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HOMOTOPY TRANSFER AND SELF-DUAL SCHUR MODULES*

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We consider the free 2-nilpotent graded Lie algebra $\mathfrak{g}$ generated in degree one by a finite dimensional vector space V. We recall the beautiful result that the cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ of $\mathfrak{g}$ with trivial coefficients carries a $GL(V)$-representation having only the Schur modules V_λ with self-dual Young diagrams $\{\lambda : \lambda = \lambda'\}$ in its decomposition into $GL(V)$-irreducibles (each with multiplicity one). The homotopy transfer theorem due to Tornike Kadeishvili allows one to equip the cohomology of the Lie algebra $\mathfrak{g}$ with a structure of homotopy commutative algebra.

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1. HOMOTOPY ALGEBRAS A_∞ AND C_∞

We start by recalling the definition of homotopy associative algebra. For a pedagogical introduction to the subject we send the reader to the textbook of J.-L. Loday and B. Valette [6].

Definition 1. *A homotopy associative algebra, or A_∞-algebra over $\mathbb{K}$ is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)*

$$m_n : A^{\otimes n} \to A, \quad \deg(m_n) = 2 - n, \quad n \geqslant 1$$

satisfying the Stasheff identities $\mathbf{SI(n)}$ for $n \geqslant 1$

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad \mathbf{SI(n)}$$

where the sum runs over all decompositions $n = r + s + t$.

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*Talk given by Todor Popov.

Throughout the text we assume the Koszul sign rule $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. We define the shuffle product by the expression $(a_1 \otimes \ldots \otimes a_p) \sqcup (a_{p+1} \otimes \ldots \otimes a_{p+q}) = \sum\limits_{\sigma \in Sh_{p,q}} \operatorname{sgn}(\sigma) a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}$, the sum running over shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q)$.

Definition 2 (see, e.g., [4]). *A homotopy commutative algebra C_∞, or C_∞-algebra is an A_∞ algebra $\{A, m_n\}$ with the additional condition: each operation m_n vanishes on shuffles*

$$m_n((a_1 \otimes \ldots \otimes a_p) \sqcup (a_{p+1} \otimes \ldots \otimes a_n)) = 0, \quad 1 \leqslant p \leqslant n-1. \tag{1}$$

In particular for m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞-algebra such that $m_n = 0$ for $n \geqslant 3$ is a supercommutative Differential Graded Algebra (DGA for short).

Morphism of two A_∞-algebras A and B is a family of graded maps $f_n : A^{\otimes n} \to B$ for $n \geq 1$ with $\deg f_n = 1 - n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes r}) = \sum_{1 \leqslant q \leqslant n} (-1)^S m_q(f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_q}),$$

where the sum is on all decompositions $i_1 + \ldots + i_q = n$, and the sign on RHS is determined by $S = \sum\limits_{k=1}^{q-1} (q-k)(i_k - 1)$. The morphism f is a *quasi-isomorphism of A_∞-algebras* if f_1 is a quasi-isomorphism. It is strict if $f_i = 0$ for $i \geqslant 1$. The identity morphism on A is the strict morphism f such that f_1 is the identity of A.

A morphism of C_∞-algebras is a morphism of A_∞-algebras with components vanishing on shuffles $f_n((a_1 \otimes \ldots \otimes a_p) \sqcup (a_{p+1} \otimes \ldots \otimes a_n)) = 0$, $1 \leqslant p \leqslant n-1$.

2. HOMOTOPY TRANSFER THEOREM

Lemma 1 (see, e.g., [6]). *Every cochain complex (A, d) of vector spaces over a field $\mathbb{K}$ has its cohomology $H^\bullet(A)$ as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex (A, d) such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$, where H^n is the cohomology and B^n is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \to A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n \oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \underset{i}{\overset{p}{\rightleftarrows}} H^n$ are chain homomorphisms (satisfying the additional conditions $hh = 0$, $hi = 0$ and $ph = 0$). With these choices done, the complex $(H^\bullet(A), 0)$ is a deformation retract of (A, d)

$$h \circlearrowright (A,d) \underset{i}{\overset{p}{\rightleftarrows}} (H^\bullet(A), 0), \quad pi = Id_{H^\bullet(A)}, \quad ip - Id_A = dh + hd. \quad (2)$$

Let now (A, d, μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d. The cochain complexes (A, d) and their contraction $H^\bullet(A)$ are homotopy equivalent, but the associative strucure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞-structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$.

