*ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА*
*2013. Т. 44. ВЫП. 2*

# DELTA GRAVITY

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We present a model of the gravitational field based on two symmetric tensors. The equations of motion of test particles are derived. Massive particles do not follow a geodesic, but massless-particles trajectories are null geodesics of an effective metric. Outside matter, the predictions of the model coincide exactly with general relativity, so all classical tests are satisfied. In cosmology, we get accelerated expansion without a cosmological constant. Additionally, we study the quantization of the model. The main result being that the effective action is finite and receives one-loop corrections only.

PACS: 04.20.-q

## INTRODUCTION

General Relativity (GR) works very well at the macroscopic scales [1]. Its quantization has proved to be difficult, though. It is nonrenormalizable, which prevents its unification with the other forces of Nature. Trying to make sense of quantum GR is the main physical motivation of String Theories [2]. Moreover, recent discoveries in cosmology [3, 4] have revealed that the most part of matter is in the form of unknown matter (dark matter, DM) and that the dynamics of the expansion of the Universe is governed by a mysterious component that accelerates the later stages of the expansion (dark energy, DE). Although GR is able to accommodate both DM and DE, the interpretation of the dark sector in terms of fundamental theories of elementary particles is problematic [5]. Although some candidates exist that could play the role of DM, none have been detected yet. Also, an alternative explanation based on the modification of the dynamics for small accelerations cannot be ruled out [6].

In GR, DE can be explained if a small cosmological constant ($\Lambda$) is present. At the later stages of the evolution of the Universe, $\Lambda$ will dominate the expansion, explaining the acceleration. Such small $\Lambda$ is very difficult to generate in Quantum Field Theory (QFT) models, because in this models $\Lambda$ is the vacuum energy, which is usually very large.

In recent years, there has been various proposals to explain the observed acceleration of the Universe. They involve the inclusion of some additional field like in quintessence, chameleon, vector dark energy or massive gravity;

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addition of higher order terms in the Einsten–Hilbert action, like $f(R)$ theories and Gauss–Bonnet terms; modification of gravity on large scales by introduction of extra dimensions. For a review, see [7].

Less widely explored, but interesting possibilities, are the search for non-trivial ultraviolet fixed points in gravity (asymptotic safety [9]) and the notion of induced gravity [10]. The first possibility uses exact renormalization-group techniques [11], and lattice and numerical techniques such as Lorentzian triangulation analysis [12]. Induced gravity proposed that gravitation is a residual force produced by other interactions.

In recent papers [13, 14], a field theory model explores the emergence of geometry by the spontaneous symmetry breaking of a larger symmetry where the metric is absent. Previous works in this direction can be found in [15,16] and [17].

In this paper, we will review the results of [21, 31]. The main observation is that GR is finite on shell at one loop [18]. In [19, 20], we presented a type of gauge theories, $\delta$ gauge theories (DGT). The main properties of DGT are: 1) The classical equations of motion are satisfied in the full quantum theory. 2) They live at one loop. 3) They are obtained through the extension of the former symmetry of the model introducing an extra symmetry that we call $\delta$ symmetry, since it is formally obtained as the variation of the original symmetry. When we apply this prescription to GR, we obtain $\delta$ gravity. Quantization of $\delta$ gravity is discussed in [21].

The impact of dark energy on cosmological observations can be expressed in terms of a fluid equation of state $p = w(R)\rho$, which is to be determined studying its influence on the large-scale structure and dynamics of the Universe.

In this paper we follow the same approach. So, we will not include the matter dynamics, except by demanding that the energy-momentum tensor of the matter fluid is covariantly conserved. This is required in order to respect the symmetries of the model.

The main properties of this model at the classical level are: a) It agrees with GR, outside the sources and with adequate boundary conditions. In particular, the causal structure of delta gravity in vacuum is the same as in general relativity. So, all standard tests are satisfied automatically. b) When we study the evolution of the Universe, it predicts acceleration without a cosmological constant or additional scalar fields. The Universe ends in a Big Rip, similar to the scenario considered in [23]. c) The scale factor agrees with the standard cosmology at early times and shows acceleration only at late times. Therefore we expect that density perturbations should not have large corrections.

It should be remarked that $\delta$ gravity is not a metric model of gravity because massive particles do not move on geodesics. Only massless particles move on null geodesics of a linear combination of both tensor fields.

Upon quantization, the model exhibits several remarkable properties. The effective action receives one-loop corrections only. Moreover, all possible divergences of one-particle irreducible graphs vanish. Thus $\delta$ gravity is a finite quantum field theory.

It was noticed in [20] that the Hamiltonian of delta models is not bounded from below. Phantoms cosmological models [22, 23] also have this property. Although it is not clear whether this problem will subsist in a diffeomorphism invariant model as delta gravity or not, we mention some ways out of the difficulty at the end.

In Sec. 1, we write the action defining the model and the corresponding symmetries. Section 2 discusses the motion of particles in the model. In Sec. 3, we define proper time and distances. In Sec. 4, we obtain the Newtonian limit. In Sec. 5, we solve the equations of the model for Friedman–Robertson–Walker metric. In Sec. 6, we find the red shift. In Sec. 7, we define luminosity distance. In Sec. 8, we fit the equations of Sec. 5 to the supernova Ia data. Section 9 contains a preliminary discussion of dark matter. In Sec. 10, we find the exact effective action for a general $\delta$ model. Section 11 contains the computation of divergences in the effective action of $\delta$ gravity, showing that it is a finite quantum field theory, in vacuum and without a cosmological constant. In Sec. 12, we exhibit the existence of ghosts in the model in a particular gauge. Section 13 deals with the simplest nonvanishing quantum corrections to the effective action of $\delta$ gravity. Section 14 contains the conclusions and brief discussions of open problems. In Appendix A, we review $\delta$ symmetries. In Appendix B, we discuss a simpler model: a delta harmonic oscillator, to illustrate the boundedness of the Hamiltonian.

## 1. DEFINITION OF DELTA GRAVITY

In this section we define the action as well as the symmetries of the model and derive the equations of motion.

We use the metric convention of [8]. The action of $\delta$ gravity is

$$\begin{aligned}S(g,\tilde{g},\lambda) = \int d^dx\sqrt{-g}\left(-\frac{1}{2\kappa}R+\mathcal{L}_M\right)+ \\ +\kappa_2\int\left[\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+\kappa T_{\mu\nu}\right]\sqrt{-g}\tilde{g}^{\mu\nu}d^dx+ \\ +\kappa_2\kappa\int\sqrt{-g}\,(\lambda^{\mu;\nu}+\lambda^{\nu;\mu})\,T_{\mu\nu}\,d^dx.\end{aligned} \quad (1)$$

Here $\kappa = 8\pi G/c^4$; $\kappa_2$ is an arbitrary constant, and $T_{\mu\nu} := -\dfrac{2}{\sqrt{-g}}\dfrac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}$ is the energy-momentum tensor of matter; $R_{\mu\nu}$ is the Ricci's tensor, and $R$ is the curvature scalar of $g_{\mu\nu}$; $\tilde{g}^{\mu\nu}$ is a two-contravariant tensor under general coordinate transformations.

The action (1) is obtained by applying the prescription contained in [19, 20]. That is, we add to the action of general relativity, the variation of it and consider the variation $\delta g_{\mu\nu} = \tilde{g}_{\mu\nu}$ as a new field. Similarly, the symmetries we write below are obtained as variation of the infinitesimal general coordinate transformations where the variation of the infinitesimal parameter $\delta\xi_0^\rho = \xi_1^\rho$ is the infinitesimal parameter of the new transformation $\delta$. The last term in (1) is needed to implement the condition $T^{\mu\nu}_{;\nu} = 0$ as an equation of motion in order to implement the $\delta$ symmetry (2) off shell. This term is not needed in vacuum (see Secs. 11–13).

Action (1) is invariant under the following transformations ($\delta$):

$$\begin{aligned} \delta g_{\mu\nu} &= g_{\mu\rho}\xi^\rho_{0,\nu} + g_{\nu\rho}\xi^\rho_{0,\mu} + g_{\mu\nu,\rho}\xi_0^\rho = \xi_{0\mu;\nu} + \xi_{0\nu;\mu}, \\ \delta\tilde{g}_{\mu\nu}(x) &= \xi_{1\mu;\nu} + \xi_{1\nu;\mu} + \tilde{g}_{\mu\rho}\xi^\rho_{0,\nu} + \tilde{g}_{\nu\rho}\xi^\rho_{0,\mu} + \tilde{g}_{\mu\nu,\rho}\xi_0^\rho, \\ \delta\lambda_\mu &= -\xi_{1\mu} + \lambda_\rho\xi^\rho_{0,\mu} + \lambda_{\mu,\rho}\xi_0^\rho. \end{aligned} \tag{2}$$

From now on, we will fix the gauge $\lambda_\mu = 0$. This gauge preserves general coordinate transformations but fixes completely the extra symmetry with parameter $\xi_{1\mu}$.

*Equations of motion.* Varying $g_{\mu\nu}$, we get

$$S^{\gamma\sigma} + \frac{1}{2}(R\tilde{g}^{\gamma\sigma} - g_{\mu\nu}\tilde{g}^{\mu\nu}R^{\gamma\sigma}) - \frac{1}{2}g^{\gamma\sigma}\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\nabla_\nu\tilde{g}^{\mu\nu}\right)_{,\mu} + \frac{1}{4}g^{\gamma\sigma}\frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\alpha\beta}\nabla_\beta(g_{\mu\nu}\tilde{g}^{\mu\nu})\right)_{,\alpha} = \kappa\frac{\delta T_{\mu\nu}}{\delta g_{\gamma\sigma}}\tilde{g}^{\mu\nu}, \tag{3}$$

where $S^{\gamma\sigma} = (U^{\sigma\beta\gamma\rho} + U^{\gamma\beta\sigma\rho} - U^{\sigma\gamma\beta\rho})_{;\rho\beta}$, $U^{\alpha\beta\gamma\rho} = (1/2)[g^{\gamma\rho}(\tilde{g}^{\beta\alpha} - (1/2)g^{\alpha\beta}\times g_{\mu\nu}\tilde{g}^{\mu\nu})]$.

