ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2012. Т. 43. ВЫП. 5

INTEGRABLE STRING MODELS OF WZNW MODEL TYPE WITH CONSTANT $SU(2)$, $SO(3)$, $SP(2)$, AND $SU(3)$ TORSION AND HYDRODYNAMIC CHAINS

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The integrability of string model of WZNW model type with constant $SU(2)$, $SO(3)$, $SP(2)$ torsion is investigated. The closed boson string model in the background gravity and antisymmetric B-field is considered as integrable system in terms of initial chiral currents. The model is considered under assumption that internal torsion related with metric of Riemann–Cartan space and external torsion related with antisymmetric B-field (anti)coincide. New equation of motion and exact solution of this equation were obtained for string model with constant $SU(2)$, $SO(3)$, $SP(2)$ torsion. New equations of motion and new Poisson brackets (PB) for infinite dimensional hydrodynamic chains were obtained for string model with constant $SU(n)$, $SO(n)$, $SP(n)$ torsion for $n \to \infty$.

PACS: 02.20.Sv; 11.30.Rd; 11.40.-q; 21.60.Fw

INTRODUCTION

The closed string model in the background gravity $g_{ab}(\phi)$ and antisymmetric fields $B_{ab}(\phi)$ in the conformal $g_{\alpha\beta} = \mathrm{e}^{\phi}\,\eta_{\alpha\beta}$ and light-cone gauge is describe by the Lagrangian

$$L = \frac{1}{2}\int\limits_0^{2\pi}\left[\sqrt{-g}g^{\alpha\beta}g_{ab}(\phi)\frac{\partial\phi^a}{\partial x^\alpha}\frac{\partial\phi^b}{\partial x^\beta} + \epsilon^{\alpha\beta}B_{ab}\frac{\partial\phi^a}{\partial x^\alpha}\frac{\partial\phi^b}{\partial x^\beta}\right]dx.$$

Here $g_{ab}(\phi(t,x))$ is the metric tensor of curve n-dimensional space $\phi^a(x+2\pi) = \phi^a(x)$, $(a,b = 1,2,\ldots,n)$, $g_{ab}(\phi) = g_{ba}(\phi)$, $B_{ab}(\phi) = -B_{ba}(\phi)$, $g^{\mu\nu}$ is the metric tensor of flat space, tangent space to curve space in point $\phi(t,x)$ and $\mu,\nu = 1,2,\ldots,n$. Both metrics can have the arbitrary signature. $g_{\alpha\beta}(t,x)$ — metric tensor of curve 2D space, $(\alpha,\beta = 0,1)$. In the repers formalism $g_{ab}(\phi) = e^\mu_a(\phi)e^\nu_b(\phi)g_{\mu\nu}$. In the conformal gauge $g_{\alpha\beta} = \mathrm{e}^{\theta(t,x)}\,\eta_{\alpha\beta}$ Lagrangian does not

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depend on the field $\theta(t,x)$. Let us introduce new variables to obtain the first-order equation instead of the second-order one

$$J_0^\mu(\phi)=e_\mu^a(\phi)[p_a-B_{ab}(\phi)\phi^{'b}],\quad J_1^\mu(\phi)=e_a^\mu\phi^{'a}.$$

Here canonical momentum is

$$p_a(t,x)=g_{ab}(\phi)\dot{\phi}^b+B_{ab}\phi^{'b},\quad \dot{\phi}^a=\frac{\partial\phi^a}{\partial t},\quad \phi^{'a}=\frac{\partial\phi^a}{\partial x}.$$

Equations of motion in new variables are

$$\partial_0J_1^\mu-\partial_1J_0^\mu=C_{\nu\lambda}^\mu(\phi)J_0^\nu J_1^\lambda,\quad \partial_0J_0^\mu-\partial_1J_1^\mu=-H_{\nu\lambda}^\mu(\phi)J_0^\nu J_1^\lambda. \tag{1}$$

