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EXTREMAL VECTORS FOR VERMA-TYPE FACTOR-REPRESENTATIONS OF $U_q(sl(3,\mathbb{C}))$

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To analyze the reducibility of the Verma modules one often needs to find the extremal vectors of the given representations. On the example of algebra $U_q(sl(3,\mathbb{C}))$ we study how the set of extremal vectors is affected when we factorize the original representation and give their explicit formula.

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INTRODUCTION

The importance of finding all extremal vectors for reducibility analysis is well known (see, e.g. [1–4]). We are interested in the case of factor-representations of the algebra $U_q(sl(3,\mathbb{C}))$. Let q be not root of unity. We start with the algebra $U_q\big(gl(3)\big)$. This algebra is generated by the elements $\mathbf{L}_i = q^{\mathbf{E}_{ii}}$, $i = 1, 2, 3$, $\mathbf{L}_i^{-1}$, $\mathbf{E}_{12}$, $\mathbf{E}_{23}$, $\mathbf{E}_{21}$, and $\mathbf{E}_{32}$ which fulfill the relations

$$\mathbf{L}_i\mathbf{L}_j = \mathbf{L}_j\mathbf{L}_i,$$
$$\mathbf{L}_i\mathbf{E}_{jk} = q^{\delta_{ij}-\delta_{ik}}\mathbf{E}_{jk}\mathbf{L}_i,$$
$$\big[\mathbf{E}_{i,i+1},\mathbf{E}_{j+1,j}\big] = \frac{\mathbf{L}_i\mathbf{L}_{i+1}^{-1}-\mathbf{L}_i^{-1}\mathbf{L}_{i+1}}{q-q^{-1}}\,\delta_{ij},$$

$$\mathbf{E}_{12}\mathbf{E}_{23}^2-(q+q^{-1})\mathbf{E}_{23}\mathbf{E}_{12}\mathbf{E}_{23}+\mathbf{E}_{23}^2\mathbf{E}_{12}=0,$$
$$\mathbf{E}_{23}\mathbf{E}_{12}^2-(q+q^{-1})\mathbf{E}_{12}\mathbf{E}_{23}\mathbf{E}_{12}+\mathbf{E}_{12}^2\mathbf{E}_{23}=0,$$
$$\mathbf{E}_{21}\mathbf{E}_{32}^2-(q+q^{-1})\mathbf{E}_{32}\mathbf{E}_{21}\mathbf{E}_{32}+\mathbf{E}_{32}^2\mathbf{E}_{21}=0,$$
$$\mathbf{E}_{32}\mathbf{E}_{21}^2-(q+q^{-1})\mathbf{E}_{21}\mathbf{E}_{32}\mathbf{E}_{21}+\mathbf{E}_{21}^2\mathbf{E}_{32}=0.$$

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It is a Chevalley basis for this algebra. We will need a Cartan–Weyl basis, so we define

$$\mathbf{E}_{13} = \mathbf{E}_{12}\mathbf{E}_{23} - q^{-1}\mathbf{E}_{23}\mathbf{E}_{12}, \quad \mathbf{E}_{31} = \mathbf{E}_{32}\mathbf{E}_{21} - q\mathbf{E}_{21}\mathbf{E}_{32}.$$

Now we denote

$$|n_1, n_2\rangle = \mathbf{E}_{32}^{n_1}\mathbf{E}_{31}^{n_2}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The algebra $U_q\big(sl(3)\big)$ is then defined by the following relations:

$$\mathbf{E}_{32}|n_1, n_2\rangle = |n_1 + 1, n_2\rangle,$$

$$\mathbf{E}_{31}|n_1, n_2\rangle = q^{n_1}|n_1, n_2 + 1\rangle,$$

$$\mathbf{E}_{21}|n_1, n_2\rangle = -q^{-1}[n_1]_q|n_1 - 1, n_2 + 1\rangle + q^{-n_1+n_2}|n_1, n_2\rangle\mathbf{E}_{21},$$