Varying $\tilde{g}^{\mu\nu}$, we get the Einstein equation

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \kappa T_{\mu\nu} = 0. \tag{4}$$

Varying $\lambda_\mu$, we get: $T^{\mu\nu}_{;\nu} = 0$.

Covariant derivatives as well as raising and lowering of indices are defined using $g_{\mu\nu}$. Notice that outside the sources ($T_{\mu\nu} = 0$), a solution of (3) is $\tilde{g}^{\mu\nu} = \lambda g^{\mu\nu}$, for a constant $\lambda$, since $g^{\mu\nu}_{;\rho} = 0$ and $R_{\mu\nu} = 0$. We will have $\tilde{g}^{\mu\nu} = g^{\mu\nu}$, assuming that both fields satisfy the same boundary conditions far from the sources.

The equation for $\tilde{g}^{\mu\nu}$ is linear and of the second order in the derivatives.

## 2. PARTICLE MOTION IN THE GRAVITATIONAL FIELD

We are aware of the presence of the gravitational field through its effects on test particles. For this reason, here we discuss the coupling of a test particle to a background gravitational field, such that the action of the particle is invariant under (2).

In $\delta$ gravity, we postulate the following action for a test particle:

$$S_p = -m \int dt \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + \kappa_2' \int d^n y \sqrt{-g} \mathcal{T}_{\mu\nu} \left( \tilde{g}^{\mu\nu} + \lambda^{\mu;\nu} + \lambda^{\nu;\mu} \right),$$

where $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor of the test particle

$$\mathcal{T}_{\mu\nu}(y) = \frac{m}{2\sqrt{-g}} \int dt \frac{\dot{x}_\mu \dot{x}_\nu}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} \delta(y-x),$$

$\kappa_2' = \kappa_2 \kappa$ is a dimensionless constant. That is,

$$S_p = m \int \frac{dt}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} \left( g_{\mu\nu} + \frac{\kappa_2{}'}{2} \bar{g}_{\mu\nu} \right) \dot{x}^\mu \dot{x}^\nu, \tag{5}$$

where $\bar{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \lambda^{\mu;\nu} + \lambda^{\nu;\mu}$. Notice that $S_p$ is invariant under (2) and $t$-parameterizations.

From now on, we work in the gauge $\lambda_\mu = 0$.

Since far from the sources, we must have free particles in Minkowski space, i.e., $g_{\mu\nu} \sim \eta_{\mu\nu}, \tilde{g}_{\mu\nu} \sim \eta_{\mu\nu}$, it follows that we are describing the motion of a particle of mass $m' = m(1 + \kappa_2'/2')$.

Since in vacuum $\tilde{g}^{\mu\nu} = g^{\mu\nu}$, the equation of motion for test particles is the same as Einstein's. Moreover, the equation of motion is independent of the mass of the particle.

In order to include massless particles, we prefer to use the action [24]

$$L = \frac{1}{2} \int dt \Bigg( v m^2 - v^{-1} \left( g_{\mu\nu} + \kappa_2' \bar{g}_{\mu\nu} \right) \dot{x}^\mu \dot{x}^\nu + \\ + \frac{m^2 + v^{-2} \left( g_{\mu\nu} + \kappa_2' \bar{g}_{\mu\nu} \right) \dot{x}^\mu \dot{x}^\nu}{2 v^{-3} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} \left( m^2 + v^{-2} g_{\lambda\rho} \dot{x}^\lambda \dot{x}^\rho \right) \Bigg). \tag{6}$$

This action is invariant under reparameterizations:

$$x'(t') = x(t), \quad dt' v'(t') = dt v(t), \quad t' = t - \varepsilon(t). \tag{7}$$

The equation of motion for $v$ is

$$v = -\frac{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}{m}. \tag{8}$$

Replacing (8) into (6), we get back (5).

Let us consider first the massive case. Using (7) we can fix the gauge $v = 1$. Introducing $m\,dt = d\tau$, we get the action

$$L_1 = \frac{1}{2}m\int d\tau\Bigg(1 - (g_{\mu\nu} + \kappa_2\bar{g}_{\mu\nu})\,\dot{x}^\mu\dot{x}^\nu + \\ + \frac{1 + (g_{\mu\nu} + \kappa_2\bar{g}_{\mu\nu})\,\dot{x}^\mu\dot{x}^\nu}{2g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\left(1 + g_{\lambda\rho}\dot{x}^\lambda\dot{x}^\rho\right)\Bigg) \quad (9)$$

plus the constraint obtained from the equation of motion for $v$

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1. \quad (10)$$

From $L_1$ the equation of motion for massive particles is derived. We define: $\bar{\mathfrak{g}}_{\mu\nu} = g_{\mu\nu} + (\kappa_2'/2)\bar{g}_{\mu\nu}$,

$$\frac{d(\dot{x}^\mu\dot{x}^\nu\bar{\mathfrak{g}}_{\mu\nu}\dot{x}^\beta g_{\alpha\beta} + 2\dot{x}^\beta\bar{\mathfrak{g}}_{\alpha\beta})}{d\tau} - \frac{1}{2}\dot{x}^\mu\dot{x}^\nu\bar{\mathfrak{g}}_{\mu\nu}\dot{x}^\beta\dot{x}^\gamma g_{\beta\gamma,\alpha} - \dot{x}^\mu\dot{x}^\nu\bar{\mathfrak{g}}_{\mu\nu,\alpha} = 0. \quad (11)$$

We will discuss the motion of massive particles elsewhere.

The action for massless particles is

$$L_0 = \frac{1}{4}\int dt\left(-v^{-1}\left(g_{\mu\nu} + \kappa_2\bar{g}_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu\right). \quad (12)$$

In the gauge $v = 1$, we get

$$L_0 = -\frac{1}{4}\int dt\,(g_{\mu\nu} + \kappa_2\bar{g}_{\mu\nu})\,\dot{x}^\mu\dot{x}^\nu \quad (13)$$

plus the equation of motion for $v$ evaluated at $v = 1$: $(g_{\mu\nu} + \kappa_2'\bar{g}_{\mu\nu})\,\dot{x}^\mu\dot{x}^\nu = 0$.

So, the massless particle moves on a null geodesic of $\mathfrak{g}_{\mu\nu} = g_{\mu\nu} + \kappa_2'\bar{g}_{\mu\nu}$.

## 3. DISTANCES AND TIME INTERVALS

In this section, we define the measurement of time and distances in the model.

In GR, the geodesic equation preserves the proper time of the particle along the trajectory. Equation (11) satisfies the same property: Along the trajectory $\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}$ is constant. Therefore we define proper time using the original metric $g_{\mu\nu}$,

$$d\tau = \sqrt{-g_{\mu\nu}\,dx^\mu\,dx^\nu} = \sqrt{-g_{00}}\,dx^0(dx^i = 0). \quad (14)$$

Following [25], we consider the motion of light rays along infinitesimally near trajectories and (14) to get the three-dimensional metric:

$$dl^2 = \gamma_{ij}\,dx^i\,dx^j, \quad \gamma_{ij} = \frac{g_{00}}{\mathfrak{g}_{00}}\left(\mathfrak{g}_{ij} - \frac{\mathfrak{g}_{0i}\mathfrak{g}_{0j}}{\mathfrak{g}_{00}}\right). \quad (15)$$

That is, we measure proper time using the metric $g_{\mu\nu}$ but the space geometry is determined by both metrics. In this model, massive particles do not move on geodesics of a four-dimensional metric. Only massless particles move on a null geodesic of $\mathfrak{g}_{\mu\nu}$. So, delta gravity is not a metric theory.

## 4. THE NEWTONIAN LIMIT

The motion of a nonrelativistic particle in a weak static gravitational field is obtained using

$$g_{\mu\nu} = \mathrm{diag}\,(-1 - 2U\epsilon,\, 1 - 2U\epsilon,\, 1 - 2U\epsilon,\, 1 - 2U\epsilon), \tag{16}$$

which solves the Einstein equations to the first order in $\epsilon$ if $\nabla^2 U = (1/2)\kappa\rho$.

The solution for $\tilde{g}_{\mu\nu}$ is

$$\tilde{g}_{\mu\nu} = \mathrm{diag}\,(\epsilon\tilde{U},\, 1 + \epsilon(\tilde{U} - 2U),\, 1 + \epsilon(\tilde{U} - 2U),\, 1 + \epsilon(\tilde{U} - 2U)). \tag{17}$$

Solving (3), to the first order in $\epsilon$ we get $\nabla^2 \tilde{U} = (1/2)\kappa\rho$.

To recover the Minkowsky metric far from the sources, $\rho \to 0$, we must require there: $U \to 0$, $\tilde{U} \to -\epsilon^{-1}$.

Equation (11) implies $d^2x^i/dt^2 = -\phi_{,i}$ with $\phi = U - \kappa_2'(2U + \tilde{U})$.

The Newtonian potential satisfies $\nabla^2\phi = (\kappa/2)(1 - 3\kappa_2')\rho$, $|\kappa_2'| \ll 1$. The whole effect is a small redefinition of the Newton constant.