Here $C^{\mu\nu\lambda}$ is the torsion:

$$C_{\nu\lambda}^\mu=\frac{\partial e_a^\mu}{\partial x^b}(e_\nu^b e_\lambda^a-e_\nu^a e_\lambda^b),\quad H_{abc}=\frac{\partial B_{ab}}{\partial\phi^c}+\frac{\partial B_{ca}}{\partial\phi^b}+\frac{\partial B_{bc}}{\partial\phi^a}. \tag{2}$$

The function H_{abc} is total antisymmetric function on a, b, c. Let us consider commutation relation function J_α^μ, $\alpha=0,1$ on the phase space on the PB:

$$\begin{aligned}
&\{J_0^\mu(x),J_0^\nu(y)\}=C_\lambda^{\mu\nu}(\phi)J_0^\lambda(x)\delta(x-y)+H_\lambda^{\mu\nu}(\phi)J_1^\lambda(x)\delta(x-y),\\
&\{J_0^\mu(x),J_1^\nu(y)\}=C_\lambda^{\mu\nu}(\phi)J_1^\lambda(x)\delta(x-y)+g^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),\\
&\{J_1^\mu(x),J_1^\nu(y)\}=0.
\end{aligned}$$

Let us introduce chiral variables:

$$U^\mu=\delta^{\mu\nu}J_{0\nu}+J_1^\mu,\quad V^\mu=\delta^{\mu\nu}J_{0\nu}-J_1^\mu.$$

The chiral variables satisfy the following relations under PB [1]:

$$\begin{aligned}
\{U^\mu(x),U^\nu(y)\}&=\frac{1}{2}[(3C_\lambda^{\mu\nu}+H_\lambda^{\mu\nu})U^\lambda-(C_\lambda^{\mu\nu}+H_\lambda^{\mu\nu})V^\lambda]\times\\
&\qquad\times\delta(x-y)+\delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),\\
\{V^\mu(x),V^\nu(y)\}&=\frac{1}{2}[(3C_\lambda^{\mu\nu}-H_\lambda^{\mu\nu})V^\lambda-(C_\lambda^{\mu\nu}-H_\lambda^{\mu\nu})U^\lambda]\times\\
&\qquad\times\delta(x-y)-\delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),\\
\{U^\mu(x),V^\nu(y)\}&=\frac{1}{2}[(C_\lambda^{\mu\nu}+H_\lambda^{\mu\nu})U^\lambda++(C_\lambda^{\mu\nu}-H_\lambda^{\mu\nu})V^\lambda]\delta(x-y).
\end{aligned}$$

Here function $H_{\mu\nu\lambda}(\phi)$ is additional external torsion. These PBs form algebra if: 1) $C_\lambda^{\mu\nu}=0$, $H_\lambda^{\mu\nu}=0$ and functions $U^\mu(x)$ are Abelian currents; 2) $C_\lambda^{\mu\nu}$, $H_\lambda^{\mu\nu}$

are structure constants $f_{\lambda}^{\mu\nu}$ of Lie algebra, and the functions $U^{\mu}(x)$ are chiral currents. Here are two possibilities to simplify this algebra:

$$\begin{aligned}
&1)\ H_{\lambda}^{\mu\nu} = -C_{\lambda}^{\mu\nu},\\
&\quad\{V^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu}(2V^{\lambda} - U^{\lambda})\delta(x-y) - \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),\\
&\quad\{U^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu}V^{\lambda}\delta(x-y),\\
&\quad\{U^{\mu}(x), U^{\nu}(y)\} = C_{\lambda}^{\mu\nu}U^{\lambda}\delta(x-y) + \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),\\
&2)\ H_{\lambda}^{\mu\nu} = C_{\lambda}^{\mu\nu},\\
&\quad\{U^{\mu}(x), U^{\nu}(y)\} = C_{\lambda}^{\mu\nu}(2U^{\lambda} - V^{\lambda})\delta(x-y) + \delta^{ab}\frac{\partial}{\partial x}\delta(x-y),\\
&\quad\{V^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu}V^{\lambda} - \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),\\
&\quad\{U^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu}U^{\lambda}\delta(x-y).
\end{aligned} \tag{3}$$