$$\mathbf{K}_1|n_1, n_2\rangle = q^{n_1-n_2}|n_1, n_2\rangle\mathbf{K}_1,$$

$$\mathbf{K}_2|n_1, n_2\rangle = q^{-2n_1-n_2}|n_1, n_2\rangle\mathbf{K}_2,$$

$$\mathbf{E}_{12}|n_1, n_2\rangle = -q[n_2]_q|n_1 + 1, n_2 - 1\rangle\mathbf{K}_1 + |n_1, n_2\rangle\mathbf{E}_{12},$$

$$\mathbf{E}_{23}|n_1, n_2\rangle = \frac{q^{-n_1-n_2+1}[n_1]_q}{q - q^{-1}}\,|n_1 - 1, n_2\rangle\mathbf{K}_2 - \\ - \frac{q^{n_1+n_2-1}[n_1]_q}{q - q^{-1}}\,|n_1 - 1, n_2\rangle\mathbf{K}_2^{-1} + q^{n_2-1}[n_2]_q|n_1, n_2 - 1\rangle\mathbf{E}_{21}\mathbf{K}_2^{-1},$$

$$\mathbf{E}_{13}|n_1, n_2\rangle = \frac{q^{-2n_1-n_2+1}[n_2]_q}{q - q^{-1}}\,|n_1, n_2 - 1\rangle\mathbf{K}_1\mathbf{K}_2 - \\ - \frac{q^{n_2-1}[n_2]_q}{q - q^{-1}}\,|n_1, n_2 - 1\rangle\mathbf{K}_1^{-1}\mathbf{K}_2^{-1} + q^{-n_1-n_2}[n_1]_q|n_1 - 1, n_2\rangle\mathbf{E}_{12}\mathbf{K}_2,$$

where $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{E}_{12}$ a $\mathbf{E}_{21}$ form a subalgebra $U_q\big(gl(2)\big)$ with Casimir operator $\mathbf{K}_1\mathbf{K}_2^2 = \mathbf{L}_1\mathbf{L}_2\mathbf{L}_3^{-2}$. One of the representations of this subalgebra we get putting $|n_3\rangle = \mathbf{E}_{21}^{n_3}$. Now we factorize this taking $\mathbf{E}_{12} \mapsto 0$, $\mathbf{K}_1 \mapsto q^{\lambda_1}$ and $\mathbf{K}_2 \mapsto q^{\lambda_2}$. We get the following representation:

$$\mathbf{E}_{21}|n_3\rangle = |n_3 + 1\rangle,$$

$$\mathbf{K}_1|n_3\rangle = q^{\lambda_1-2n_3}|n_3\rangle, \quad \mathbf{K}_2|n_3\rangle = q^{\lambda_2+n_3}|n_3\rangle,$$

$$\mathbf{E}_{12}|n_3\rangle = [n_3]_q[\lambda_1 - n_3 + 1]_q|n_3 - 1\rangle,$$

where λ_1, $\lambda_2 \in \mathbb{C}$ a $n_3 \in \mathbb{N}_0$.

Combining with previous relations we finally get the Verma module of $U_q\big(sl(3)\big)$ which is given by the following relations:

$$\begin{aligned}
\mathbf{E}_{32}|n_1,n_2,n_3\rangle &= |n_1+1,n_2,n_3\rangle,\\
\mathbf{E}_{31}|n_1,n_2,n_3\rangle &= q^{n_1}|n_1,n_2+1,n_3\rangle,\\
\mathbf{E}_{21}|n_1,n_2,n_3\rangle &= q^{-n_1+n_2}|n_1,n_2,n_3+1\rangle - q^{-1}[n_1]_q|n_1-1,n_2+1,n_3\rangle,\\
\mathbf{K}_1|n_1,n_2,n_3\rangle &= q^{\lambda_1+n_1-n_2-2n_3}|n_1,n_2,n_3\rangle,\\
\mathbf{K}_2|n_1,n_2,n_3\rangle &= q^{\lambda_2-2n_1-n_2+n_3}|n_1,n_2,n_3\rangle,\\
\mathbf{E}_{12}|n_1,n_2,n_3\rangle &= [n_3]_q[\lambda_1-n_3+1]_q|n_1,n_2,n_3-1\rangle-\\
&\qquad - q^{\lambda_1-2n_3+1}[n_2]_q|n_1+1,n_2-1,n_3\rangle,\\
\mathbf{E}_{23}|n_1,n_2,n_3\rangle &= [n_1]_q[\lambda_2-n_1-n_2+n_3+1]_q|n_1-1,n_2,n_3\rangle+\\
&\qquad + q^{-\lambda_2+n_2-n_3-1}[n_2]_q|n_1,n_2-1,n_3+1\rangle,\\
\mathbf{E}_{13}|n_1,n_2,n_3\rangle &= q^{-n_1}[n_2]_q[\lambda_1+\lambda_2-n_1-n_2-n_3+1]_q|n_1,n_2-1,n_3\rangle+\\
&\qquad + q^{\lambda_1-n_1-n_2+n_3}[n_1]_q[n_3]_q[\lambda_1-n_3+1]_q|n_1-1,n_2,n_3-1\rangle,
\end{aligned}$$

where λ_1, $\lambda_2 \in \mathbb{C}$ a n_1, n_2, $n_3 \in \mathbb{N}_0$. For $\lambda_1 = J \in \mathbb{N}_0$ we have

$$J \in \mathbb{N}_0, \quad \lambda_2 \in \mathbb{C}, \quad n_1,\, n_2 \in \mathbb{N}_0, \quad n_3 = 0,\, 1,\, \ldots,\, J.$$

1. THE EXTREMAL VECTORS

Now we are interested in the set of extremal vectors. We have three positive roots of $sl(3)$, $\alpha_1 = (2,-1)$, $\alpha_2 = (-1,2)$ and $\alpha_3 = \alpha_1+\alpha_2$ corresponding to the generators $\mathbf{E}_{12}$ (i.e., n_3), $\mathbf{E}_{23}$ (i.e., n_1) and $\mathbf{E}_{13}$ (n_2). We denote $\lambda = (\lambda_1, \lambda_2)$ and $\mathcal{V}^\lambda = U_q\big(sl(3)\big)\mathbf{v}_0$. The space $\mathcal{V}^\lambda$ is decomposed to $\oplus_\mu \mathcal{V}^\lambda_\mu$, where $\mu = (\mu_1, \mu_2)$ and $\mathcal{V}^\lambda_\mu = \big\{\mathbf{v} \in \mathcal{V}^\lambda;\ \mathbf{K}_1\mathbf{v} = q^{\mu_1}\mathbf{v},\quad \mathbf{K}_2\mathbf{v} = q^{\mu_2}\mathbf{v}\big\}$. In order to vector $|n_1,n_2,n_3\rangle = \mathbf{E}_{32}^{n_1}\mathbf{E}_{31}^{n_2}\mathbf{E}_{21}^{n_3}$ be an element from the space $\mathcal{V}^\lambda_\mu$, we get the condition $\mu = \lambda - n_1\alpha_2 - n_2\alpha_3 - n_3\alpha_1$. Let us now denote $\delta = (1/2)(\alpha_1+\alpha_2+\alpha_3)$, $\lambda = \widehat{\lambda} - \delta$, $\mu = \widehat{\mu} - \delta$. If the vector $\mathbf{v}^\lambda_\mu$ is an extremal vector from the space $\mathcal{V}^\lambda_\mu$, i.e., if $\mathbf{v}^\lambda_\mu$ is nonzero vector such that $\mathbf{E}_{12}\mathbf{v}^\lambda_\mu = \mathbf{E}_{23}\mathbf{v}^\lambda_\mu = \mathbf{E}_{13}\mathbf{v}^\lambda_\mu = 0$, we must have $\widehat{\mu} = s_w\big(\widehat{\lambda}\big)$ for some w which is an element of the Weyl group of the algebra $sl(3)$. Let Verma module $\mathcal{V}^{(\lambda_1,\lambda_2)}$ is generated by the vector $\mathbf{v}_0$. The Weyl group has six elements. Analyzing these cases separately, we get the following results:

1. If $\lambda_1,\ \lambda_2,\ \lambda_1+\lambda_2+1 \notin \mathbb{N}_0$, there is only one extremal vector $\mathbf{v}_0$.