Gravitational red shift experiments can be used to put bounds on $\kappa_2'$. According to (14), the shift in frequency of a source located at $x_1$, compared to the same source located at $x_2$, due to the change in gravitational potential is: $(\nu_2 - \nu_1)/\nu_1 = \phi_N(x_2) - \phi_N(x_1)$, where $\phi_N$ is the usual Newtonian potential, computed with $\kappa$ as the Newton constant. From [26], we get $\Delta\nu/\nu = (1 + (2.5 \pm 70)\cdot 10^{-6})(\varphi_S - \varphi_E + \ldots)$, where $\varphi_S$ is the gravitational potential at the spacecraft position and $\varphi_E$ is the gravitational potential on the Earth. $\ldots$ accounts for additional effects not related to the gravitational potential. We can ascribe the uncertainty of the experiment to $\kappa_2'$, to get the bound

$$|\kappa_2'| < 24 \cdot 10^{-6}.$$

This bound is conservative because the Newton constant itself has a larger error [27]: $G = (6.67428 \pm 0.00067)\cdot 10^{-11}$ m$^3$/kGs$^2$.

In our description of the evolution of the Universe, the value of $\kappa_2'$ is not important, so we will keep it arbitrary for the time being.

## 5. FRIEDMAN–ROBERTSON–WALKER (FRW) METRIC

In this section, we discuss the equations of motion for the Universe described by the FRW metric. We use spatial curvature equal to zero to agree with cosmological observations.

Here we will deal only with a perfect fluid, since rotational and translational invariance implies that the energy-momentum tensor of the Universe has this form. The energy-momentum tensor for a perfect fluid is [8]

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)U_\mu U_\nu, g^{\mu\nu}U_\mu U_\nu = -1, \tag{18}$$

then

$$\frac{\delta T_{\mu\nu}}{\delta g_{\gamma\sigma}}\tilde{g}^{\mu\nu} = p\tilde{g}^{\gamma\sigma} + \frac{1}{2}(p+\rho)(U^\gamma U_\nu \tilde{g}^{\sigma\nu} + U^\sigma U_\nu \tilde{g}^{\gamma\nu}). \tag{19}$$

In this case, assuming flat three-dimensional metric:

$$\begin{aligned}
-ds^2 &= dt^2 - R(t)^2\left\{dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right\},\\
-d\tilde{s}^2 &= \tilde{A}(t)dt^2 - \tilde{B}(t)\left\{dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right\}.
\end{aligned}$$

Using (11), (14), we can check that these are co-mobile coordinates and the proper time interval $d\tau$ for a co-moving clock is just $dt$, so $t$ is the time measured in the rest frame of a co-moving clock. Equations (3), (19) give

$$\begin{aligned}
-\dot{R}\dot{\tilde{B}} - \frac{1}{2}pR\tilde{B} + \frac{1}{2}R^{-1}\dot{R}^2\tilde{B} - \frac{1}{6}\rho R^3\tilde{A} + \frac{3}{2}R\dot{R}^2\tilde{A} = 0,\\
-p\tilde{B} - 2\ddot{\tilde{B}} - R^{-2}\dot{R}^2\tilde{B} + 2R^{-1}\ddot{R}\tilde{B} + 2R^{-1}\dot{R}\dot{\tilde{B}} +\\
+\rho R^2\tilde{A} + \dot{R}^2\tilde{A} + 2R\dot{R}\dot{\tilde{A}} + 2R\tilde{A}\ddot{R} = 0.
\end{aligned} \tag{20}$$

Einstein's equations are

$$\frac{3((d/dt)R)^2}{R^2} = \kappa\rho, \quad 2R\left(\frac{d^2}{dt^2}R\right) + \left(\frac{d}{dt}R\right)^2 = -\kappa R^2 p.$$

We use the equation of state $p = w\rho$, to get, for $w \neq -1$:

$$\begin{aligned}
R &= R_0 t^{\frac{2}{3(1+w)}}, \quad \tilde{A} = 3wl_2 t^{(\frac{w-1}{w+1})},\\
\tilde{B} &= R_0^2 l_2 t^b, \quad b = \frac{4}{3w+3} + \frac{w-1}{w+1},
\end{aligned} \tag{21}$$

$l_2$ is a free parameter.

## 6. RED SHIFT

To make the usual connection between red shift and the scale factor, we consider light waves traveling from $r = r_1$ to $r = 0$, along the $r$ direction with fixed $\theta, \phi$. Photons move on a null geodesic of $\mathfrak{g}$:

$$0 = -(1 + \kappa_2' \tilde{A})\, dt^2 + (R^2 + \kappa_2' \tilde{B})(dr^2 + r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2), \tag{22}$$

so,

$$\int_{t_1}^{t_0} dt \sqrt{\frac{1 + \kappa_2' tA}{R^2 + \kappa_2' tB}} = r_1. \tag{23}$$

A typical galaxy will have fixed $r_1$, $\theta_1$, $\phi_1$. If a second wave crest is emitted at $t = t_1 + \delta t_1$ from $r = r_1$, it will reach $r = 0$ at $t_0 + \delta t_0$, where

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} dt \sqrt{\frac{1 + \kappa_2' tA}{R^2 + \kappa_2' tB}} = r_1.$$

Therefore, for $\delta t_1, \delta t_0$ small, which is true for light waves, we have

$$\delta t_0 \sqrt{\frac{1 + \kappa_2' tA}{R^2 + \kappa_2' tB}}(t_0) = \delta t_1 \sqrt{\frac{1 + \kappa_2' tA}{R^2 + \kappa_2' tB}}(t_1). \tag{24}$$

Introduce

$$\tilde{R}(t) = \sqrt{\frac{R^2 + \kappa_2' tB}{1 + \kappa_2' tA}}(t).$$

We get $\dfrac{\delta t_0}{\delta t_1} = \dfrac{\tilde{R}(t_0)}{\tilde{R}(t_1)}$. A crucial point is that, according to equation (14), $\delta t$ measure the change in proper time. That is, $\dfrac{\nu_1}{\nu_0} = \dfrac{\tilde{R}(t_0)}{\tilde{R}(t_1)}$, where $\nu_0$ is the light frequency detected at $r = 0$ corresponding to a source emission at frequency $\nu_1$. Or in terms of the red shift parameter $z$, defined as the fractional increase of the wavelength $\lambda$:

$$z = \frac{\tilde{R}(t_0)}{\tilde{R}(t_1)} - 1 = \frac{\lambda_0 - \lambda_1}{\lambda_1}. \tag{25}$$

We see that $\tilde{R}$ replaces the usual scale factor $R$ in the computation of $z$.

## 7. LUMINOSITY DISTANCE

Let us consider a mirror of radius $b$ that is receiving light from a distant source. The photons that reach the mirror are inside a cone of half-angle $\varepsilon$ with origin at the source.

Let us compute $\varepsilon$. The light path of rays coming from a far away source at $\mathbf{x}_1$ is given by $\mathbf{x}(\rho) = \rho\hat{n} + \mathbf{x}_1$; $\rho > 0$ is a parameter and $\hat{n}$ is the direction of the light ray. The path reaches us at $\mathbf{x} = 0$ for $\rho = |\mathbf{x}_1| = r_1$. Write $\hat{n} = -\hat{x}_1 + \boldsymbol{\varepsilon}$. Since $\hat{n}$, $\hat{x}_1$ have modulus 1, $\varepsilon = |\boldsymbol{\varepsilon}| \ll 1$ is precisely the angle between $-\mathbf{x}_1$ and $\hat{n}$ at the source. The impact parameter is the proper distance of the path from the origin, when $\rho = |\mathbf{x}_1|$. The proper distance is determined by the 3-dimensional metric (15). That is, $b = \tilde{R}(t_0)r_1\theta = \tilde{R}(t_0)r_1\varepsilon$, i.e., $\varepsilon = \dfrac{b}{\tilde{R}(t_0)r_1}$.

Then the solid angle of the cone is $\pi\varepsilon^2 = \dfrac{A}{r_1^2\tilde{R}(t_0)^2}$, where $A = \pi b^2$ is the proper area of the mirror. The fraction of all isotropically emitted photons that reach the mirror is $f = \dfrac{A}{4\pi r_1^2\tilde{R}(t_0)^2}$. Each photon carries an energy $h\nu_1$ at the source and $h\nu_0$ at the mirror. Photons emitted at intervals $\delta t_1$ will arrive at intervals $\delta t_0$. We have $\dfrac{\nu_1}{\nu_0} = \dfrac{\tilde{R}(t_0)}{\tilde{R}(t_1)}$, $\dfrac{\delta t_0}{\delta t_1} = \dfrac{\tilde{R}(t_0)}{\tilde{R}(t_1)}$. Therefore the power at the mirror is $P_0 = L\dfrac{\tilde{R}(t_1)^2}{\tilde{R}(t_0)^2}f$, where $L$ is the luminosity of the source. The apparent luminosity is $l = \dfrac{P_0}{A} = L\dfrac{\tilde{R}(t_1)^2}{\tilde{R}(t_0)^2}\dfrac{1}{4\pi r_1^2\tilde{R}(t_0)^2}$. In Euclidean space, the luminosity decreases with distance $d$ according to $l = L/4\pi d^2$. This permits one to define the luminosity distance: $d_L = \sqrt{L/4\pi l} = \tilde{R}(t_0)^2\dfrac{r_1}{\tilde{R}(t_1)}$. Using (23) we can write this in terms of the red shift:

$$d_L = (1+z)\int_0^z \frac{dz'}{\tilde{H}(z')}, \quad \tilde{H} = \frac{\dot{\tilde{R}}}{\tilde{R}}. \tag{26}$$

## 8. SUPERNOVA Ia DATA

The supernova Ia data gives $m$ (apparent or effective magnitude) as a function of $z$. This is related to distance $d_L$ by $m = M + 5\,\log\,(d_L/10pc)$. Here $M$ is common to all supernova, and $m$ changes with $d_L$ alone.

