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NAMBU–POISSON DYNAMICS WITH SOME APPLICATIONS

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Short introduction in NPD with several applications to (in)finite dimensional problems of mechanics, hydrodynamics, M-theory and quanputing is given.

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Nabu — Babylonian God
of Wisdom and Writing.

The Hamiltonian mechanics (HM) is in the fundamentals of mathematical description of the physical theories [1]. But HM is in a sense blind; e.g., it does not make a difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) [2] and (super)integrable Hamiltonian systems (with maximal number of the integrals of motion).

Nambu mechanics (NM) [3, 4] is a proper generalization of the HM, which makes the difference between dynamical systems with different numbers of integrals of motion explicit (see, e.g., [5]).

1. HAMILTONIZATION OF DYNAMICAL SYSTEMS

Let us consider a general dynamical system described by the following system of the ordinary differential equations [6]:

$$\dot{x}_n = v_n(x), \quad 1 \leqslant n \leqslant N, \tag{1}$$

$\dot{x}_n$ stands for the total derivative with respect to the parameter t.

When the number of the degrees of freedom is even, and

$$v_n(x) = \varepsilon_{nm} \frac{\partial H_0}{\partial x_m}, \quad 1 \leqslant n, \quad m \leqslant 2M, \tag{2}$$

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the system (1) is Hamiltonian one and can be put in the form

$$\dot{x}_n = \{x_n, H_0\}_0, \tag{3}$$

where the Poisson bracket is defined as

$$\{A, B\}_0 = \varepsilon_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = A \frac{\overleftarrow{\partial}}{\partial x_n} \varepsilon_{nm} \frac{\overrightarrow{\partial}}{\partial x_m} B, \tag{4}$$

and summation rule under repeated indices has been used.

Let us consider the following Lagrangian:

$$L = (\dot{x}_n - v_n(x))\psi_n \tag{5}$$

and the corresponding equations of motion

$$\dot{x}_n = v_n(x), \quad \dot{\psi}_n = -\frac{\partial v_m}{\partial x_n} \psi_m. \tag{6}$$

The system (6) extends the general system (1) by linear equation for the variables ψ. The extended system can be put in the Hamiltonian form [7]

$$\dot{x}_n = \{x_n, H_1\}_1, \quad \dot{\psi}_n = \{\psi_n, H_1\}_1, \tag{7}$$

where first-level (order) Hamiltonian is

$$H_1 = v_n(x)\psi_n \tag{8}$$

and (first-level) bracket is defined as

$$\{A, B\}_1 = A \left(\frac{\overleftarrow{\partial}}{\partial x_n} \frac{\overrightarrow{\partial}}{\partial \psi_n} - \frac{\overleftarrow{\partial}}{\partial \psi_n} \frac{\overrightarrow{\partial}}{\partial x_n} \right) B. \tag{9}$$

Note that when the Grassmann grading [8] of the conjugated variables x_n and ψ_n is different, the bracket (9) is known as Buttin bracket [9].

In the Faddeev–Jackiw formalism [10] for the Hamiltonian treatment of systems defined by first-order Lagrangians, i.e., by a Lagrangian of the form

$$L = f_n(x)\dot{x}_n - H(x), \tag{10}$$

motion equations

$$f_{mn}\dot{x}_n = \frac{\partial H}{\partial x_m}, \tag{11}$$

for the regular structure function f_{mn}, can be put in the explicit Hamiltonian (Poisson; Dirac) form

$$\dot{x}_n = f^{-1}_{nm} \frac{\partial H}{\partial x_m} = \{x_n, x_m\} \frac{\partial H}{\partial x_m} = \{x_n, H\}, \tag{12}$$

where the fundamental Poisson (Dirac) bracket is

$$\{x_n, x_m\} = f^{-1}_{nm}, \quad f_{mn} = \partial_m f_n - \partial_n f_m. \tag{13}$$