The chiral currents U^{μ} in the first case and V^{μ} in the second case form Kac–Moody algebras. The chiral currents $U^{\mu}(x)$ are generators of translation on the curve space

$$\delta\phi^{a}(x) = \{\phi^{a}(x), c^{\mu}U_{\mu}(x)\} = c^{\mu}e_{\mu}^{a}(\phi) = c^{a}(\phi).$$

Simultaneously, they are generators of group transformations with structure constant $C_{\lambda}^{\mu\nu}$ in the tangent space.

1. INTEGRABLE STRING MODEL WITH CONSTANT TORSION

To construct integrable system we must have hierarchy of Hamiltonians and find hierarchy of PB brackets. This way is more simple if the dynamical system has some group structure. Let torsion $C_{bc}^{a}(\phi)) \neq 0$ and C_{abc} are structure constants of the Lie algebra. In bi-Hamiltonian approach to integrable string models with constant torsion [2–5] we have considered the conserved primitive chiral currents $C_n(U(x))$, as local fields of the Riemann manifold. The primitive and nonprimitive local charges of invariant chiral currents form the hierarchy of new Hamiltonians. The primitive invariant currents are densities of Casimir operators. The nonprimitive currents are functions of primitive currents. Commutation relations show that currents U^{μ} form closed algebra. Therefore, we will consider PB of right chiral currents U^{μ} and Hamiltonians constructed only of right currents. The constant torsion does not contribute to equation of motion, but it gives possibility to introduce group structure and to introduce symmetric structure constant.

Let t_a are the generators $SU(n)$, $SO(n)$, $SP(n)$ Lie algebras:

$$[t_\mu, t_\nu] = 2if_{\mu\nu\lambda}t_\lambda, \quad \mu = 1, \dots, n^2 - 1. \tag{4}$$

There is additional relation for generators Lie algebra in defining matrix representation. There is the following relation for symmetric double product generators of $SU(n)$ algebra:

$$\{t_\mu, t_\nu\} = \frac{4}{n}\delta_{\mu\nu} + 2d_{\mu\nu\lambda}t_\lambda. \tag{5}$$

Here $d_{\mu\nu\lambda}$ is the total symmetric structure constant tensor. The similar relation for total symmetric triple product of $SO(n)$ and $SP(n)$ algebras has form:

$$t_{(\mu}t_\nu t_{\lambda)} = v^{\rho}_{\mu\nu\lambda}\, t_\rho. \tag{6}$$

Here $v_{\rho\mu\nu\lambda}$ is the total symmetric structure constant tensor. The invariant chiral currents for $SU(n)$ group can be constructed as a product of invariant symmetric tensors and initial chiral currents U^μ:

$$\begin{aligned} d_{(\mu_1\cdots\mu_n)} &= d^{k_1}_{(\mu_1\mu_2}d^{k_2}_{\mu_3 k_1}\cdots d^{k_{n-3}}_{\mu_{m-1}\mu_M)}, \quad d_{\mu_1\mu_2} = \delta_{\mu_1\mu_2}, \\ C_n(U(x)) &= d_{(\mu_1\cdots\mu_n)}U_{\mu_1}U_{\mu_2}\cdots U_{\mu_n}, \quad C_2 = \delta_{\mu\nu}U^\mu U^\nu. \end{aligned} \tag{7}$$

The invariant chiral currents for $SO(n)$, $SP(n)$ group can be constructed as a product of invariant symmetric constant tensor and initial chiral currents:

$$\begin{aligned} v_{(\mu_1\cdots\mu_{2n})} &= v^{\nu_1}_{(\mu_1\mu_2\mu_3}v^{\nu_1\nu_2}_{\mu_4\mu_5}\cdots v^{\nu_{2n-3}\mu_{2n-2}\mu_{2n-1}\mu_{2n})}, \quad v_{\mu_1\mu_2} = \delta_{\mu_1\mu_2}, \\ C_{2n} &= v_{\mu 1\cdots\mu_{2n}}U^{\mu_1}\cdots U^{\mu_{2n}}, \quad C_2 = \delta_{\mu_1\mu_2}U^{\mu_1}U^{\mu_2}. \end{aligned} \tag{8}$$