We compare $\delta$ gravity to general relativity (GR) with a cosmological constant:

$$H^2 = H_0^2(\Omega_m(1+z)^3 + (1-\Omega_m)), \quad \Omega_\Lambda = 1 - \Omega_m.$$

Notice that $\tilde{A} = 0$ for $w = 0$ in (21). So, it seems that we cannot fit the supernova data. However, $w = 0$ is not the only component of the Universe. The massless particles that decoupled earlier still remain. It means that the true $w$ is between $0 \leqslant w < 1/3$, but very close to $w = 0$. So, we will fit the data with $w = 0.1$, 0.01, 0.001 and see how sensitive the predictions are to the value of $w$.

Using the data from Essence [28], we notice that $R^2$ test changes very little for the chosen sequence of $w$s. Each fit determines the best $l_2$ for a given $w$. In this way, we see that $l_2$ scales like $l_2 \sim a/3w$, $a$ being independent of $w$. As an approximation to the limit $w = 0$, we get

$$\tilde{R}(t) = R(t)\frac{\sqrt{a}}{\sqrt{a-t}}, \tag{27}$$

$\sqrt{1/3w}$ renormalizes the derivative of $\tilde{R}$ at $t = 0$. It is not divergent, because for $t \to 0$, $w \to 1/3$, $a$ is a free parameter determined by the best fit to the data.

Of course, the complete model must include the contribution of normal matter ($w = 0$) plus relativistic matter ($w = 1/3$). But, at later times, the data should tend to (27).

Let us fit the data to the simple scaling model (27).

We get

$\Omega_m = 0.22 \pm 0.03, M = 43.29 \pm 0.03$, $\chi^2$(per point) $= 1.0328$, general relativity;

$a = 2.21 \pm 0.12, M = 43.45 \pm 0.06$, $\chi^2$(per point) $= 1.0327$, delta gravity;

$\delta$ gravity with nonrelativistic (NR) matter alone gives a fit to the data as good as GR with NR matter plus a cosmological constant.

According to the fit to data, a Big Rip will happen at $t = 2.21049$ in unities of $t_0$ (today). It is a similar scenario as in [23].

Finally, we want to point out that since for $t \to 0$, we have $w \to 1/3$, then $\tilde{R}(t) = R(t)$. Therefore the accelerated expansion is slower than (27) when we include both matter and radiation in the model.

## 9. DARK MATTER

Several years ago, the astronomers were able to measure the speed of individual stars around the center of the galaxies [3]. Surprisingly, such speeds $v(r)$ as a function of the distance $r$ to the galactic center, did not follow the Kepler law. Most of the galactic mass was assumed to be in the form of stars, which concentrate near the galactic center. So, the expectation was that the speed of rotation of stars far from the center will decrease as $r^{-1/2}$. The observation shows that $v(r)$ approaches a constant $v_0$ far from the center. A natural way to explain the observed velocities was to assume the existence of extra mass that cannot be seen but interacts gravitationally (dark matter, DM). Additional support for the existence of DM comes from the study of galaxy stability against gravitational collapse: The form of the galaxy that we can see (luminous part) is not gravitationally stable unless we assume the existence of a spherically symmetric halo that we cannot see.

From observation, we get that 80–90% of the galactic mass is DM.

However, the physical nature of dark matter is not known yet.

Most people think that DM is made of particles that interact weekly with normal matter. Until recently the standard cosmological scenario was the so-called $\Lambda$ CDM model. That is, the evolution of the Universe is governed by a cosmological constant $\Lambda$ that produces the accelerated expansion of the Universe (dark energy) [4, 29] and nonrelativistic particles (cold DM) that were the seeds of the galaxies. However, recent simulations of the neighborhood of the Milky Way [30] have challenged the CDM paradigma. They proposed instead that DM particles are warm, with a rest mass of 1 keV.

There is an alternative to DM that is gaining some support: MOND [6]. The main idea of MOND involves a modification of Newton Second Law for accelerations below a critical acceleration $a_0$. In this way, the constant speed $v_0$ of individual stars far from the galactic center is explained. Therefore, according to MOND, DM particles do not exist.

Since DM particles have not been detected yet and even their existence is challenged in some models, in this section we want to explore a different scenario to understand the properties of galaxies. Preliminary studies of the solutions of DG in vacuum have shown that it contains extra degrees of freedom that produces an additional Newtonian potential far from the sources.

In fact, far from a source the gravitational field corresponds to the Schwarzschild solution: point-like source, spherically symmetric.

The exact solution is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\dfrac{a}{r}\right) & 0 & 0 & 0 \\ 0 & \dfrac{1}{1-\dfrac{a}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin(\theta)^2 \end{pmatrix}, \tag{28}$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -\left(1-\dfrac{a}{r}+\dfrac{ba}{r}\right) & 0 & 0 & 0 \\ 0 & \dfrac{1}{1-\dfrac{a}{r}} - \dfrac{ab}{r\left(1-\dfrac{a}{r}\right)^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin(\theta)^2 \end{pmatrix}. \tag{29}$$

Boundary condition is $g_{\mu\nu} \sim \eta_{\mu\nu}\tilde{g}^{\mu\nu} \sim \eta^{\mu\nu}$ for $r \to \infty$. Notice that still there are two arbitrary constants.

**9.1. Newtonian Potential for Massive Particles**

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{00} = \frac{a}{r}, a = 2M,$$
$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad \tilde{h}_{00} = \frac{a(1-b)}{r}, \quad a(1-b) = 2M'.$$

The Newtonian potential is

$$\phi = -\left(\left(\frac{1}{1+(\kappa_2/2)'} - \frac{1}{2}\right) h_{00} + \frac{(\kappa_2/2)'}{1+(\kappa_2/2)'}\tilde{h}_{00}\right) = -\frac{M_T}{r}.$$

So, the total mass of the source is

$$M_T = M - \frac{\kappa_2' bM}{1+(\kappa_2/2)'}.$$

So, the dark matter mass is

$$M_{\mathrm{DM}} = -\frac{\kappa_2' bM}{1+(\kappa_2/2)'}. \tag{30}$$

$M$ is the mass coming from the fluid density in Einstein equations. $b$ is a new constant to accomodate DM.

**9.2. Photons and Gravitational Lensing.** The photon trajectory is given by

$$\left[-\left(1-\frac{a}{r}\right) - \kappa_2'\left(1-\frac{a}{r}+\frac{ba}{r}\right)\right] dt^2 +$$
$$+\left[\frac{1}{1-a/r} + \kappa_2'\left(\frac{1}{1-a/r} - \frac{ab}{r(1-a/r)^2}\right)\right] dr^2 = 0,$$
$$\left[-1+\frac{1}{r}\left(a - \frac{\kappa_2' ba}{1+\kappa_2'}\right)\right] dt^2 + \left[1+\frac{1}{r}\left(a-\frac{\kappa_2' ba}{1+\kappa_2'}\right)\right] dr^2 = 0.$$

So, according to photons:

$$M_T = M - \frac{\kappa_2' bM}{1+\kappa_2'}. \tag{31}$$

Notice that photons and massive particles see different $M_T$, but since $\kappa_2'$ is very small, this difference is hard to detect.

To determine if $\delta$ gravity can describe dark matter, we must be able to compute the speed of stars rotating around the center of galaxies. This work is in progress.

## 10. EFFECTIVE ACTION FOR A GENERIC $\delta$ MODEL

In this section, we derive the exact effective action for a generic $\delta$ model.

We start by considering a model based on a given action $S_0[\phi_I]$, where $\phi_I$ are generic fields, then we add to it a piece that is equal to a $\delta$ variation with respect to the fields, and we let $\delta\phi_J = \tilde{\phi}_J$, so that we have

$$S[\phi, \tilde{\phi}] = S_0[\phi] + \kappa_2 \int d^4x \frac{\delta S_0}{\delta \phi_I(x)}[\phi]\tilde{\phi}_I(x), \tag{32}$$

with $\kappa_2$ — an arbitrary constant, and the indexes $I$ can represent any kind of indexes. For more details of the definition of $\delta$, please, see Appendix A. This new defined action shows the standard structure used to define any modified element or function for $\delta$-type models, for example, the gauge fixing and Faddeev–Popov. Next, we verify that this form of action is indeed the correct one for $\delta$ gravity and so is invariant to the new general coordinate transformation.

We saw that the classical action for a $\delta$ model is (32). This in turn implies that we now have two fields to be integrated in the generating functional of Green functions:

$$\begin{aligned} Z(j,\tilde{j}) = \mathrm{e}^{iW(j,\tilde{j})} = \int \mathcal{D}\phi\,\mathcal{D}\tilde{\phi}\times \\ \times \exp\left[i\left(S_0 + \int d^Nx \frac{\delta S_0}{\delta\phi_I}\tilde{\phi}_I + \int d^Nx (j_I(x)\,\phi_I(x) + \tilde{j}_I(x)\tilde{\phi}_I(x))\right)\right]. \end{aligned} \tag{33}$$

We can readily appreciate that, because of the linearity of the exponent on $\tilde{\phi}_J$, what we have is the integral representation of a Dirac delta function, so that our modified model, once integrated over $\tilde{\phi}_J$, gives a model with a constraint making the original model live on shell:

$$Z(j,\tilde{j}) = \int \mathcal{D}\phi \exp\left[i\left(S_0 + \int d^Nx j_I(x)\phi_I(x)\right)\right]\delta\left(\frac{\delta S_0}{\delta\phi_I(x)} + \tilde{j}_I(x)\right). \tag{34}$$

A first glance at equation (34) could lead us to believe that this model is purely classical. But we can see by doing a simple analysis that this is not so. For this, we follow [32] (see also [33]).