2. If $\lambda_1 \in \mathbb{N}_0$ and $\lambda_1+\lambda_2+1 \notin \mathbb{N}_0$, there are just two extremal elements: $\mathbf{v}_0$ and

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)} = \mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0.$$

3. If $\lambda_2 \in \mathbb{N}_0$ and $\lambda_1+\lambda_2+1 \notin \mathbb{N}_0$, there are just two extremal elements: $\mathbf{v}_0$ and

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)} = \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0.$$

4. If $\lambda_1+\lambda_2+1 \in \mathbb{N}_0$ and $\lambda_1,\ \lambda_2 \notin \mathbb{N}_0$, there are just two extremal elements: $\mathbf{v}_0$ and

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{\lambda_1+\lambda_2+2} (-1)^n q^{n(\lambda_1+2\lambda_2-n+2)}[n]_q! \begin{bmatrix}\lambda_2+1\\ n\end{bmatrix}_q \begin{bmatrix}\lambda_1+\lambda_2+2\\ n\end{bmatrix}_q \times \mathbf{E}_{32}^{\lambda_1+\lambda_2-n+2}\mathbf{E}_{31}^{n}\mathbf{E}_{21}^{\lambda_1+\lambda_2-n+2}\mathbf{v}_0.$$

5. If $\lambda_1 \in \mathbb{N}_0$ and $\lambda_2 = -1$, there are just three extremal elements: $\mathbf{v}_0$,

$$\mathbf{v}^{(\lambda_1,-1)}_{(-\lambda_1-2,\lambda_1)} = \mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0,$$
$$\mathbf{v}^{(\lambda_1,-1)}_{(-1,-\lambda_1-2)} = \mathbf{E}_{32}^{\lambda_1+1}\mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0 = \mathbf{E}_{32}^{\lambda_1+1}\mathbf{v}^{(\lambda_1,-1)}_{(-\lambda_1-2,\lambda_1)}.$$

6. If $\lambda_2 \in \mathbb{N}_0$ and $\lambda_1 = -1$, there are just three extremal elements: $\mathbf{v}_0$,

$$\mathbf{v}^{(-1,\lambda_2)}_{(\lambda_2,-\lambda_2-2)} = \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0,$$
$$\mathbf{v}^{(-1,\lambda_2)}_{(-\lambda_2-2,-1)} = \mathbf{E}_{21}^{\lambda_2+1}\mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0 = \mathbf{E}_{21}^{\lambda_2+1}\mathbf{v}^{(-1,\lambda_2)}_{(\lambda_2,-\lambda_2-2)}.$$

7. If $\lambda_1 \in \mathbb{N}$ and $\lambda_2 = -2, -3, \ldots, -\lambda_1-1$, there are just four extremal elements: $\mathbf{v}_0$,

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)} = \mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0,$$
$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{\lambda_1+\lambda_2+2} (-1)^n q^{n(\lambda_1+2\lambda_2-n+2)}[n]_q! \begin{bmatrix}\lambda_2+1\\ n\end{bmatrix}_q \begin{bmatrix}\lambda_1+\lambda_2+2\\ n\end{bmatrix}_q \times \mathbf{E}_{32}^{\lambda_1+\lambda_2-n+2}\mathbf{E}_{31}^{n}\mathbf{E}_{21}^{\lambda_1+\lambda_2-n+2}\mathbf{v}_0,$$
$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_2,-\lambda_1-\lambda_2-3)} = \mathbf{E}_{32}^{\lambda_1+\lambda_2+2}\mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0 = \mathbf{E}_{32}^{\lambda_1+\lambda_2+2}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)} \sim \sim \mathbf{E}_{21}^{-\lambda_2-1}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)}.$$

8. If $\lambda_2 \in \mathbb{N}$ and $\lambda_1 = -2, -3, \ldots, -\lambda_2 - 1$, there are just four extremal elements: $\mathbf{v}_0$,

