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MODULI MATRICES OF THE VACUA AND WALLS ON $SO(2N)/U(N)$

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We construct parallel domain walls on the $SO(2N)/U(N)$ manifold by using the moduli matrices, which were originally constructed in the Grassmann manifold. We propose a method to impose a quadratic constraint to the moduli matrices. This talk is based on arXiv:1103.1490.

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We study parallel domain walls of the massive nonlinear sigma model (NLSM) on the $SO(2N)/U(N)$ manifold by using the moduli matrices [1]. Discrete vacua can be induced by a mass term. The Bogomol'nyi–Parasad–Sommerfield (BPS) solutions describe walls interpolating the vacua. We use the moduli matrices, which are the coefficients of the vacua and the BPS solutions. We show that the moduli matrices are on $SO(2N)/U(N)$. We discuss the moduli matrices of domain walls.

The massive Lagrangian with four supersymmetries can be obtained in three dimensions by dimensional reduction of the $\mathcal{N}=1$ massless NLSM [2] in four dimensions, which is Kählerian. The Lagrangian is obtained by imposing an F-term constraint to the Grassmann manifold $G_{2N,N}$. We only consider the bosonic part of the Lagrangian:

$$\mathcal{L}_{\mathrm{bos}\,3D} = -|D_m\phi_a^i|^2 - |i\phi_a^j M_j^i - i\Sigma_a^b\phi_b^i|^2 + |F_a^i|^2 + \frac{1}{2}(D_a^b\phi_b^i\bar{\phi}_i^a - D_a^a) + \\ + \Big((F_0)^{ab}\phi_b^i J_{ij}\phi_a^{Tj} + (\phi_0)^{ab}F_b^i J_{ij}\phi_a^{Tj} + (\phi_0)^{ab}\phi_b^i J_{ij}F_a^{Tj} + \mathrm{c.c.}\Big), \quad (1)$$

with

$$J = \mathbf{1}\otimes\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_j^i = \mathrm{diag}\,(m_1, m_2, \ldots, m_N)\otimes\sigma_3. \quad (2)$$

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The index m denotes three-dimensional spacetime coordinates. The indices i, j are for flavor numbers $(i, j = 1, \ldots, 2N)$, and the indices a, b are for color numbers $(a, b = 1, \ldots, N)$. J is an invariant tensor of $O(N)$, and $M_i{}^j$ is the Cartan matrix of $SO(2N)$. The components m_i $(i = 1, \ldots, N)$ are real and positive parameters with a condition $m_i > m_{i+1}$.

We consider the case that fields are static and all the fields depend only on the x_1 coordinate. We also take account of the Poincaré invariance on the two-dimensional worldvolume of walls. The BPS equation can be derived from the Bogomol'nyi completion of the Hamiltonian as

$$(D\phi)^i_a - (\phi^j_a M^i_j - \Sigma^b_a \phi^i_b) = 0. \tag{3}$$

The equation can be solved by introducing two complex matrix functions $S^b_a(x)$ and $f^i_a(x)$ defined by

$$\Sigma^b_a - iv^b_a = (S^{-1}\partial S)^b_a, \quad \phi^i_a = (S^{-1})^b_a f^i_b. \tag{4}$$

The BPS solutions to (3) are

$$\phi^i_a = (S^{-1})^b_a H^j_{0b} (\mathrm{e}^{Mx})^i_j. \tag{5}$$

Σ, v, and ϕ are invariant under the transformation

$$S'^{\,b}_a = V^c_a S^b_c, \quad H'^{\,i}_{0a} = V^c_a H^i_{0c}, \tag{6}$$

where $V \in GL(N, \mathbf{C})$. The V defines an equivalent class of the sets of the matrix functions and the moduli matrices (S, H_0). This is called the worldvolume symmetry. The D-term and F-term constraints of the Lagrangian (1) become

$$H^i_{0a} (\mathrm{e}^{2Mx})^j_i H^{\dagger\, b}_{0\,j} = (S\bar{S})^b_a \equiv \Omega^b_a, \tag{7}$$

$$H^i_{0a} J_{ij} H^{Tj}_b = 0. \tag{8}$$

Equations (6) and (8) are the definition of $SO(2N)/U(N)$. Thus the moduli matrices H_0, which parameterize the moduli space of the vacua and domain walls are on $SO(2N)/U(N)$.

For $N = 2$ case, the moduli matrices for the vacua of (1) are obtained by the relation (5) as

$$H_{0\langle 1\rangle} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, H_{0\langle 2\rangle} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

As J in (2) is invariant under $O(2N)$, the vacuum condition we get from (1) allows the vacua related to the other vacua by parity. The half of the vacua, therefore have been removed. There are two vacua, hence there exists only one wall, which is an elementary wall.