Let $\varphi_I$ solve the classical equation of motion:

$$\left.\frac{\delta S_0}{\delta\phi_I(x)}\right|_{\varphi_I} + \tilde{j}_I(x) = 0. \tag{35}$$

We have

$$\delta\left(\frac{\delta S_0}{\delta\phi_I(x)} + \tilde{j}_I(x)\right) = \overset{-1}{\det}\left(\left.\frac{\delta^2 S_0}{\delta\phi_I(x)\,\delta\phi_J(y)}\right|_{\varphi_I}\right)\delta(\phi_I - \varphi_I). \tag{36}$$

Therefore

$$Z(j,\tilde{j}) = \int \mathcal{D}\phi \, \exp\left[i(S_0 + \int d^N x j_I(x)\phi_I(x))\right] \delta\left(\frac{\delta S_0}{\delta \phi_I(x)} + \tilde{j}_I(x)\right),$$
$$= \exp\left[i(S_0(\varphi) + \int d^N x j_I(x)\varphi_I(x))\right] \overset{-1}{\det}\left(\left.\frac{\delta^2 S_0}{\delta\phi_I(x)\delta\phi_J(y)}\right|_{\varphi_I}\right). \tag{37}$$

Notice that $\varphi$ is a functional of $\tilde{j}$. The generating functional of connected Green functions is

$$W(j,\tilde{j}) = S_0(\varphi) + \int d^N x j_I(x)\varphi_I(x) + i\,\mathrm{Tr}\left(\log\left(\left.\frac{\delta^2 S_0}{\delta\phi_I(x)\delta\phi_J(y)}\right|_{\varphi_I}\right)\right). \tag{38}$$

Define

$$\Phi_I(x) = \frac{\delta W}{\delta j_I(x)},$$
$$= \varphi_I(x)$$
$$\tilde{\Phi}_I(x) = \frac{\delta W}{\delta \tilde{j}_I(x)}.$$

The effective action is defined by

$$\Gamma(\Phi,\tilde{\Phi}) = W(j,\tilde{j}) - \int d^N x \left\{ j_I(x)\Phi_I(x) + \tilde{j}_I(x)\,\tilde{\Phi}_I(x)\right\}.$$

We get, using equations (35) and (38):

$$\Gamma(\Phi,\tilde{\Phi}) = S_0(\Phi) + \int d^N x \frac{\delta S_0}{\delta\Phi_I(x)}\tilde{\Phi}_I(x) + i\,\mathrm{Tr}\left(\log\left(\frac{\delta^2 S_0}{\delta\Phi_I(x)\delta\Phi_J(y)}\right)\right). \tag{39}$$

This is the exact effective action for $\delta$ theories. In this proof, it is assumed that all the relevant steps for fixing the gauge have been made in (33), so $S_0$ includes the gauge fixing and Faddeev–Popov Lagrangian. More details can be found in [21].

Comparing equation (16.42) of [32] with equation (39), we see that the one-loop contribution to the effective action of $\delta$ theories is exact, and the $\delta$ modified model lives only to one loop because higher corrections simply do not exist. Finally, it is twice the one-loop contribution of the original theory from which the $\delta$ model was derived. This results from having doubled the number of degrees of freedom. We also see that this term does not depend on the $\tilde{\phi}_I$ fields.

## 11. DIVERGENT PART OF THE EFFECTIVE ACTION IN DELTA GRAVITY IN VACUUM AND WITHOUT A COSMOLOGICAL CONSTANT

In the previous section, we demonstrated that the quantum corrections to the effective action do not depend on the tilde fields, in this case $\tilde{g}_{\mu\nu}$. On the other hand, renormalization theory tells us that its divergent corrections can only be local terms. So, by power counting and invariance of the background field effective action under general coordinate transformations, we know that the divergent part to $L$ loops is [21, 34, 35]

$$\Delta S^L_{\text{div}} \propto \int d^4x\sqrt{-g}\, R^{L+1}, \tag{40}$$

where $R^{L+1}$ is any scalar contraction of $(L+1)$ Riemann tensors. As our model lives only to one loop, then

$$L^{\text{div}}_Q = \sqrt{-g}(a_1 R^2 + a_2 R_{\alpha\beta}R^{\alpha\beta}). \tag{41}$$

We do not use $R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda}$ because we have the topological identity in four dimensions:

$$\sqrt{-g}\left(R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda} - 4R_{\alpha\beta}R^{\alpha\beta} + R\right) = \text{total derivative}. \tag{42}$$

The divergent part of the effective action in our model (i.e., $a_1$ and $a_2$ in (41)), was calculated in [21]. It is

$$\begin{aligned} L^{\text{div}}_{Q,\text{grav}} &= \sqrt{-g}\frac{\hbar c}{\varepsilon}\left(\frac{7}{12}R^2 + \frac{7}{6}R_{\alpha\beta}R^{\alpha\beta}\right), \\ L^{\text{div}}_{Q,\text{ghost}} &= -2\sqrt{-g}\frac{\hbar c}{\varepsilon}\left(\frac{17}{60}R^2 + \frac{7}{30}R_{\alpha\beta}R^{\alpha\beta}\right), \\ L^{\text{div}}_{Q} &= \sqrt{-g}\frac{\hbar c}{\varepsilon}\left(\frac{1}{60}R^2 + \frac{7}{10}R_{\alpha\beta}R^{\alpha\beta}\right), \end{aligned} \tag{43}$$

with $\varepsilon = 8\pi^2(N-4)$. $N$ is the dimension of space-time in dimensional regularization. When we compare with the usual result in gravitation [34, 36], we can see that we obtain twice the divergent term of general relativity. Divergences also double in Yang–Mills [37].

Moreover, since Einstein's equations of motion are exactly valid at the quantum level,

$$\left(\frac{\delta\Gamma(g,\tilde{g})}{\delta\tilde{g}_{\mu\nu}}\right) = R^{\mu\nu} = 0, \tag{44}$$

where $\Gamma(g,\tilde{g})$ is the effective action in the background field method. It follows that the contribution of (43) to the equation of motion vanishes:

$$\hbar c\left[\frac{\sqrt{-g}}{\varepsilon}\left(\frac{1}{2}g^{\mu\nu}\left(\frac{1}{60}R^2+\frac{7}{10}R_{\alpha\beta}R^{\alpha\beta}\right)+\right.\right.$$
$$\left.\left.+\ \frac{1}{30}R\frac{\delta R}{\delta g_{\mu\nu}}+\frac{7}{10}R_{\alpha\beta}\frac{\delta R^{\alpha\beta}}{\delta g_{\mu\nu}}+\frac{7}{10}R^{\alpha\beta}\frac{\delta R_{\alpha\beta}}{\delta g_{\mu\nu}}\right)\right]_{R_{\alpha\beta}=0}=0. \quad (45)$$

Therefore, $\delta$ gravity is a finite model of gravitation if we do not have matter and a cosmological constant. The finiteness of our model implies that Newton's constant does not run at all, neither with time nor energy scale, which would be supported by the very stringent experimental bounds set on its change [38, 39]. We must notice that this model is finite only in four dimensions because we need (42). Moreover, in more dimensions there could appear more terms in (41), that contains $R^{\mu_1\mu_2\cdots\mu_N}$ with $N$ — the dimension of space, that give a nonzero contribution to the equations of motion.

In spite of these apparent successes, there seems to be a problem with this model, namely is the possible existence of ghosts. This issue will be dealt with in the next section.

## 12. GHOSTS

In this section, we discuss the fact that our model has ghosts, as well as the lost of unitarity due to them. In order to proceed with this endeavor, we first write the action (1) in vacuum, i.e., $\mathcal{L}_M=0, T_{\mu\nu}=0, \lambda^\mu=0$, and putting the backgrounds equal to the Minkowski metric $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ and $\tilde{g}_{\mu\nu}=\eta_{\mu\nu}+\tilde{h}_{\mu\nu}$, and calculate from it the canonical conjugate momenta to the quantum fields. It is important to notice that a gauge has been chosen (49), (50). Thus, it is possible to show that under these conditions and in this gauge, the quantum fields obey the wave equation, and an expansion in plane waves is possible where the Fourier coefficients are promoted to creation and annihilation operators much in the same way as can be done for the electromagnetic potential. We use the canonical commutation relations for fields and momenta to work out the corresponding canonical commutation relations for the creation and annihilation operators. We also show first the Hamiltonian in terms of fields and momenta and then in terms of annihilation and creation operators.

We obtain

$$S[h,\tilde{h}]=$$
$$=-\frac{1}{2\kappa}\int d^4xP^{((\alpha\beta)(\mu\nu))}\left(\frac{(1-\kappa_2)}{2}\partial_\rho h_{\alpha\beta}\partial^\rho h_{\mu\nu}+\kappa_2\partial_\rho\tilde{h}_{\alpha\beta}\partial^\rho h_{\mu\nu}\right), \quad (46)$$

where now

$$P^{((\alpha\beta)(\mu\nu))} = \frac{1}{4}\left(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu}\right), \tag{47}$$

and the equations of motion for the fields are

$$\partial^2 h_{\mu\nu} = 0, \quad \partial^2 \tilde{h}_{\mu\nu} = 0, \tag{48}$$

with $\partial^2 = \eta^{\rho\lambda}\partial_\rho\partial_\lambda$. This corresponds to the wave equation with energy $E_{\mathbf{p}} = |\mathbf{p}|$. Here we notice that in order to obtain these equations, we have made use of a particular gauge fixing term

$$h^{\nu}_{\mu,\nu} - \frac{1}{2}h^{\nu}_{\nu,\mu} = 0, \tag{49}$$

$$\tilde{h}^{\nu}_{\mu,\nu} - \frac{1}{2}\tilde{h}^{\nu}_{\nu,\mu} = 0. \tag{50}$$