The system (6) is an important example of the first-order regular Hamiltonian systems. Indeed, in the new variables,

$$y^1_n = x_n, \quad y^2_n = \psi_n, \tag{14}$$

Lagrangian (5) takes the following first-order form:

$$\begin{gathered} L = (\dot{x}_n - v_n(x))\psi_n \Rightarrow \frac{1}{2}(\dot{x}_n\psi_n - \dot{\psi}_n x_n) - v_n(x)\psi_n = \frac{1}{2} y^a_n \varepsilon^{ab} \dot{y}^b_n - H(y) = \\ = f^a_n(y)\dot{y}^a_n - H(y), \quad f^a_n = \frac{1}{2} y^b_n \varepsilon^{ba}, \quad H = v_n(y^1) y^2_n, \\ f^{ab}_{nm} = \frac{\partial f^b_m}{\partial y^a_n} - \frac{\partial f^a_n}{\partial y^b_m} = \varepsilon^{ab}\delta_{nm}; \end{gathered} \tag{15}$$

corresponding motion equations and the fundamental Poisson bracket are

$$\dot{y}^a_n = \varepsilon_{ab}\delta_{nm}\frac{\partial H}{\partial y^b_m} = \{y^a_n, H\}, \quad \{y^a_n, y^b_m\} = \varepsilon_{ab}\delta_{nm}. \tag{16}$$

To the canonical quantization of this system corresponds

$$[\hat{y}^a_n, \hat{y}^b_m] = i\hbar\varepsilon_{ab}\delta_{nm}, \quad \hat{y}^1_n = y^1_n, \quad \hat{y}^2_n = -i\hbar\frac{\partial}{\partial y^1_n}. \tag{17}$$

In this quantum theory, classical part, motion equations for y^1_n, remain classical.

1.1. Modified Bochner–Killing–Yano (MBKY) Structures. Now we return to our extended system (6) and formulate conditions for the integrals of motion $H(x, \psi)$

$$H = H_0(x) + H_1 + \ldots + H_N, \tag{18}$$

where

$$H_n = A_{k_1 k_2 \ldots k_n}(x)\psi_{k_1}\psi_{k_2}\ldots\psi_{k_N}, \quad 1 \leqslant n \leqslant N, \tag{19}$$

we are assuming Grassmann valued ψ_n and the tensor $A_{k_1 k_2 \ldots k_n}$ are skew-symmetric. For integrals (18) we have

$$\dot{H} = \left\{\sum_{n=0}^{N} H_n, H_1\right\} = \sum_{n=0}^{N}\{H_n, H_1\} = \sum_{n=0}^{N}\dot{H}_n = 0. \tag{20}$$

Now we see, that each term in the sum (18) must be conserved separately. In particular for Hamiltonian systems (2), zeroth, H_0, and first-level H_1, (8), Hamiltonians are integrals of motion. For $n = 0$

$$\dot{H}_0 = H_{0,k}v_k = 0, \tag{21}$$

for $1 \leqslant n \leqslant N$ we have

$$\dot{H}_n = \dot{A}_{k_1 k_2 \cdots k_n} \psi_{k_1} \psi_{k_2} \cdots \psi_{k_N} + A_{k_1 k_2 \cdots k_n} \dot{\psi}_{k_1} \psi_{k_2} \cdots \psi_{k_N} + \ldots + A_{k_1 k_2 \cdots k_n} \psi_{k_1} \psi_{k_2} \cdots \dot{\psi}_{k_N} = (A_{k_1 k_2 \cdots k_n, k} v_k - A_{k k_2 \cdots k_n} v_{k_1, k} - \ldots - A_{k_1 \cdots k_{n-1} k} v_{k_n, k}) \psi_{k_1} \psi_{k_2} \cdots \psi_{k_N} = 0, \quad (22)$$

and there is one-to-one correspondence between the existence of the integrals (19) and the existence of the nontrivial solutions of the following equations:

$$\frac{D}{Dt} A_{k_1 k_2 \cdots k_n} = A_{k_1 k_2 \cdots k_n, k} v_k - A_{k k_2 \cdots k_n} v_{k_1, k} - \ldots - A_{k_1 \cdots k_{n-1} k} v_{k_n, k} = 0. \quad (23)$$