The invariant chiral currents for $SU(2)$, $SO(3)$, $SP(2)$ have the form: $C_{2n} = (C_2)^n$. Another family of the invariant symmetric currents J_n, based on the invariant symmetric chiral currents on simple Lie groups, is realized as symmetric trace of n-product chiral currents $U(x) = t_\mu U^\mu$, $\mu = 1, \dots, n^2 - 1$:

$$J_n = \mathrm{Sym}\,\mathrm{Tr}\,(U \dots U). \tag{9}$$

These invariant currents are polynomials of product basic chiral currents $C_k,\ k = 2, 3, \dots, k$. The commutation relations for chiral curents have the form:

$$\{C_m(x), C_n(y)\} = W_{mn}(y)\frac{\partial}{\partial y}\delta(y - x) - W_{nm}(x)\frac{\partial}{\partial x}\delta(x - y).$$

Hamiltonian function $W_{mn}(x)$ for finite dimensional $SU(n)$, $SO(n)$, $SP(n)$ group has the form:

$$W_{mn}(x) = \frac{n-1}{m+n-2}\sum_k a_k C_{m+n-2,k}(x), \quad \sum_{k=0} a_k = mn. \tag{10}$$

Here the invariant total symmetric currents $C_{n,k},\ k = 1, 2\ldots$ are new currents, which are polynomials of product basic invariant currents $C_{n_1}C_{n_2}\cdots C_{n_n}$, $n_1 + \ldots + n_n = n$. They can be obtained by the calculation of the total symmetric currents J_n using the different replacements of the double product (5) for the $SU(n)$ group and of the triple product (6) for the $SO(n)$, $SP(n)$ groups in the expressions for the invariant currents J_n. This PB can be rewritten as PB of hydrodynamic type

$$\{C_m(x), C_n(y)\} = -\frac{n-1}{m+n-2}\frac{\partial}{\partial x}\sum_k C_{m+n-2,k}(x)\delta(x-y) - \\ -\sum_k C_{m+n-2,k}(x)\frac{\partial}{\partial x}\delta(x-y).$$

Here are only $l = n-1$ primitive invariant tensors for $SU(n)$ algebra, $l = (n-1)/2$ for $SO(n)$ algebra and $l = n/2$ for $SP(n)$ algebra. Higher invariant currents C_n for $n \geqslant l+1$ are nonprimitive currents and they are polynomials of primitive currents. The corresponding nonprimitive chiral currents — the charges — are not Casimir operators. The expressions for these polynomials [4] are obtained from the generating function

$$\det(1-\lambda t_\mu U^\mu) = \exp\left[\mathrm{Tr}\left(\ln(1-\lambda U)\right)\right] = \exp\left(-\sum_{n=2}^{\infty}\frac{\lambda^n}{n}J_n\right).$$

2. EQUATION OF MOTION FOR $SU(2)$, $SO(3)$, $SP(2)$ TORSION

Here is one primitive invariant tensor on $SU(2)$, $SO(3)$, $SP(2)$ algebras. The invariant nonprimitive tensors for $n \geqslant 2$ are functions of primitive tensor. Let us introduce the local chiral currents based on the invariant symmetric polynomials on $SU(2)$, $SO(3)$, $SP(2)$ Lie group:

$$C_2(U) = \delta_{ab}U^aU^b, \quad C_{2n}(U) = (\delta_{ab}U^aU^b)^n, \quad n = 1, 2, \ldots$$

$$\{C_2(x), C_2(y)\} = 2[C_2(y)\partial_y\delta(y-x) - C_2(x)\partial_x\delta(x-y)].$$