It is well known that for a diffeomorfism-invariant Lagrangian, the canonical Hamiltonian is zero. This is so in delta gravity as well as in general relativity: the total Hamiltonian is a linear combination of the first-class constraints (see [40]). After gauge fixing, the Hamiltonian is

$$H = \int d^3x \left(\frac{2\kappa}{\kappa_2}P^{-1}_{((\alpha\beta)(\mu\nu))}\left(\tilde{\Pi}^{\alpha\beta}\Pi^{\mu\nu} - \frac{(1-\kappa_2)}{2\kappa_2}\tilde{\Pi}^{\alpha\beta}\tilde{\Pi}^{\mu\nu}\right)\right) + \\ + \int d^3x \left(\frac{\kappa_2}{2\kappa}P^{((\alpha\beta)(\mu\nu))}\left(\partial_i\tilde{h}_{\alpha\beta}\partial_i h_{\mu\nu} + \frac{(1-\kappa_2)}{2\kappa_2}\partial_i h_{\alpha\beta}\partial_i h_{\mu\nu}\right)\right), \tag{51}$$

with

$$P^{-1}_{((\alpha\beta)(\mu\nu))} = \eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\beta}\eta_{\mu\nu} = 4P_{((\alpha\beta)(\mu\nu))}, \tag{52}$$

and where the conjugate momenta are

$$\Pi^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\dot{h}_{\mu\nu}} = \frac{1}{2\kappa}P^{((\alpha\beta)(\mu\nu))}\left((1-\kappa_2)\dot{h}_{\alpha\beta} + \kappa_2\dot{\tilde{h}}_{\alpha\beta}\right), \tag{53}$$

$$\tilde{\Pi}^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\dot{\tilde{h}}_{\mu\nu}} = \frac{\kappa_2}{2kappa}P^{((\alpha\beta)(\mu\nu))}\dot{h}_{\alpha\beta}. \tag{54}$$

We can write our fields $h$ and $\tilde{h}$ in the following way:

$$h_{\mu\nu}(\mathbf{x},t) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\mathbf{p}}}} \times$$
$$\times \left[\chi^{(AB)}_{(\mu\nu)}(\mathbf{p}) a_{(AB)}(\mathbf{p})\, \mathrm{e}^{ip\cdot x} + \chi^{(AB)}_{(\mu\nu)}(\mathbf{p}) a^{+}_{(AB)}(\mathbf{p})\, \mathrm{e}^{-ip\cdot x}\right]|_{p_0=E_{\mathbf{p}}}, \tag{55}$$
$$\tilde{h}_{\mu\nu}(\mathbf{x},t) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\mathbf{p}}}} \times$$
$$\times \left[\chi^{(AB)}_{(\mu\nu)}(\mathbf{p}) \tilde{a}_{(AB)}(\mathbf{p})\, \mathrm{e}^{ip\cdot x} + \chi^{(AB)}_{(\mu\nu)}(\mathbf{p}) \tilde{a}^{+}_{(AB)}(\mathbf{p})\, \mathrm{e}^{-ip\cdot x}\right]|_{p_0=E_{\mathbf{p}}},$$

where $\chi^{(AB)}_{(\mu\nu)}(\mathbf{p})$ is a polarization tensor, and $a_{(AB)}(\mathbf{p})$ and $\tilde{a}_{(AB)}(\mathbf{p})$ are promoted to annihilation operators when we quantize it; $a^{+}_{(AB)}(\mathbf{p})$ and $\tilde{a}^{+}_{(AB)}(\mathbf{p})$ correspond to the creation operators; $A$ and $B$ are indices of polarization that work like Lorentz indices, that is, they go from 0 to 3 and are moved up and down with $\eta^{AB}$. As these indices are presented symmetrically, we will have ten polarization tensors, enough to make a complete basis. For quantization of the model, we must impose the canonical commutation relations, the only nonvanishing commutators are

$$[h_{\mu\nu}(t,\mathbf{x}), \Pi^{\alpha\beta}(t,\mathbf{y})] = [\tilde{h}_{\mu\nu}(t,\mathbf{x}), \tilde{\Pi}^{\alpha\beta}(t,\mathbf{y})] = i\delta^{\alpha\beta}_{\mu\nu}\delta^3(\mathbf{x}-\mathbf{y}). \tag{56}$$

When expressed using (55) the nonvanishing commutators are

$$[a^{AB}(\mathbf{p}), \tilde{a}^{+}_{CD}(\mathbf{p}')] = [\tilde{a}^{AB}(\mathbf{p}), a^{+}_{CD}(\mathbf{p}')] = \frac{4\kappa}{\kappa_2}\delta^{AB}_{CD}\delta^3(\mathbf{p}-\mathbf{p}'), \tag{57}$$
$$[\tilde{a}^{AB}(\mathbf{p}), \tilde{a}^{+}_{CD}(\mathbf{p}')] = -\frac{4\kappa(1-\kappa_2)}{\kappa_2^2}\delta^{AB}_{CD}\delta^3(\mathbf{p}-\mathbf{p}'). \tag{58}$$

There is a slight subtlety in calculating the above commutators. Basically, the expression that appears at one stage of the calculus is

$$\sum_{ABCD} \chi^{(AB)}_{(\mu\nu)} P^{(\alpha\beta)}_{(\gamma\epsilon)} \chi^{(\gamma\epsilon)}_{CD} = \sum_{ABCD} \chi^{(AB)}_{(\mu\nu)} \frac{1}{2}\delta^{(\alpha\beta)}_{(\gamma\epsilon)} \chi^{(\gamma\epsilon)}_{CD} - \frac{1}{4}\eta^{\alpha\beta}\chi^{(AB)}_{(\mu\nu)}\,\mathrm{Tr}\,(\chi), \tag{59}$$

and since we have the completeness relation

$$\sum_{ABCD} \chi^{(AB)}_{(\mu\nu)} \chi^{(\alpha\beta)}_{(CD)} \delta^{(CD)}_{(AB)} = \delta^{(\alpha\beta)}_{(\mu\nu)}, \tag{60}$$

we must impose $\mathrm{Tr}\,(\chi) = 0$, which in turn means that $\mathrm{Tr}\,(h) = \mathrm{Tr}\,(\tilde{h}) = 0$. This can always be done, because the gauge fixing being used does not fix the gauge freedom entirely, and this further condition can be imposed (see [41]).

The Hamiltonian expressed in terms of creation and annihilation operators is

$$H = \int \frac{d^3p}{4\kappa} E_{\mathbf{p}} \left( (1-\kappa_2) a^{+}_{AB} a^{AB} + \kappa_2 a^{+}_{AB} \tilde{a}^{AB} + \kappa_2 \tilde{a}^{+}_{AB} a^{AB} \right), \tag{61}$$

where we have subtracted an infinite constant. Looking at this Hamiltonian, we notice that it has cross-products of operators, which obscures its physical interpretation. Something analogous happens when we observe the commutators (57) and (58), and so it is difficult to define their action over states. Because of this, we redefine our annihilation (and therefore also the creation) operators, for which we return to our action (46), defining:

$$\begin{aligned} h_{\mu\nu} &= A\bar{h}^1_{\mu\nu} + B\bar{h}^2_{\mu\nu}, \\ \tilde{h}_{\mu\nu} &= C\bar{h}^1_{\mu\nu} + D\bar{h}^2_{\mu\nu}, \end{aligned} \tag{62}$$

where $A$, $B$, $C$, and $D$ are real constants, so that the new fields, $\bar{h}^1$ and $\bar{h}^2$, are real fields. When replacing this in (46), we obtain

$$\begin{aligned} S[\bar{h}^1, \bar{h}^2] &= \frac{1}{2\kappa} \int d^4x P^{((\alpha\beta)(\mu\nu))} \times \\ &\times \left( \frac{A}{2}(A - \kappa_2 A + 2\kappa_2 C)\bar{h}^1_{\alpha\beta}\partial^2 \bar{h}^1_{\mu\nu} + + \frac{B}{2}(B - \kappa_2 B + 2\kappa_2 D)\bar{h}^2_{\alpha\beta}\partial^2 \bar{h}^2_{\mu\nu} \right) + \\ &+ P^{((\alpha\beta)(\mu\nu))}(AB - \kappa_2 AB + \kappa_2 AD + \kappa_2 BC)\bar{h}^1_{\alpha\beta}\partial^2 \bar{h}^2_{\mu\nu}. \end{aligned} \tag{63}$$

With the objective of decoupling the new fields, we make the last term in (63) null. It can be demonstrated that imposing the above criteria, it is inevitable that one (and only one) of two fields will be a ghost. We make the choice of $\bar{h}^2$ as the corresponding ghost. Taking the above considerations plus the condition that (63) has the usual form of an action with real fields, we impose that the coefficients of the first and second terms in it are $1/2$ and $-1/2$, respectively. This means

$$\begin{aligned} A &= B, \\ C &= \frac{1 - (1-\kappa_2)B^2}{2\kappa_2 B}, \\ D &= -\frac{1 + (1-\kappa_2)B^2}{2\kappa_2 B}, \end{aligned} \tag{64}$$

where $B$ is left as an arbitrary real constant. Here we make the point that, if we had chosen $\bar{h}^1$ as the ghost, then the real constants change such that $C \leftrightarrow D$.