For $n = 1$ the system (23) gives

$$A_{k_1, k} v_k - A_k v_{k_1, k} = 0 \quad (24)$$

and this equation has at list one solution, $A_k = v_k$. If we have two (or more) independent first order integrals

$$H_1^{(1)} = A_k^1 \Psi_k; \quad H_1^{(2)} = A_k^2 \Psi_k, \ldots, \quad (25)$$

we can construct corresponding (reducible) second (or higher) order MBKY tensor(s)

$$H_2 = H_1^{(1)} H_1^{(2)} = A_k^1 A_l^2 \Psi_k \Psi_l = A_{kl} \Psi_k \Psi_l,$$
$$H_M = H_1^{(1)} \cdots H_M^{(M)} = A_{k_1 \cdots k_M} \Psi_{k_1} \cdots \Psi_{k_M}, \quad (26)$$
$$A_{k_1 \ldots k_M} = \{A_{k_1}^{(1)} \cdots A_{k_M}^{(M)}\}, \quad 2 \leqslant M \leqslant N,$$

where under the bracket operation, $\{B_{k_1, \ldots, k_N}\} = \{B\}$ we understand complete antisymmetrization. The system (23) defines a generalization of the Bochner–Killing–Yano structures of the geodesic motion of the point particle, for the case of the general (1) (and extended (6)) dynamical systems. Having $A_M, 2 \leqslant M \leqslant N$ independent MBKY structures, we can construct corresponding second order Killing tensors and Nambu–Poisson dynamics. In the superintegrable case, we have maximal number of the motion integrals, $N - 1$.

The structures defined by the system (23) we call the Modified Bochner–Killing–Yano structures or MBKY structures for short, [11].

1.2. Point Vortex Dynamics (PVD). PVD can dy defined (see, e.g., [12, 13]) as the following first order system:

$$\dot{z}_n = i \sum_{m \neq n}^{N} \frac{\gamma_m}{z_n^* - z_m^*}, \quad z_n = x_n + i y_n, \quad 1 \leqslant n \leqslant N. \quad (27)$$

Corresponding first order Lagrangian, Hamiltonian, momenta, Poisson brackets and commutators are

$$L = \sum_n \frac{i}{2}\gamma_n (z_n \dot{z}_n^* - \dot{z}_n z_n^*) - \sum_{n\neq m} \gamma_n\gamma_m \ln|z_n - z_m|,$$
$$H = \sum_{n\neq m} \gamma_n\gamma_m \ln|z_n - z_m| = \frac{1}{2}\sum_{n\neq m}\gamma_n\gamma_m(\ln(z_n - z_m) + \ln(p_n - p_m)),$$
$$p_n = \frac{\partial L}{\partial \dot{z}_n} = -\frac{i}{2}\gamma_n z_n^*, \quad p_n^* = \frac{\partial L}{\partial \dot{z}_n^*} = \frac{i}{2}\gamma_n z_n, \tag{28}$$
$$\{p_n, z_m\} = \delta_{nm}, \quad \{p_n^*, z_m^*\} = \delta_{nm}, \quad \{x_n, y_m\} = \delta_{nm},$$
$$[p_n, z_m] = -i\hbar\delta_{nm} \Rightarrow [x_n, y_m] = -i\frac{\hbar}{\gamma_n}\delta_{nm}.$$

So, quantum vortex dynamics corresponds to the noncommutative space. It is natural to assume that vortex parameters are quantized as

$$\gamma_n = \frac{\hbar}{a^2}n, \quad n = \pm 1, \pm 2, \ldots, \tag{29}$$

and a is a characteristic (fundamental) length.