We will consider $C_2(x)$ as a local field on the Riemann space of chiral currents. As Hamiltonians we choose functions

$$H_n = \frac{1}{2(n+1)}\int_0^{2\pi} C_2^{n+1}(y)\,dy. \tag{11}$$

The equation of motion for density of the first Casimir operator has the form:

$$\frac{\partial C_2}{\partial t_n} + (2n+1)(C_2)^n\frac{C_2}{\partial x} = 0. \tag{12}$$

The equation for currents C_2^n is the following:

$$\frac{\partial C_2^n}{\partial \tau_n} + (C_2)^n \frac{C_2^n}{\partial x} = 0, \quad \tau_n = (2n+1)t_n.$$

This equation is the inviscid Burgers equation. The solution of the last equation is defined by the Lambert function:

$$C_2^n = \frac{1}{i\tau_n} W(i\tau_n \,\mathrm{e}^{a+ix}).$$

Consequently solution for the first Casimir operator is

$$C_2(t_n, x) = \left[\frac{1}{i(2n+1)t_n} W(i(2n+1)t_n \,\mathrm{e}^{a+ix})\right]^{1/n}. \tag{13}$$

The equation of motion for initial chiral current U^μ is defined by PB (3) and Hamiltonian (11):

$$\frac{\partial U^\mu}{\partial t_n} = \partial_x [U^\mu (UU)^n], \quad \mu = 1, 2, 3. \tag{14}$$

It is possible to rewrite this equation as a linear nonhomogeneous equation using solution (13), which diagonalizes Eq. (12):

$$\frac{\partial z^\mu}{\partial t_n} = \partial_x z^\mu + \partial_x f(t_n, x), \quad z^\mu = \ln U^\mu, \quad f = C_2^n.$$

3. EQUATION OF MOTION FOR $SU(3)$ TORSION

The invariant chiral currents $C_2(U)$, $C_3(U)$ form closed system. The non-primitive currents have the form:

$$C_{2n} = C_2^n, \quad C_{2n+1} = C_2^{n-1} C_3.$$

The algebra of corresponding charges is not Abelian, but charges C_{2n} form invariant subalgebra. The currents C_2 and C_3 are local coordinates on the Riemann space, and invariant currents C_{2n} are densities of Hamiltonians. Equation of motion for C_3 is the following:

$$\frac{\partial C_3(x)}{\partial t_n} = -2C_2^n \partial_x C_3 - 6C_3 \partial_x C_2^n.$$

In terms of variables $g = \ln C_3$, $f = C_2^n$ it is a linear equation

$$\frac{\partial g}{\partial_n} + 2f\partial_x g + 6\partial_x f = 0.$$

4. INFINITE DIMENSIONAL HYDRODYNAMIC CHAIN

In the case, if dimension of matrix representation n is not ended, all the chiral currents are primitive currents. The algebra of PB for chiral currents has the form:

$$\{C_m(x), C_n(y)\} = W_{mn}(y)\frac{\partial}{\partial y}\delta(y-x) - W_{nm}(x)\frac{\partial}{\partial x}\delta(x-y),$$
$$W_{mn}(x) = \frac{mn(n-1)}{m+n-2}C_{m+n-2}(x).$$

This PB satisfies the skew-symmetric condition and Jacobi identity impose conditions on the Hamiltonian function $W_{mn}(x)$:

$$(W_{kp} + W_{pk})\frac{\partial}{\partial C_k}W_{mn} = (W_{km} + W_{mk})\frac{\partial}{\partial C_k}W_{pn},$$
$$\frac{\partial}{\partial x}W_{kp}\frac{\partial}{\partial C_k}W_{nm} = \frac{\partial}{\partial x}W_{km}\frac{\partial}{\partial C_k}W_{np}.$$