Theorem 1 (Kadeishvili [4]). *Let (A, d, μ) be a (commutative) DGA over a field $\mathbb{K}$. There exists a A_∞-algebra (C_∞-algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$-quasi-isomorphism $f_i : (\otimes^i H^\bullet(A), \{m_i\}) \to (A, \{d, \mu, 0, 0, \ldots\})$ such that the inclusion $f_1 = i : H^\bullet(A) \to A$ is a cocycle-choosing homomorphism of cochain complexes. The differential on $H^\bullet(A)$ is zero $m_1 = 0$ and m_2 is the associative operation induced by the multiplication on A. The resulting structure is unique up to quasi-isomorphism.*

3. HOMOLOGY AND COHOMOLOGY OF THE LIE ALGEBRA $\mathfrak{g}$

Let $\mathfrak{g}$ be the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ generated by the finite dimensional vector space V over the ground field $\mathbf{K}$ of characteristics zero. The Lie bracket on $\mathfrak{g}$ reads

$$[x, y] := x \wedge y \quad \text{when} \quad x, y \in V \quad \text{and} \quad [x, y] := 0 \quad \text{otherwise.}$$

We define the homology with trivial coefficients of the Lie algebra $\mathfrak{g}$ through the Chevalley–Eilenberg complex $C_\bullet(\mathfrak{g}) = (\bigwedge^\bullet \mathfrak{g}, \partial_\bullet)$ having differential $\partial_n : \bigwedge^n \mathfrak{g} \to \bigwedge^{n-1} \mathfrak{g}$,

$$\partial_n(x_1 \wedge \ldots \wedge x_n) = \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n, \quad p > 0. \quad (3)$$

The homology $H_n(\mathfrak{g}, \mathbb{K})$ of the Lie algebra $\mathfrak{g}$ is the homology space of the complex $C(\mathfrak{g})$

$$H_n(\mathfrak{g}, \mathbb{K}) := H_n(C_\bullet(\mathfrak{g})), \quad H_n(C_\bullet(\mathfrak{g})) = \ker \partial_n / \mathrm{im} \partial_{n+1}.$$

The differential ∂ is induced by the Lie bracket $[\cdot, \cdot] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ of the graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. It identifies a pair of degree-1 generators $e_i, e_j \in \mathfrak{g}_1$ with one degree-2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \mathfrak{g}_2$. In more detail the chain degrees read

$$\bigwedge^n \mathfrak{g} = \bigwedge^n \left(V \oplus \bigwedge^2 V\right) = \bigoplus_{s+r=n} \bigwedge^s \left(\bigwedge^2 V\right) \otimes \bigwedge^r V \tag{4}$$

and differentials $\partial_{n=r+s} : \bigwedge^s(\bigwedge^2 V) \otimes \bigwedge^r V \to \bigwedge^{s+1}(\bigwedge^2 V) \otimes \bigwedge^{r-2} V$ are given by

$$\begin{aligned} \partial_n : \quad & e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \otimes e_1 \wedge \ldots \wedge e_r \mapsto \\ \mapsto & \sum_{i<j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \otimes e_1 \wedge \ldots \wedge \hat{e_i} \wedge \ldots \wedge \hat{e_j} \wedge \ldots \wedge e_r. \end{aligned}$$

The differential ∂ commutes with the $GL(V)$-action, thus the homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ is also a $GL(V)$-module; its decomposition into irreducible polynomial representations V_λ (the so-called Schur modules) is given by the following beautiful result.

Theorem 2 (Józefiak and Weyman [3], Sigg [7]). *The homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ decomposes into irreducible $GL(V)$-modules*

$$H_n(\mathfrak{g}, \mathbb{K}) = H_n(\bigwedge^\bullet \mathfrak{g}, \partial_\bullet) \cong \bigoplus_{\lambda : \lambda = \lambda'} V_\lambda, \tag{5}$$

where the sum is over the self-dual Young diagrams $\{\lambda : \lambda = \lambda'\}$ such that $n = (1/2)(|\lambda| + r(\lambda))$.