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)} = \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0,$$

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{\lambda_1+\lambda_2+2} (-1)^n q^{n(\lambda_1+2\lambda_2-n+2)} [n]_q! \begin{bmatrix} \lambda_2+1 \\ n \end{bmatrix}_q \begin{bmatrix} \lambda_1+\lambda_2+2 \\ n \end{bmatrix}_q \times \mathbf{E}_{32}^{\lambda_1+\lambda_2-n+2}\mathbf{E}_{31}^n\mathbf{E}_{21}^{\lambda_1+\lambda_2-n+2}\mathbf{v}_0,$$

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-\lambda_2-3,\lambda_1)} = \mathbf{E}_{32}^{-\lambda_1-1}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} \sim \mathbf{E}_{21}^{\lambda_1+\lambda_2+2}\mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0 = \mathbf{E}_{21}^{\lambda_1+\lambda_2+2}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)}.$$

9. If $\lambda_1, \lambda_2 \in \mathbb{N}_0$, there are six extremal elements: $\mathbf{v}_0$,

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)} = \mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0,$$

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)} = \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0,$$

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_2,-\lambda_1-\lambda_2-3)} = \mathbf{E}_{32}^{\lambda_1+\lambda_2+2}\mathbf{E}_{21}^{\lambda_1+1}\mathbf{v}_0 = \mathbf{E}_{32}^{\lambda_1+\lambda_2+2}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-2,\lambda_1+\lambda_2+1)},$$

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_1-\lambda_2-3,\lambda_1)} = \mathbf{E}_{21}^{\lambda_1+\lambda_2+2}\mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0 = \mathbf{E}_{21}^{\lambda_1+\lambda_2+2}\mathbf{v}^{(\lambda_1,\lambda_2)}_{(\lambda_1+\lambda_2+1,-\lambda_2-2)},$$

$$\mathbf{v}^{(\lambda_1,\lambda_2)}_{(-\lambda_2-2,-\lambda_1-2)} = \sum_{n=0}^{\lambda_2+1} (-1)^n q^{n(\lambda_1+2\lambda_2-n+2)} [n]_q! \begin{bmatrix} \lambda_2+1 \\ n \end{bmatrix}_q \begin{bmatrix} \lambda_1+\lambda_2+2 \\ n \end{bmatrix}_q \times \mathbf{E}_{32}^{\lambda_1+\lambda_2-n+2}\mathbf{E}_{31}^n\mathbf{E}_{21}^{\lambda_1+\lambda_2-n+2}\mathbf{v}_0.$$

On the other hand, in the case of factor-representation when $\lambda_1 = J \in \mathbb{N}_0$, we get the following:

1. If $J + \lambda_2 + 1 \notin \mathbb{N}_0$, there is only one extremal vector $\mathbf{v}_0$.
2. If $\lambda_2 = -1$, there is only one extremal vector $\mathbf{v}_0$.
3. If $J \in \mathbb{N}$ and $\lambda_2 = -2, -3, \ldots, -J - 1$, there are two extremal vectors: $\mathbf{v}_0$ and

$$\mathbf{v}^{(J,\lambda_2)}_{(-\lambda_2-2,-J-2)} = \sum_{n=\lambda_2+2}^{J+\lambda_2+2} (-1)^n q^{n(J+2\lambda_2-n+2)} [n]_q! \begin{bmatrix} \lambda_2+1 \\ n \end{bmatrix}_q \begin{bmatrix} J+\lambda_2+2 \\ n \end{bmatrix}_q \times \mathbf{E}_{32}^{J+\lambda_2-n+2}\mathbf{E}_{31}^n\mathbf{E}_{21}^{J+\lambda_2-n+2}\mathbf{v}_0.$$

4. If $\lambda_2 \in \mathbb{N}_0$, there are two extremal vectors: $\mathbf{v}_0$ and $\mathbf{v}^{(J,\lambda_2)}_{(J+\lambda_2+1,-\lambda_2-2)} = \mathbf{E}_{32}^{\lambda_2+1}\mathbf{v}_0$.

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