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THE COULOMB PROBLEM IN SUPERSTRONG B: ATOMIC LEVELS AND CRITICAL NUCLEI CHARGES

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The spectrum of atomic levels of hydrogen-like ions originating from the lowest Landau level in an external homogeneous superstrong magnetic field is obtained. The influence of the screening of the Coulomb potential on the values of critical nuclear charges is studied.

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To Dmitry Kazakov in occasion of the 60th birthday and in memory of good times which we had together in Dubna, Moscow and all over the world.

INTRODUCTION

We will discuss the modification of the Coulomb law and atomic spectra in superstrong magnetic field. The talk is based on papers [1–3], see also [4].

1. $D = 2$ QED

Let us consider two-dimensional QED with massive charged fermions. The electric potential of the external point-like charge equals

$$\mathbf{\Phi}(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)}, \tag{1}$$

where $\Pi(k^2)$ is the one-loop expression for the photon polarization operator

$$\Pi(k^2) = 4g^2 \left[\frac{1}{\sqrt{t(1+t)}} \ln\left(\sqrt{1+t} + \sqrt{t}\right) - 1 \right] \equiv -4g^2 P(t), \tag{2}$$

and $t \equiv -k^2/4m^2$, $[g] = $ mass.

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In the coordinate representation for $k = (0, k_\parallel)$ we obtain

$$\mathbf{\Phi}(z) = 4\pi g \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ik_\parallel z}\, dk_\parallel/2\pi}{k_\parallel^2 + 4g^2 P(k_\parallel^2/4m^2)}. \tag{3}$$

With the help of the interpolating formula

$$\overline{P}(t) = \frac{2t}{3+2t} \tag{4}$$

the accuracy of which is better than 10% for $0 < t < \infty$ we obtain

$$\mathbf{\Phi} = 4\pi g \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ik_\parallel z}\, dk_\parallel/2\pi}{k_\parallel^2 + 4g^2(k_\parallel^2/2m^2)/(3 + k_\parallel^2/2m^2)} = $$
$$= \frac{4\pi g}{1 + 2g^2/3m^2}\left[-\frac{1}{2}|z| + \frac{g^2/3m^2}{\sqrt{6m^2+4g^2}}\exp\left(-\sqrt{6m^2+4g^2}|z|\right)\right]. \tag{5}$$

In the case of heavy fermions ($m \gg g$), the potential is given by the tree level expression; the corrections are suppressed as g^2/m^2.

In the case of light fermions ($m \ll g$):

$$\mathbf{\Phi}(z)\big|_{m\ll g} = \begin{cases} \pi\,\mathrm{e}^{-2g|z|}, & z \ll \dfrac{1}{g}\ln\left(\dfrac{g}{m}\right), \\ -2\pi g\left(\dfrac{3m^2}{2g^2}\right)|z|, & z \gg \dfrac{1}{g}\ln\left(\dfrac{g}{m}\right), \end{cases} \tag{6}$$

$m = 0$ corresponds to the Schwinger model; photon gets a mass due to a photon polarization operator with massless fermions.

2. ELECTRIC POTENTIAL OF THE POINT-LIKE CHARGE IN $D = 4$ IN SUPERSTRONG B

We need an expression for the polarization operator in the external magnetic field B. It simplifies greatly for $B \gg B_0 \equiv m_e^2/e$, where m_e is the electron mass and we use Gauss units, $e^2 = \alpha = 1/137\ldots$ The following results were obtained in [2]:

$$\mathbf{\Phi}(k) = \frac{4\pi e}{k_\parallel^2 + k_\perp^2 + \dfrac{2e^3 B}{\pi}\exp\left(-\dfrac{k_\perp^2}{2eB}\right) P\left(\dfrac{k_\parallel^2}{4m^2}\right)}, \tag{7}$$

$$\mathbf{\Phi}(z) = 4\pi e \int \frac{\mathrm{e}^{ik_\parallel z}\, dk_\parallel\, d^2k_\perp/(2\pi)^3}{k_\parallel^2 + k_\perp^2 + \dfrac{2e^3B}{\pi}\exp\left(-k_\perp^2/(2eB)\right)(k_\parallel^2/2m_e^2)/(3 + k_\parallel^2/2m_e^2)} =$$

$$= \frac{e}{|z|}\left[1 - \mathrm{e}^{-\sqrt{6m_e^2}|z|} + \mathrm{e}^{-\sqrt{(2/\pi)e^3B + 6m_e^2}|z|}\right]. \quad (8)$$