Thus, the action we are finally left with is

$$S[\bar{h}^1, \bar{h}^2] = \frac{1}{2\kappa} \int d^4x P^{((\alpha\beta)(\mu\nu))} \left( \frac{1}{2} \bar{h}^1_{\alpha\beta} \partial^2 \bar{h}^1_{\mu\nu} - \frac{1}{2} \bar{h}^2_{\alpha\beta} \partial^2 \bar{h}^2_{\mu\nu} \right). \tag{65}$$

Following this same line of reasoning, we can find the annihilation operators for $\bar{h}^1$ and $\bar{h}^2$:

$$b^1_{AB}(\mathbf{p}) = \frac{1 + B^2(1 - \kappa_2)}{2B} a_{AB}(\mathbf{p}) + \kappa_2 B \tilde{a}_{AB}(\mathbf{p}), \tag{66}$$

$$b^2_{AB}(\mathbf{p}) = \frac{1 - B^2(1 - \kappa_2)}{2B} a_{AB}(\mathbf{p}) - \kappa_2 B \tilde{a}_{AB}(\mathbf{p}), \tag{67}$$

where we have used (62). It can be verified that the only nonvanishing commutators are now

$$[b^{1(AB)}(\mathbf{p}), b^{1+}_{CD}(\mathbf{p}')] = 4\kappa \delta^{AB}_{CD} \delta^3(\mathbf{p} - \mathbf{p}'), \tag{68}$$

$$[b^{2(AB)}(\mathbf{p}), b^{2+}_{CD}(\mathbf{p}')] = -4\kappa \delta^{AB}_{CD} \delta^3(\mathbf{p} - \mathbf{p}'). \tag{69}$$

These commutators indicate that $b^1$ and $b^2$ have a vanishing inner product, and that $b^2$ is the annihilation operator for the ghost. On the other hand, the Hamiltonian expressed in terms of these operators is

$$H = \int \frac{d^3p}{4\kappa} E_p \left( b^{1+}_{AB} b^{1AB} - b^{2+}_{AB} b^{2AB} \right). \tag{70}$$

Due to the existence of the ghost, it is possible that this model will not be unitary. To analyze this in greater depth, it is necessary to do a more profound study of the $S$ matrix, but to do this for gravitation is a colossal task that would take us beyond the original scope of this work. On the other hand, the existence of ghost or phantom fields has been proposed by some authors to explain the accelerated expansion of the Universe [22,42–45], a feature that our model presents [31]. The problem with these models is that, when they are quantized, either there is a loss of unitarity or there is negative energy, which means loss of stability. Looking at (65), we find that the propagators of $\bar{h}^1$ and $\bar{h}^2$ are, respectively,

$$-2\kappa P^{-1}_{((\alpha\beta)(\mu\nu))} \frac{i}{p^2 - i\varepsilon}, \tag{71}$$

$$2\kappa P^{-1}_{((\alpha\beta)(\mu\nu))} \frac{i}{p^2 \pm i\varepsilon}, \tag{72}$$

where the sign $\pm$ in the phantom propagator, $\bar{h}^2$, will decide whether unitarity and negative energy solutions or nonunitary and positive energy solutions will be present in the model [44].

The advantage that our model has against other models that use scalar fields for the phantoms is that, being a gauge model, the possibility remains open of fixing a gauge in which the model is unitary, keeping the model's good attributes, as in the BRST canonical quantization [47].

It is important to indicate that the existence of ghosts is a general feature of all delta theories and not only subscribed to delta gravity, as can clearly be seen in [37] (see there Appendix B, where the Hamiltonian of the model is not bounded from below).

The fact that our model has ghosts permits us to avoid a no-go theorem [48, 49] on the impossibility of having models with more than one consistent interacting gravitons (spin-two fields). Thus, in our case, we have a model with two interacting gravitons, but with a Hamiltonian not bounded from below (instability) as is exhibited by (70).

On the other hand, as a possible solution to the case of instability, we may consider $\delta$ supergravity, which may solve the unboundedness from below of the Hamiltonian. The last argument comes from the fact that in supersymmetry one defines the Hamiltonian as the square of an Hermitian charge, making it positive definite [50, 51].

Additionally, a delta model has more symmetries than the original model. This permits one to bound the Hamiltonian for a fixed value of a conserved quantity. See Appendix B.

Having explained the problem that our model has, now we discuss the new physics that our model might predict. For this, we will analyze the type of some finite quantum corrections and how the simplest of these affect the equations of motion of the model.

## 13. FINITE QUANTUM CORRECTIONS

The finite quantum corrections to our modified model of gravity can be separated into two groups. The first are the nonlocal terms, which are characterized by the presence of a logarithm, in the form [52]:

$$\sqrt{-g}R_{\mu\nu}\ln\left(\frac{\nabla^2}{\mu^2}\right)R^{\mu\nu}, \quad \sqrt{-g}R\ln\left(\frac{\nabla^2}{\mu^2}\right)R, \tag{73}$$

whith $\nabla^2 = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, $\nabla_\beta$ being the covariant derivative. There are no terms like the above ones but quadratic in the Riemann tensor because these terms always occur like

$$\frac{1}{\epsilon} + \ln\left(\frac{\nabla^2}{\mu^2}\right), \tag{74}$$

and it is known that the terms that appear with the pole are purely Ricci tensors and Ricci scalars [34, 36] (see Eq. (43), too), which in turn is due to (42). Now, when looking at the quantum corrections and Eq. (44), we need to care about the variations of (73) with respect to $g_{\mu\nu}$. Taking this into consideration, for the nonlocal terms we have

$$\begin{aligned}
\delta\left(\sqrt{-g}\right) R_{\mu\nu} \ln\left(\frac{\nabla^2}{\mu^2}\right) R^{\mu\nu} &= 0, \\
\sqrt{-g} R_{\mu\nu} \delta\left(\ln\left(\frac{\nabla^2}{\mu^2}\right) R^{\mu\nu}\right) &= 0, \\
\sqrt{-g}\delta(R_{\mu\nu}) \ln\left(\frac{\nabla^2}{\mu^2}\right) R^{\mu\nu} &= 0, \\
\delta\left(\sqrt{-g}\right) R \ln\left(\frac{\nabla^2}{\mu^2}\right) R &= 0, \\
\sqrt{-g} R \delta\left(\ln\left(\frac{\nabla^2}{\mu^2}\right) R\right) &= 0, \\
\sqrt{-g}\delta(R) \ln\left(\frac{\nabla^2}{\mu^2}\right) R &= 0,
\end{aligned} \tag{75}$$

because our model lives on shell, i.e., $R_{\mu\nu} \equiv 0$ and $R \equiv 0$. So, we see that the only relevant quantum corrections will come from the second group, that is, from the local terms that correspond to a series expansion in powers of the curvature tensor. The linear term is basically $R$, which corresponds to the original action, and the quadratic terms when taking into account their contribution are null due to (42). The next terms to consider are cubic in the Riemann tensor. In principle, any power of the curvature tensor will appear, but we now want to discuss only the cubic ones because they are the simpler to be dealt with [53]. The most general form of these corrections is

$$\begin{aligned}
L_Q^{\mathrm{fin}} = \sqrt{-g}\Big(& c_1 R_{\mu\nu\lambda\sigma} R^{\alpha\beta\lambda\sigma} R^{\mu\nu}_{\alpha\beta} + c_2 R^{\mu\nu}_{\lambda\sigma} R^{\lambda\beta}_{\mu\alpha} R^{\alpha\sigma}_{\nu\beta} + \\
& + c_3 R_{\mu\nu} R^{\mu\alpha\beta\gamma} R^{\nu}_{\alpha\beta\gamma} + c_4 R R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa}\Big).
\end{aligned} \tag{76}$$

This type of corrections will affect the equations of motion for $\tilde{g}_{\mu\nu}$. So, using (3), we obtain

$$S^{\mu\nu} = -\frac{1}{\kappa_2}\left(M^{(\mu\nu)} + c_1 N^{(\mu\nu)} + c_2 B^{(\mu\nu)} + 3\left\{D_\rho, D\sigma\right\} E^{[\sigma\mu][\nu\rho]}\right), \tag{77}$$

with

$$M^{(\mu\nu)} = \frac{1}{2}\big(D_\alpha D^\nu A^{(\alpha\mu)} + D_\alpha D^\mu A^{(\alpha\nu)} - D_\alpha D^\alpha A^{(\mu\nu)} - g^{\mu\nu} D_\alpha D_\beta A^{(\alpha\beta)}\big), \tag{78}$$

$$A^{(\mu\nu)} = c_3 R^{\mu\alpha\beta\gamma} R^{\nu}_{\sim\alpha\beta\gamma} + c_4 g^{\mu\nu} R^{\alpha\beta\gamma\epsilon} R_{\alpha\beta\gamma\epsilon}, \tag{79}$$

$$N^{(\mu\nu)} = \frac{1}{2} g^{\mu\nu} R_{\rho\epsilon\lambda\sigma} R^{\lambda\sigma\alpha\beta} R_{\alpha\beta}^{\sim\sim\sim\rho\epsilon} + 3 R_{\rho\epsilon\lambda\sigma} R_{\alpha}^{\sim\nu\epsilon\rho} R^{\alpha\mu\lambda\sigma}, \tag{80}$$

$$B^{(\mu\nu)} = \frac{1}{2} g^{\mu\nu} R_{\rho\epsilon\lambda\sigma} R^{\rho\alpha\lambda\beta} R^{\sim\sigma\epsilon}_{\alpha\sim\sim\beta} + 3 R_{\rho\epsilon\lambda\sigma} R^{\nu\sigma\rho}_{\sim\sim\sim\beta} R^{\mu\epsilon\beta\lambda}, \tag{81}$$

$$E^{[\sigma\mu][\nu\rho]} = c_1 R^{\sigma\mu}_{\sim\sim\alpha\beta} R^{\alpha\beta\nu\rho} + \frac{1}{2} c_2 \big(R^{\nu\sim\sigma}_{\sim\alpha\sim\beta} R^{\rho\beta\alpha\mu} - R^{\rho\sim\sigma}_{\sim\alpha\sim\beta} R^{\nu\beta\alpha\mu}\big), \tag{82}$$

where $S^{\mu\nu}$ was defined in Sec. 1. Obviously, if we do not have quantum corrections, i.e., $c_1 = c_2 = c_3 = c_4 = 0$, (77) is transformed in (3). It is possible to demonstrate that one solution to (3) is $\tilde{g}_{\mu\nu} = g_{\mu\nu}$, a fact that is necessary so that the predictions of the original theory of Einstein–Hilbert are still fulfilled in vacuum. This means, the solution of (77) must come to be small perturbations to $g_{\mu\nu}$.