By duality, one has the cochain complex $\mathrm{Hom}_{\mathbb{K}}(C(\mathfrak{g}), \mathbb{K}) = (\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ which is a (super)commutative DGA. The cohomology $H^n(\mathfrak{g}, \mathbb{K})$ with trivial coefficients is calculated by the complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$

$$H^n(\mathfrak{g}, \mathbb{K}) := H^n(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet).$$

Here the coboundary map $\delta^n : \bigwedge^n \mathfrak{g}^* \to \bigwedge^{n+1} \mathfrak{g}^*$ is transposed* to the differential ∂_{n+1}

$$\begin{aligned} \delta^n : \quad & e^*_{i_1 j_1} \wedge \ldots \wedge e^*_{i_s j_s} \otimes e^*_1 \wedge \ldots \wedge e^*_r \mapsto \\ \mapsto & \sum_{k=1}^{s} \sum_{i_k < j_k} (-1)^{i+j} e^*_{i_1 j_1} \wedge \ldots \wedge \hat{e}^*_{i_k j_k} \wedge \ldots \wedge e^*_{i_s j_s} \otimes e^*_{i_k} \wedge e^*_{j_k} \wedge e^*_1 \wedge \ldots \wedge \ldots \wedge e^*_r, \end{aligned} \tag{6}$$

it is (up to a conventional sign) a continuation of the dualization of the Lie bracket $\delta^1 := [\cdot, \cdot]^* : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ by the Leibniz rule.

*In the presence of metric one has $\delta := \partial^*$ (see below).

Proposition 1 [2]. *The cohomology $H^\bullet(\mathfrak{g},\mathbb{K})$ of the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \bigwedge^2 V$ is a homotopy commutative algebra. The C_∞-algebra $H^\bullet(\mathfrak{g},\mathbb{K})$ is generated in degree 1, i.e., in $H^1(\mathfrak{g},\mathbb{K})$, by the operations m_2 and m_3.*

Sketch of the Proof. By Lemma 1 the commutative DGA $(\bigwedge^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$ has a deformation retract $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ thus from the Kadeishvili homotopy transfer theorem 1 follows that the cohomology $H^\bullet(\mathfrak{g},\mathbb{K})$ is a C_∞-algebra.

To prove that the C_∞-algebra $H^\bullet(\mathfrak{g},\mathbb{K})$ is generated by m_2 and m_3 we will need convenient choice of the homotopy h, the projection p and the inclusion i in the deformation retract (2).

Let us choose a metric $g(\ ,\ ) = \langle\ ,\ \rangle$ on the vector space V and an orthonormal basis $\langle e_i, e_j\rangle = \delta_{ij}$. The choice induces a metric on $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$. In presence of metric g, the differential δ is identified with the adjoint of ∂, $\delta :\overset{g}{=} \partial^*$ (see Eq. (5)) while ∂ plays the role of homotopy. The deformation retract of the complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ takes the following form [7]:

$$pi = Id_{H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)}, \quad ip - Id_{\bigwedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta, \quad \delta^* \overset{g}{=} \partial.$$

Here the projection p identifies the subspace $\ker\delta \cap \ker\delta^*$ with $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\mathrm{im}\delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

Due to the isomorphisms $H^n(\mathfrak{g},\mathbb{K}) \cong H^*_n(\mathfrak{g},\mathbb{K})$ (i.e., $\mathrm{Tor}_n^{U\mathfrak{g}}(\mathbb{K},\mathbb{K}) \cong \mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})$ in the category of graded algebras [1]) induced by $V \overset{g}{\cong} V^*$, the theorem 2 implies the decomposition

$$H^n(\mathfrak{g},\mathbb{K}) \cong H^n(\wedge\mathfrak{g}^*, \delta) \cong \oplus_{\lambda:\lambda=\lambda'} V_\lambda,$$

where the sum is over the self-dual Young diagrams λ such that $n = (1/2)(|\lambda| + r(\lambda))$.

We were able to show in [2] that with the use of the explicit expressions [5] for the operations $m_2(x,y) := p\mu(i(x), i(y))$ and $m_3(x,y,z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$ one can generate all the elements in $H^\bullet(\mathfrak{g},\mathbb{K})$ by the degree one elements $H^1(\mathfrak{g},\mathbb{K})$. $\square$

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