For $B \ll 3\pi m^2/e^3$ the potential is Coulomb up to small corrections:

$$\mathbf{\Phi}(z)\big|_{e^3B \ll m_e^2} = \frac{e}{|z|}\left[1 + O\left(\frac{e^3B}{m_e^2}\right)\right], \quad (9)$$

analogously to $D = 2$ case with substitution $e^3B \to g^2$.

For $B \gg 3\pi m_e^2/e^3$ we obtain

$$\mathbf{\Phi}(z) =$$

$$= \begin{cases} \dfrac{e}{|z|}\,\mathrm{e}^{(-\sqrt{(2/\pi)e^3B}|z|)}, & \dfrac{1}{\sqrt{(2/\pi)\,e^3B}} \ln\left(\sqrt{\dfrac{e^3B}{3\pi m_e^2}}\right) > |z| > \dfrac{1}{\sqrt{eB}}, \\ \dfrac{e}{|z|}(1 - \mathrm{e}^{(-\sqrt{6m_e^2}|z|)}), & \dfrac{1}{m_e} > |z| > \dfrac{1}{\sqrt{(2/\pi)e^3B}} \ln\left(\sqrt{\dfrac{e^3B}{3\pi m_e^2}}\right), \\ \dfrac{e}{|z|}, & |z| > \dfrac{1}{m_e}, \end{cases} \quad (10)$$

$$V(z) = -e\mathbf{\Phi}(z). \quad (11)$$

The close relation of the radiative corrections at $B \gg B_0$ in $D = 4$ to the radiative corrections in $D = 2$ QED allows one to prove that just like in $D = 2$ case higher loops are not essential (see, for example, [5]).

3. HYDROGEN ATOM IN THE MAGNETIC FIELD

For $B > B_0 = m_e^2/e$, the spectrum of Dirac equation consists of ultrarelativistic electrons with only one exception: the electrons from the lowest Landau level (LLL, $n = 0$, $\sigma_z = -1$) are nonrelativistic. So we will find the spectrum of electrons from LLL in the screened Coulomb field of the proton.

The wave function of electron from LLL is

$$R_{0m}(\rho) = \left[\pi(2a_H^2)^{1+|m|}(|m|!)\right]^{-1/2} \rho^{|m|}\,\mathrm{e}^{(im\varphi - \rho^2/(4a_H^2))}, \quad (12)$$

where $m = 0, -1, -2$ is the projection of the electron orbital momentum on the direction of the magnetic field.

For $a_H \equiv 1/\sqrt{eB} \ll a_B = 1/(m_e e^2)$ the adiabatic approximation is applicable and the wave function looks like

$$\Psi_{n0m-1} = R_{0m}(\rho)\chi_n(z), \tag{13}$$

where $\chi_n(z)$ satisfy the one-dimensional Schrödinger equation:

$$\left[-\frac{1}{2m_e}\frac{d^2}{dz^2} + U_{\text{eff}}(z)\right]\chi_n(z) = E_n\chi_n(z). \tag{14}$$

Since screening occurs at very short distances it is not important for odd states, for which the effective potential is

$$U_{\text{eff}}(z) = -e^2 \int \frac{|R_{0m}(\rho)|^2}{\sqrt{\rho^2+z^2}} d^2\rho. \tag{15}$$

It equals the Coulomb potential for $|z| \gg a_H$ and is regular at $z = 0$.

Thus the energies of the odd states are

$$E_{\text{odd}} = -\frac{m_e e^4}{2n^2} + O\left(\frac{m_e^2 e^3}{B}\right), \quad n = 1, 2, \ldots, \tag{16}$$

and for the superstrong magnetic fields $B > m_e^2/e^3$ they coincide with the Balmer series with high accuracy.