Delta gravity will provide finite answers for the constants $c_i$. Due to the general structure of the finite quantum corrections, they will be relevant only at very short distances and strong curvatures. So, the natural scenario to test the predictions of the model is the inflationary epoch of the Universe. The computation of the $c_i$ and the phenomenological implications of quantum $\delta$ gravity will be discussed elsewhere.

## 14. CONCLUSIONS AND OPEN PROBLEMS

Delta gravity agrees with GR when $T_{\mu\nu} = 0$, imposing the same boundary conditions for both tensor fields. In particular, the causal structure of delta gravity in vacuum is the same as in GR, since in this case the action (5) is proportional to the geodesic action in GR.

We recover the Newtonian approximation.

In a homogeneous and isotropic universe, we get accelerated expansion without a cosmological constant or additional scalar fields.

The computation of Post Newtonian Parameters (PNP) is in progress, but we do not expect large departures from general relativity, because the Newtonian limit is the right one, as explained in Sec. 5. Moreover, the interstellar space has very small matter densities, so $\delta$ gravity must give GR values for the PNPs (see comments after Eq. (4)). Additionally, please notice that all $\tilde{g}$ contributions are multiplied by the small parameter $\kappa_2'$ of the order of $10^{-5}$ or less, so they are very small in the Solar System.

Stellar evolution will not be changed from its Newtonian description, unless density of matter becomes very large. Even at the densities of white dwarfs, the Poisson equation for the gravitational potential suffices. (See, for instance, [8], Ch. 11.3.); $\delta$ gravity implies it, as it is shown in Sec. 5. Higher densities which are present in neutron stars may provide new tests of $\delta$ gravity, since there we have to use the whole nonlinear Einstein equations and the corresponding $\delta$-gravity equations. But for the inner regions of massive stars, data is very scarce.

Notice that equation (21) implies that $\tilde{R} = R$ at the beginning of the Universe, when $w = 1/3$, corresponding to ultrarelativistic matter. That is, the accelerated expansion started at a later time, which is needed if we want to recover the observational data of density perturbations and growth of structures in the Universe. An earlier acceleration of the expansion would prevent the growth of density perturbations.

Work is in progress to compute the growth of density perturbations and the anisotropies in the CMBR. The comparison of these calculations with the considerable amount of astronomical data that will be available in the near future will be a very stringent test of the present gravitational model.

It was noticed in [20] that the Hamiltonian of delta models is not bounded from below. See also Sec. 12. Phantoms cosmological models [22, 23] also have this property. Although it is not clear whether this problem will subsist in a diffeomorphism invariant model as delta gravity or not, we want to mention some ways out of the difficulty.

a) Delta gravity is a gauge theory. Moreover, it is diffeomorphism invariant. Thus, the canonical Hamiltonian vanishes identically. It may be possible to truncate the Hilbert space, using the BRST formalism, to define a model with a Hamiltonian bounded from below. This is a difficult task that goes far beyond the present paper, but should be pursued in a future work.

b) In a supersymmetric model, we have $H = Q^2$, where $H$ is the Hamiltonian and $Q$ is the Hermitian supersymmetry charge. Thus, the Hamiltonian is bounded from below. So, we expect that a delta-supergravity model has a Hamiltonian bounded from below.

c) A delta model has more symmetries than the original model. This permits one to bound the Hamiltonian for a given value of a conserved quantity. We explain this in Appendix B.

**Acknowledgements.** The author wants to thank Prof. I. Kondrashuk for the kind invitation to write an article for the special issue dedicated to the 60th birthday of Prof. D. I. Kazakov. The work is partially supported by Fondecyt 1110378. The author wants to thank R. Avila and P. González for several useful remarks. He acknowledges interesting conversations with L. Infante, G. Palma, M. Bañados, and A. Clocchiatti. In particular, he wants to thank A. Clocchiatti for pointing out the data in [28].

## Appendix A
## REVIEW OF $\delta$ SYMMETRIES

Assume we have a group of transformations acting on the variables $y$ with infinitesimal parameters $\epsilon$, that is

$$\delta y^i = \Lambda^i_\alpha(y)\epsilon^\alpha. \tag{83}$$

We define the $\delta$ transformation by

$$\delta \bar{y}^i = \Lambda^i_\alpha(y)_{,j}\bar{y}^j\epsilon^\alpha + \Lambda^i_\alpha(y)\bar{\epsilon}^\alpha, \tag{84}$$

$k_{,i} = \partial k/\partial y^i$.

Notice that we have introduced a new field $\bar{y}^i$ and a new transformation with parameter $\bar{\epsilon}^\alpha$.

It is easy to see that (83), (84) form a closed algebra.

An invariant action under the extended symmetry is built in the same way. We assume that

$S(y)$ is invariant under (83)

$$\frac{\delta S}{\delta y^i}\Lambda^i_\alpha(y) = 0, \forall y, \quad \text{all} \quad \alpha, \tag{85}$$

then

$$\bar{S}(y,\bar{y}) = S(y) + \frac{\delta S}{\delta y^i}\bar{y}^i$$

is invariant under (83), (84).

Proof:

$$\begin{aligned}\delta\bar{S}(y,\bar{y}) = \frac{\delta S}{\delta y^i}\Lambda^i_\alpha(y)\epsilon^\alpha + \frac{\delta^2 S}{\delta y^i \delta y^j}\Lambda^j_\alpha(y)\epsilon^\alpha\bar{y}^i + \frac{\delta S}{\delta y^i}(\Lambda^i_\alpha(y)_{,j}\bar{y}^j\epsilon^\alpha + \Lambda^i_\alpha(y)\bar{\epsilon}^\alpha) = \\ = 0 + \left(\frac{\delta^2 S}{\delta y^i \delta y^j}\Lambda^j_\alpha(y)\bar{y}^i + \frac{\delta S}{\delta y^i}\Lambda^i_\alpha(y)_{,j}\bar{y}^j\right)\epsilon^\alpha + 0\bar{\epsilon}^\alpha = \\ = \left(\frac{\delta^2 S}{\delta y^i \delta y^j}\Lambda^i_\alpha(y) + \frac{\delta S}{\delta y^i}\Lambda^i_\alpha(y)_{,j}\right)\epsilon^\alpha\bar{y}^j = \left\{\frac{\delta}{\delta y^j}\left(\frac{\delta S}{\delta y^i}\Lambda^i_\alpha(y)\right)\right\}\epsilon^\alpha\bar{y}^j = 0.\end{aligned}$$

Last equality follows from equation (85).

Being careful with signs of permutations, these results are true for anti-commuting $y, \epsilon$ as well. In particular, supersymmetric transformations can be generalized to a $\delta$ symmetry.

Other generalizations are possible. Suppose we have canonical transformations generated by $\epsilon(x,p)$:

$$\delta F = (\epsilon, F), \quad \delta\bar{F} = (\epsilon, \bar{F}) + (\bar{\epsilon}, F), \tag{86}$$

Eqs. (83), (84) are particular cases of (86). $(A, B)$ is the Poisson bracket. Now we can prove the closure of the algebra in a more general context:

$$[\delta_\beta, \delta_\alpha]F = (\delta_\beta(\alpha, F) - \alpha \leftrightarrow \beta) = (\alpha, (\beta, F)) - (\beta, (\alpha, F)) = (F, (\beta, \alpha)) = \\ = ((\alpha, \beta), F) = \delta_{(\alpha,\beta)}F,$$

$$[\delta_\beta, \delta_\alpha]\bar{F} = (\delta_\beta(\alpha, \bar{F}) - \alpha \leftrightarrow \beta) = (\alpha, (\beta, \bar{F})) - (\beta, (\alpha, \bar{F})) = (\bar{F}, (\beta, \alpha)) = \\ = ((\alpha, \beta), \bar{F}) = \delta_{(\alpha,\beta)}\bar{F},$$

$$[\delta_\alpha, \delta_{\bar{\beta}}]F = 0,$$

$$[\delta_\alpha, \delta_{\bar{\beta}}]\bar{F} = (\delta_\alpha(\bar{\beta}, F) - \delta_{\bar{\beta}}(\alpha, \bar{F})) = (\bar{\beta}, (\alpha, F)) - (\alpha, (\bar{\beta}, F)) = \\ = (F, (\alpha, \bar{\beta})) = \delta_{(\bar{\beta},\alpha)}F,$$

$$[\delta_{\bar{\alpha}}, \delta_{\bar{\beta}}]\bar{F} = \delta_{\bar{\alpha}}(\bar{\beta}, F) - \bar{\alpha} \leftrightarrow \bar{\beta} = 0.$$

Replacing Poisson bracket by commutators is the realization of the algebra we used in [20].

## Appendix B
## THE DELTA HARMONIC OSCILLATOR

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 + \dot{x}\dot{y} - \omega^2 xy.$$

The canonical variables are

$$p_x = \dot{x} + \dot{y}, \quad p_y = \dot{x},$$
$$H = \dot{x}(\dot{x} + \dot{y}) + \dot{y}\dot{x} - L = H_0 + \dot{x}\dot{y} + \omega^2 xy, \tag{87}$$
$$H_0 = \frac{1}{2}p_y^2 + \frac{1}{2}\omega^2 x^2.$$

We know that $H_0$ is conserved, because the model satisfies the equations of motion for $x$. Since $H$ is conserved, we have that $Q$ is conserved, too:

$$Q = \dot{x}\dot{y} + \omega^2 xy,$$

since $H_0$ is greater or equal to zero. $H$ is bounded from below by the value of the conserved quantity $Q$.

This is a generic feature of delta models. The Hamiltonian is bounded from below, for a given value of a conserved quantity $Q$, if the Hamiltonian of the original model is bounded from below.

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