|^2, \tag{36}$$

which is a Hamiltonian flow (note the imaginary multiplier!), running along the level curves of the function H, that is identical to W. It has the obvious analytic transverse kink solution,

$$\Phi(\tau) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \left(\tanh\,\frac{\tau}{2}\right) \tag{37}$$

and its images under the tri-symmetry action.

Have I discovered another hidden G_2-structure here? Let me try the other obvious choice,

$$\widetilde{\psi} = \frac{1}{3}\tilde{\kappa}\omega\wedge d\tau + \tilde{\alpha}\operatorname{Re}\Omega \Longrightarrow d\widetilde{\psi} \sim \widetilde{\kappa}\operatorname{Im}\Omega\wedge d\tau + 2\tilde{\alpha}\,\omega\wedge\omega, \tag{38}$$

with coefficients $\tilde{\kappa}$ and $\tilde{\alpha}$ to be determined. It has not appeared in Table 1, but obeys $d * \widetilde{\psi} = 0$, which is known as a *cocalibrated* G_2-structure. But can it produce the proper torsion,

$$d\widetilde{\psi}\wedge F \sim (\widetilde{\kappa}\operatorname{Im}\Omega\wedge d\tau + 2\tilde{\alpha}\omega\wedge\omega)\wedge F \overset{!}{=} -\operatorname{Im}\Omega\wedge d\tau\wedge F? \tag{39}$$

Employing the anti-self-duality with respect to $\tilde{\psi}$,

$$*\widetilde{\psi}\wedge F = 0 \Longrightarrow \omega\wedge\omega\wedge F = 2\operatorname{Im}\Omega\wedge d\tau\wedge F, \tag{40}$$

it works out, adjusting the coefficients to $\tilde{\kappa} = 3$ and $\tilde{\alpha} = -1$. Hence, my cocalibrated G_2-structure

$$\widetilde{\psi} = \omega \wedge d\tau - \mathrm{Re}\,\Omega \qquad (41)$$

is responsible for the Hamiltonian flow. To see this directly, I import (41) into (20) and get

$$J_{ab}\,F_{ab} = 0 \quad \text{and} \quad \dot{A}_a \sim [J\,f]_{abc}\,F_{bc}. \qquad (42)$$

Again, the ansatz (26) fulfills the first relation, but the second one nicely turns into (36).