For even states the effective potential looks like

$$\tilde{U}_{\text{eff}}(z) = -e^2 \int \frac{|R_{0m}(\boldsymbol{\rho})|^2}{\sqrt{\rho^2+z^2}} d^2\rho \left[1 - \mathrm{e}^{-\sqrt{6m_e^2}\, z} + \mathrm{e}^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\, z}\right]. \tag{17}$$

Integrating the Schrödinger equation with the effective potential from $x = 0$ till $x = z$, where $a_H \ll z \ll a_B$, and equating the obtained expression for $\chi'(z)$ to the logarithmic derivative of Whittaker function — the solution of the Schrödinger equation with Coulomb potential — we obtain the following equation for the energies of even states:

$$\ln\left(\frac{H}{1+\frac{e^6}{3\pi}H}\right) = \lambda + 2\ln\lambda + 2\psi\left(1-\frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1+|m|), \tag{18}$$

where $H \equiv B/(m_e^2 e^3)$, $\psi(x)$ is the logarithmic derivative of the gamma function and

$$E = -(m_e e^4/2)\lambda^2. \tag{19}$$

The spectrum of the hydrogen atom in the limit $B \gg m_e^2/e^3$ is shown in Fig. 1.

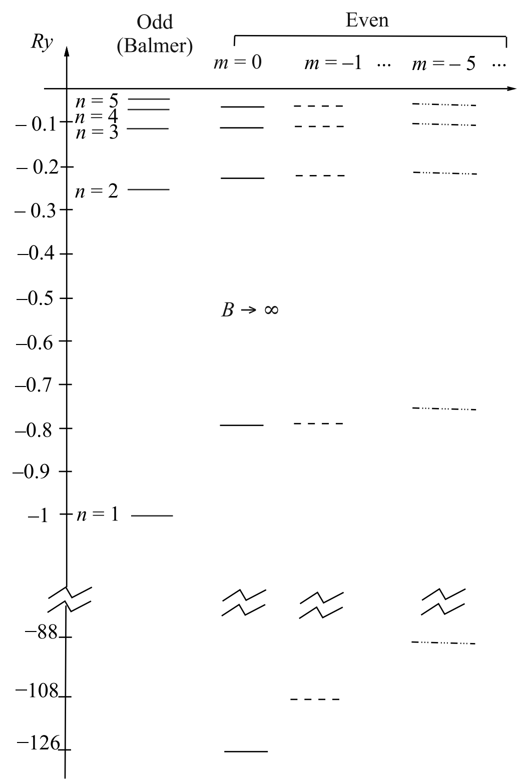

Fig. 1. Spectrum of the hydrogen atom in the limit of the infinite magnetic field. Energies are given in Rydberg units, $Ry \equiv 13.6$ eV

4. SCREENING VERSUS CRITICAL NUCLEUS CHARGE

Hydrogen-like ion becomes critical at $Z \approx 170$: the ground level reaches lower continuum, $\varepsilon_0 = -m_e$, and two e^+e^- pairs are produced from vacuum. Electrons with the opposite spins occupy the ground level, while positrons are emitted to infinity [6]. According to [7] in the strong magnetic field $Z_{\rm cr}$ diminishes: it equals approximately 90 at $B = 100B_0$; at $B = 3 \cdot 10^4 B_0$ it equals approximately 40. Screening of the Coulomb potential by the magnetic field acts in the opposite direction, and with the account of it larger magnetic fields are needed for a nucleus to become critical.

Let us parameterize bispinor which describes electron wave function in the following way:

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \varphi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{20}$$

Substituting Ψ in the Dirac equation for the electron in an external electromagnetic field we obtain

$$\left\{ \begin{array}{l} (\varepsilon - m - e\varphi)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \left(-i\bar{\sigma}\dfrac{\partial}{\partial \bar{r}} + e\bar{A}\bar{\sigma}\right)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0, \\ -\left(i\bar{\sigma}\dfrac{\partial}{\partial \bar{r}} - e\bar{A}\bar{\sigma}\right)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + (\varepsilon + m - e\varphi)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0. \end{array} \right. \tag{21}$$