2. NAMBU DYNAMICS

In the canonical formulation, the equations of motion of a physical system are defined via a Poisson bracket and a Hamiltonian, [6]. In Nambu's formulation, the Poisson bracket is replaced by the Nambu bracket with $n+1, n \geqslant 1$, slots. For $n = 1$, we have the canonical formalism with one Hamiltonian. For $n \geqslant 2$, we have Nambu–Poisson formalism, with n Hamiltonians, [3,4].

2.1. System of Three Vortexes. The system of N vortexes (27) for $N = 3$, and

$$u_1 = \ln|z_2 - z_3|^2, \quad u_2 = \ln|z_3 - z_1|^2, \quad u_3 = \ln|z_1 - z_2|^2 \tag{30}$$

reduces to the following system:

$$\dot{u}_1 = \gamma_1(\mathrm{e}^{u_2} - \mathrm{e}^{u_3}), \quad \dot{u}_2 = \gamma_2(\mathrm{e}^{u_3} - \mathrm{e}^{u_1}), \quad \dot{u}_3 = \gamma_3(\mathrm{e}^{u_1} - \mathrm{e}^{u_2}). \tag{31}$$

The system (31) has two integrals of motion

$$H_1 = \sum_{i=1}^{3} \frac{\mathrm{e}^{u_i}}{\gamma_i}, \quad H_2 = \sum_{i=1}^{3} \frac{u_i}{\gamma_i}$$

and can be presented in the Nambu–Poisson form [14]

$$\dot{u}_i = \omega_{ijk} \frac{\partial H_1}{\partial u_j} \frac{\partial H_2}{\partial u_k} = \{x_i, H_1, H_2\} = \omega_{ijk} \frac{\mathrm{e}^{u_j}}{\gamma_j} \frac{1}{\gamma_k},$$

where

$$\omega_{ijk} = \epsilon_{ijk}\rho, \rho = \gamma_1\gamma_2\gamma_3,$$

and the Nambu–Poisson bracket of the functions A, B, C on the three-dimensional phase space is

$$\{A, B, C\} = \omega_{ijk} \frac{\partial A}{\partial u_i} \frac{\partial B}{\partial u_j} \frac{\partial C}{\partial u_k}. \tag{32}$$

This system is superintegrable: for $N = 3$ degrees of freedom, we have maximal number of the integrals of motion $N - 1 = 2$.

2.2. Extended Quantum Mechanics. As an example of the infinite dimensional Nambu–Poisson dynamics, let me conside the following extension of Schrödinger quantum mechanics [15]:

$$V_t = \Delta V - \frac{V^2}{2}, \tag{33}$$

$$i\psi_t = -\Delta\psi + V\psi. \tag{34}$$

An interesting solution to the equation for the potential (34) is

$$V = \frac{4(4-d)}{r^2}, \tag{35}$$

where d is the dimension of the space. In the case of $d = 1$, we have the potential of conformal quantum mechanics.

The variational formulation of the extended quantum theory, is given by the following Lagrangian:

$$L = \left(iV_t - \Delta V + \frac{1}{2}V^2\right)\psi. \tag{36}$$

The momentum variables are

$$P_v = \frac{\partial L}{\partial V_t} = i\psi, \quad P_\psi = 0. \tag{37}$$

As Hamiltonians of the Nambu-theoretic formulation, we take the following integrals of motion:

$$\begin{aligned} H_1 &= \int d^d x \left(\Delta V - \frac{1}{2}V^2\right)\psi, \\ H_2 &= \int d^d x (P_v - i\psi), \\ H_3 &= \int d^d x P_\psi. \end{aligned} \tag{38}$$