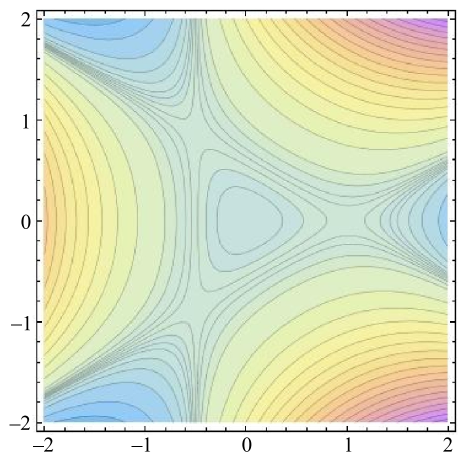


Fig. 3. Contours of the superpotential/Hamiltonian

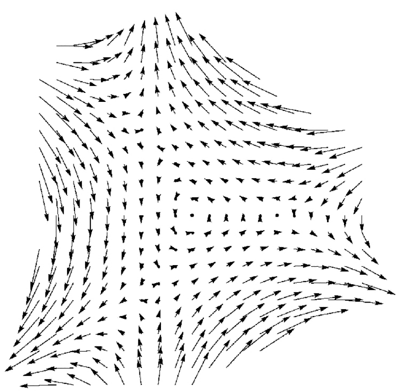


Fig. 4. Hamiltonian vector field

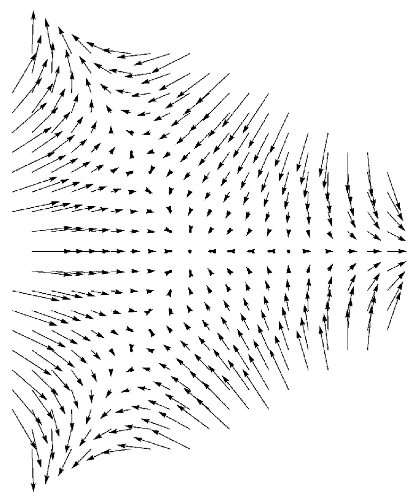


Fig. 5. Gradient vector field

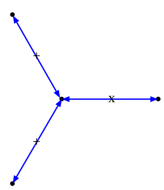
The story has an eight-dimensional twist, which can be inferred from the diagram in Sec. 1. There it is indicated that my cylinder is embedded into an 8-manifold M^8 equipped with a parallel $\mathrm{Spin}(7)$-structure Σ. It can be regarded as the cylinder over the cone over K/H. The four-form Σ descends to the cocalibrated G_2-structure $\widetilde{\psi}$, while ψ is obtained by reducing to the cone and applying a conformal transformation.

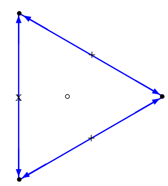
The anti-self-duality condition on M^8 represents 7 relations, which project F_8 to the 21-dimensional -1 eigenspace of $*(\Sigma\wedge\cdot)$. Contrary to the G_2 situation (34), where 7 anti-self-duality equations split to 6 flow equations and the supplementary condition $\omega\lrcorner F{=}0$, for $\mathrm{Spin}(7)$ the count precisely matches, as I have also 7 flow equations. Indeed, there is equivalence:

$$*_8 F_8 = -\Sigma\wedge F_8 \Longleftrightarrow \frac{\partial A_7(\sigma)}{\partial\sigma} = *_7(d\psi\wedge F_7(\sigma)). \tag{43}$$

5. PARTIAL SUMMARY

Let me schematically sum up the construction.





$$\Sigma\wedge F_8 \;=\; -*_8 F_8$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$\psi\wedge F = -*_7 F \qquad \text{on}\;\; \mathbb{R}\times\frac{K}{H} \qquad \widetilde{\psi}\wedge F = -*_7 F$$

$$\dot{A}_a \sim f_{abc}\,F_{bc} \qquad\qquad\qquad \dot{A}_a \sim [Jf]_{abc}\,F_{bc}$$

$$\downarrow \qquad \text{ansatz}\quad A = e^i\,I_i + e^a[\Phi\,I]_a \qquad \downarrow$$

$$\sqrt{2}\dot{\Phi} = \pm\frac{\partial W}{\partial\bar{\Phi}} \qquad\qquad\qquad \sqrt{2}\dot{\Phi} = \pm i\,\frac{\partial H}{\partial\bar{\Phi}}$$

$$\downarrow \qquad W = \frac{1}{3}(\Phi^3 + \bar{\Phi}^3) - |\Phi|^2 = H \qquad \downarrow$$

$$F(\tau) = d\tau\wedge e^a[\dot{\Phi}\,I]_a + \tfrac{1}{2}e^a\wedge e^b\big\{(|\Phi|^2{-}1)\,f^i_{ab}\,I_i + [(\bar{\Phi}^2{-}\Phi)\,f]_{abc}\,I_c\big\}$$

are G_2-instantons for Yang–Mills with torsion $D*F + (*\mathcal{H})\wedge F = 0$ from $S[A] = \int_{\mathbb{R}\times K/H} \mathrm{tr}\,\{F\wedge *F + 1/3\kappa\omega\wedge d\tau\wedge F\wedge F\}$ with $\kappa = +3$ or -1 and obey gradient/Hamiltonian flow equations for $\int_{K/H} \mathrm{tr}\,\{\omega\wedge F\wedge F\} \propto W(\Phi){+}1/3$.

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