We review the properties of walls in the moduli matrix approach. In [1], walls are constructed algebraically from elementary walls in the Grassmann manifold. By definition, an elementary wall connects the two nearest vacua of the same color index changing the flavor by one unit $i \leftarrow i+1$. An elementary wall interpolating two vacua $\langle A \rangle$ and $\langle B \rangle$ in the flavor i and $i+1$ in the same color is $H_{0\langle A \leftarrow B \rangle} = H_{0\langle A \rangle}\, \mathrm{e}^{a_i(r)}$, where $a_i(r) \equiv \mathrm{e}^r a_i (r \in \mathbf{C})$. The a_i of an elementary wall carrying tension $T_{\langle A \leftarrow B \rangle}$ is defined by

$$[cM, a_i] = c(m_i - m_{i+1})a_i = T_{\langle i \leftarrow i+1 \rangle} a_i, \tag{10}$$

where c is a constant; M is the mass matrix, and a_i is an $N_f \times N_f$ square matrix generating an elementary wall. N_f is the number of the flavors. From the first equality the mass matrix M and the matrix a_i can be interpreted as a Cartan generator and a step operator, respectively. The a_i has a nonzero component only in the $(i, i+1)$th element, which is equal to a unit.

In the $SO(2N)/U(N)$, the elementary walls changing the flavor by one unit for the same color cannot be defined consistently with (8) which stems from the F-term constraint as it can also be seen in (9). As our interest is the algebras of a_i, which are chart-independent, we construct the moduli matrices of the vacua, which keep the $SO(2N)$ isometry of the domain walls rather than defining the commutator for the algebras on the $SO(2N)/U(N)$ manifold.

Equations (6) state that H_0's are homogeneous coordinates on the Grassmann manifold. Equation (8), which stems from the F-term constraint, is a holomorphic embedding of $SO(2N)$. H_0's are therefore on the submanifold of Grassmann manifold $G_{2N,N}$ as an algebraic variety.

The moduli matrices of the vacua and walls can be obtained as follows. We enlarge the $G_{2N,N}$ by using the worldvolume symmetry (6) to embed the constraint (8) completely but only once. Then we can construct moduli matrices on the surface defined by (8). To do this, we introduce an additional element with an opposite sign in step operators a_i. We also transform the moduli matrices of the vacua, which are obtained from the vacuum condition of the Lagrangian, to the surface (8) by the worldvolume transformation (6).

As an example, we construct moduli matrices on $SO(4)/U(2)$ from (9). The wall interpolating $H_{0\langle 1 \rangle}$ and $H_{0\langle 2 \rangle}$ is an elementary wall $H_{0\langle 1 \leftarrow 2 \rangle} = H_{0\langle 1 \rangle}\, \mathrm{e}^{a(r)}$. The $a(r)$ is constrained by (8) as

$$Ja(r)^T + a(r)Ja(r)^T + a(r)J = 0. \tag{11}$$

Since the second and the fourth columns of $H_{0\langle 1 \rangle}$ are zero, the second and the fourth rows of $a(r)$ can be set to be zero. Then the $a(r)$ is uniquely determined by (11) up to an overall sign. By using the relation $a(r) = \mathrm{e}^r a$, the operator a is

$$a = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

By definition of an elemenatary wall $H_{0\langle 1\leftarrow 2\rangle} = H_{0\langle 1\rangle}\,\mathrm{e}^{a(r)}$, the other vacua are determined as

$$H'_{0\langle 2\rangle} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{13}$$

$H'_{0\langle 2\rangle}$ is related to $H_{0\langle 2\rangle}$ in (9) by the worldvolume transformation (6). $H_{0\langle 1\rangle}$ and $H'_{0\langle 2\rangle}$ are the moduli matrices of the vacua, and the matrix a is the operator generating an elementary wall interpolating them. The moduli matrices and the operators generating domain walls on the $SO(6)/U(3)$ manifold are discussed in [3].

We have used the fact that the moduli space parameterizing the $SO(2N)/U(N)$ manifold is a submanifold of $G_{2N,N}$ as an algebraic variety. We add a brief comment partially repeating what is mentioned in Sec. 4 of [2]. Unlike $\mathbf{R}^n$, algebraic varieties are restricted in $\mathbf{C}^n$. A holomorphic embedding of a compact manifold to $\mathbf{C}^n$ in general cannot be defined except for a point*. Thus one of standard approaches to define compact manifolds is a holomorphic embedding to projective spaces. The compact Hermitian symmetric spaces consist of four classical types

$$\begin{gathered} G_{N+M,M} = \frac{U(N+M)}{U(N)\times U(M)},\ \frac{SO(2N)}{U(N)},\ \frac{Sp(N)}{U(N)}, \\ Q^N = \frac{SO(N+2)}{SO(N)\times SO(2)}, \end{gathered} \tag{14}$$

and two exceptional types

$$\frac{E_6}{SO(10)\times U(1)},\ \frac{E_7}{E_6\times U(1)}. \tag{15}$$

In supersymmetric gauge theory, Q^N and $SO(2N)/U(N), Sp(N)/U(N)$ are holomorphically embedded in $\mathbf{C}P^{N+1}$ and $G_{2N,N}$, respectively. $E_6/(SO(10)\times U(1))$ and $E_7/(E_6\times U(1))$ are holomorphically embedded in $\mathbf{C}P^{26}$ and $\mathbf{C}P^{55}$, respectively [2].

*Whitney embedding theorem; Maximum modulus theorem; Liouville's theorem.

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