We invent unifying vector notation, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (\psi, P_\psi, V, P_v)$. Then it may be verified that the equations of the extended quantum theory can be put in the following Nambu-theoretic form:

$$\phi_t(x) = \{\phi(x), H_1, H_2, H_3\}, \tag{39}$$

where the bracket is defined as

$$\{A_1, A_2, A_3, A_4\} = i\varepsilon_{ijkl} \int \frac{\delta A_1}{\delta\phi_i(y)} \frac{\delta A_2}{\delta\phi_j(y)} \frac{\delta A_3}{\delta\phi_k(y)} \frac{\delta A_4}{\delta\phi_l(y)}\, dy = \\ = i \int \frac{\delta(A_1, A_2, A_3, A_4)}{\delta(\phi_1(y), \phi_2(y), \phi_3(y), \phi_4(y))}\, dy = i \det\left(\frac{\delta A_k}{\delta\phi_l}\right). \tag{40}$$

2.3. M **Theory.** The basic building blocks of M theory are membranes and $M5$-branes. Membranes are fundamental objects carrying electric charges with respect to the 3-form C-field, and $M5$-branes are magnetic solitons. The Nambu–Poisson 3-algebras appear as gauge symmetries of superconformal Chern–Simons non-Abelian theories in $2+1$E dimensions with the maximum allowed number of $N = 8$ linear supersymmetries.

The Bagger and Lambert [16] and Gustavsson [17] (BLG) model is based on a 3-algebra,

$$[T^a, T^b, T^c] = f_d^{abc} T^d, \tag{41}$$

where T^a are generators and f_{abcd} is a fully antisymmetric tensor. Given this algebra, a maximally supersymmetric Chern–Simons Lagrangian is

$$L = L_{\rm CS} + L_{\rm matter}, \\ L_{\rm CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f_{abcd}A_\mu^{ab}\partial_\nu A_\lambda^{cd} + \frac{2}{3}f_{cdag}f_{efb}^{g}A_\mu^{ab}A_\nu^{cd}A_\lambda^{ef}\right), \\ L_{\rm matter} = \frac{1}{2}B_\mu^{Ia}B_a^{\mu I} - B_\mu^{Ia}D^\mu X_a^I + +\frac{i}{2}\bar\psi^a\Gamma^\mu D_\mu\psi_a + \frac{i}{4}\bar\psi^b\Gamma_{IJ}x_c^I x_d^J\psi_a f^{abcd} - \\ - \frac{1}{12}\,{\rm tr}\,([X^I, X^J, X^K][X^I, X^J, X^K]), \quad I = 1, 2, \ldots, 8, \tag{42}$$

where A_μ^{ab} is gauge boson, ψ^a and $X^I = X_a^I T^a$ are matter fields. If $a = 1, 2, 3, 4$, then we can obtain an $SO(4)$ gauge symmetry by choosing $f_{abcd} = f\varepsilon_{abcd}, f$ being a constant. It turns out to be the only case that gives a gauge theory with manifest unitarity and $N = 8$ supersymmetry.

The action has the first order form so we can use the formalism of the first section. The motion equations for the gauge fields

$$f_{abcd}^{nm}\dot A_m^{cd}(t, x) = \frac{\delta H}{\delta A_n^{ab}(t, x)}, \quad f_{abcd}^{nm} = \varepsilon^{nm} f_{abcd} \tag{43}$$

take canonical form

$$\dot{A}_n^{ab} = f_{nm}^{abcd}\frac{\delta H}{\delta A_m^{cd}} = \{A_n^{ab}, A_m^{cd}\}\frac{\delta H}{\delta A_m^{cd}} = \{A_n^{ab}, H\},$$
$$\{A_n^{ab}(t,x), A_m^{cd}(t,y)\} = \varepsilon_{nm} f^{abcd}\delta^{(2)}(x-y). \tag{44}$$

3. DISCRETE DYNAMICAL SYSTEMS

Computers are physical devices and their behavior is determined by physical laws. The Quantum Computations [18, 19], Quantum Computing, Quanputing [20], is a new interdisciplinary field of research, which benefits from the contributions of physicists, computer scientists, mathematicians, chemists and engineers.