The Jacobi identity satisfies metric tensor $W_{mn}(U)$ for $m = p$ from compatibility condition Kronekers $\delta_{m+n-2,k}$ and $\delta_{p+n-2,k}$. This PB can be rewritten as PB of hydrodynamic type and describe the hydrodynamic chain (see [6,7] and references therein):

$$\{C_m(x), C_n(y)\} = -\frac{mn(n-1)}{m+n-2}\frac{\partial}{\partial x}C_{m+n-2}(x)\delta(x-y) - \\ - mnC_{m+n-2}(x)\frac{\partial}{\partial x}\delta(x-y).$$

The algebra of charges $\int_0^{2\pi} C_n(x)dx$ is Abelian algebra. Let us choose as Hamiltonians the Casimir operators C_n:

$$H_n = \frac{1}{n}\int_0^{2\pi} C_n(x)dx, \quad n = 2, 3\ldots \tag{15}$$

The equations of motion for densities of Casimir operators are the following:

$$\frac{\partial C_m(x)}{\partial t_n} = \frac{1}{n}\int_0^{2\pi} W_{mn}(y)\,\partial_y\delta(y-x)\,dy - \\ - \frac{1}{n}\int_0^{2\pi} W_{nm}(x)\,\partial_x\delta(x-y)\,dy = \frac{m(n-1)}{m+n-2}\partial_x C_{m+n-2}. \tag{16}$$

We can construct equations of motion for initial chiral currents U^μ using flat PB (3) and Hamiltonians H_n (11), where $C_n(x)$ are defined by (7) for $SU(\infty)$ group:

$$\frac{\partial U^\mu(x)}{\partial t_n} = \frac{1}{n}\int_0^{2\pi} dy\{U^\mu(x), C_n(y)\}_0,$$
$$\frac{\partial U_{\underline{\mu}}(x)}{\partial t_n} = \partial_x(d^{k_1}_{\mu_1\mu_2} d^{k_2}_{k_1\mu_3}\cdots d^{k_{n-3}}_{\mu_{n-1}\underline{\mu}} U^{\mu_1}(x)\cdots U^{\mu_{n-1}}(x)). \tag{17}$$

As an example we consider $n = 3$:

$$\frac{\partial U_\mu}{\partial t_3} = \partial_x(d_{\mu\nu\lambda}U^\nu U^\lambda).$$

This system is bi-Hamiltonian with Hamiltonian C_2 and metric tensor $W_{\mu\nu} = d_{\mu\nu\lambda}U^\lambda$. The Jacobi identity is satisfied by the equation

$$d_{\mu\nu\lambda}d_{\rho\sigma\phi} + d_{\mu\nu\sigma}d_{\mu\lambda\phi} + d_{\mu\nu\phi}d_{\mu\lambda\sigma} = 0$$

for $n \to \infty$. By similar manner, we can obtain equation of motion for chiral currents of $SO(n)$, $SP(n)$:

$$\frac{\partial U_{\underline{\mu}}(x)}{\partial t_n} = \partial_x(v^{k_1}_{\mu_1\mu_2\mu_3}\cdots v^{k_{2n-3}}_{\mu_{2n-2}\mu_{2n-1}\underline{\mu}} U^{\mu_1}\cdots U^{\mu_{2n-1}}). \tag{18}$$

As an example we consider $n = 4$:

$$\frac{\partial U_\mu}{\partial t_4} = \partial_x(v_{\mu\nu\lambda\rho}U^\nu U^\lambda U^\rho).$$

CONCLUSIONS

We obtained hydrodynamic PBs and hydrodynamic equations for invariant chiral currents and initial canonical currents of string model in the background gravity and antisymmetric fields. We obtained bi-Hamiltonian system in suggestion, that internal torsion, related with metric of string coordinates, and external torsion, related with antisymmetric fields, (anti)coincide. The equations of motion are obtained for string with constant $SU(2)$, $SO(3)$, $SP(n)$ torsion and for infinite hydrodynamic chains.

Acknowledgements. This work was supported by the Joint DFFD–RFBR Grant No. F40.2/040.

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