Contemporary digital computer and its logical elements can be considered as a spatial type of discrete dynamical systems [21]

$$S_n(k+1) = \Phi_n(S(k)), \tag{45}$$

where

$$S_n(k), \quad 1 \leqslant n \leqslant N(k) \tag{46}$$

is the state vector of the system at the discrete time step k. Vector S may describe the state and Φ transition rule of some Cellular Automata [22]. The system of the type (45) appears in applied mathematics as an explicit finite difference scheme approximation of the equations of the physics [23].

Definition: *We assume that the system (45) is time-reversible if we can define the reverse dynamical system*

$$S_n(k) = \Phi_n^{-1}(S(k+1)). \tag{47}$$

In this case the following matrix:

$$M_{nm} = \frac{\partial\Phi_n(S(k))}{\partial S_m(k)} \tag{48}$$

is regular, i.e., has an inverse. If the matrix is not regular, this is the case, for example, when $N(k+1) \neq N(k)$, we have an irreversible dynamical system (usual digital computers and/or corresponding irreversible gates).

Let us consider an extension of the dynamical system (45) given by the following action function:

$$A = \sum_{kn} l_n(k)(S_n(k+1) - \Phi_n(S(k))) \tag{49}$$

and corresponding motion equations

$$\begin{aligned} S_n(k+1) &= \Phi_n(S(k)) = \frac{\partial H}{\partial l_n(k)}, \\ l_n(k-1) = l_m(k)\frac{\partial \Phi_m(S(k))}{\partial S_n(k)} &= l_m(k)\, M_{mn}(S(k)) = \frac{\partial H}{\partial S_n(k)}, \end{aligned} \tag{50}$$

where

$$H = \sum_{kn} l_n(k)\, \Phi_n(S(k)) \tag{51}$$

is discrete Hamiltonian. In the regular case, we put the system (50) in an explicit form

$$\begin{aligned} S_n(k+1) &= \Phi_n(S(k)), \\ l_n(k+1) &= l_m(k) M^{-1}_{mn}(S(k+1)). \end{aligned} \tag{52}$$

From this system it is obvious that, when the initial value $l_n(k_0)$ is given, the evolution of the vector $l(k)$ is defined by evolution of the state vector $S(k)$. The equation of motion for $l_n(k)$ is linear and has an important property that linear superpositions of the solutions are also solutions.

Statement. *Any time-reversible dynamical system (e.g., a time-reversible computer) can be extended by corresponding linear dynamical system (quantum-like processor) which is controlled by the dynamical system and has a huge computational power* [20, 21, 24, 25].

3.1. (de)Coherence Criterion. For motion equations (50) in the continual approximation, we have

$$\begin{aligned} S_n(k+1) = x_n(t_k+\tau) &= x_n(t_k) + \dot{x}_n(t_k)\tau + O(\tau^2), \\ \dot{x}_n(t_k) &= v_n(x(t_k)) + O(\tau), \quad t_k = k\tau, \\ v_n(x(t_k)) &= (\Phi_n(x(t_k)) - x_n(t_k))/\tau, \\ M_{mn}(x(t_k)) &= \delta_{mn} + \tau\frac{\partial v_m(x(t_k))}{\partial x_n(t_k)}. \end{aligned} \tag{53}$$

(de)Coherence criterion: *The system is reversible, the linear (quantum, coherent, soul) subsystem exists, when the matrix M is regular,*

$$\det M = 1 + \tau \sum_n \frac{\partial v_n}{\partial x_n} + O(\tau^2) \neq 0. \tag{54}$$

For the Nambu–Poisson dynamical systems (see, e.g., [5])

$$v_n(x) = \varepsilon_{nm_1m_2\dots m_p}\frac{\partial H_1}{\partial x_{m_1}}\frac{\partial H_2}{\partial x_{m_2}}\cdots\frac{\partial H_p}{\partial x_{m_p}}, \quad p = 1, 2, 3, \dots, N-1,$$
$$\sum_n \frac{\partial v_n}{\partial x_n} \equiv \operatorname{div} v = 0. \tag{55}$$

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