Taking vector potential which describes constant magnetic field B directed along z axis in the form $\bar{A} = (-(1/2)By, (1/2)Bx, 0)$, we get

$$e\bar{A}\bar{\sigma} = -\frac{e}{2}B\begin{pmatrix} 0 & y + ix \\ y - ix & 0 \end{pmatrix} = -\frac{i}{2}eB\rho\begin{pmatrix} 0 & \mathrm{e}^{-i\theta} \\ -\mathrm{e}^{i\theta} & 0 \end{pmatrix}, \tag{22}$$

where $\rho = \sqrt{x^2 + y^2}$, $\theta \equiv \arctan(y/x)$. Analogously we obtain

$$-i\bar{\sigma}\frac{\partial}{\partial \bar{r}} = -i\begin{pmatrix} \dfrac{\partial}{\partial z} & \mathrm{e}^{-i\theta}\dfrac{\partial}{\partial \rho} - \dfrac{i\mathrm{e}^{-i\theta}}{\rho}\dfrac{\partial}{\partial \theta} \\ \mathrm{e}^{i\theta}\dfrac{\partial}{\partial \rho} + \dfrac{i\mathrm{e}^{i\theta}}{\rho}\dfrac{\partial}{\partial \theta} & -\dfrac{\partial}{\partial z} \end{pmatrix}. \tag{23}$$

Substituting two last expressions in the Dirac equation we get

$$\left\{ \begin{array}{l} (\varepsilon - m - e\varphi)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \\ -i\begin{pmatrix} \dfrac{\partial}{\partial z} & \mathrm{e}^{-i\theta}\left(\dfrac{1}{2}eB\rho + \dfrac{\partial}{\partial \rho} - \dfrac{i}{\rho}\dfrac{\partial}{\partial \theta}\right) \\ \mathrm{e}^{i\theta}\left(-\dfrac{1}{2}eB\rho + \dfrac{\partial}{\partial \rho} + \dfrac{i}{\rho}\dfrac{\partial}{\partial \theta}\right) & -\dfrac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0, \\ (\varepsilon + m - e\varphi)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \\ -i\begin{pmatrix} \dfrac{\partial}{\partial z} & \mathrm{e}^{-i\theta}\left(\dfrac{1}{2}eB\rho + \dfrac{\partial}{\partial \rho} - \dfrac{i}{\rho}\dfrac{\partial}{\partial \theta}\right) \\ \mathrm{e}^{i\theta}\left(-\dfrac{1}{2}eB\rho + \dfrac{\partial}{\partial \rho} + \dfrac{i}{\rho}\dfrac{\partial}{\partial \theta}\right) & -\dfrac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0. \end{array} \right. \tag{24}$$

Axial symmetry of electromagnetic field allows one to determine θ dependence of the functions c_i and b_i:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1(\rho, z) & \mathrm{e}^{i(M-1/2)\theta} \\ c_2(\rho, z) & \mathrm{e}^{i(M+1/2)\theta} \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1(\rho, z) & \mathrm{e}^{i(M-1/2)\theta} \\ b_2(\rho, z) & \mathrm{e}^{i(M+1/2)\theta} \end{pmatrix}, \tag{25}$$

where $M = \pm 1/2, \pm 3/2, \ldots$ is the projection of electron angular momentum on z axis. Substituting (25) into (24) we get four linear equations for four unknown functions c_i and b_i (here and below $c_1 \equiv c_1(\rho, z)$, $b_1 \equiv b_1(\rho, z) \ldots$):

$$\begin{aligned}
&(\varepsilon - m - e\varphi)c_1 + i\left(-b_{1z} - b_{2\rho} - \frac{M+1/2}{\rho}b_2 - \frac{eB\rho}{2}b_2\right) = 0, \\
&(\varepsilon - m - e\varphi)c_2 + i\left(-b_{1\rho} + \frac{M-1/2}{\rho}b_1 + \frac{eB\rho}{2}b_1 + b_{2z}\right) = 0, \\
&(\varepsilon + m - e\varphi)b_1 + i\left(-c_{1z} - c_{2\rho} - \frac{M+1/2}{\rho}c_2 - \frac{eB\rho}{2}c_2\right) = 0, \\
&(\varepsilon + m - e\varphi)b_2 + i\left(-c_{1\rho} + \frac{M-1/2}{\rho}c_1 + \frac{eB\rho}{2}c_1 + c_{2z}\right) = 0,
\end{aligned} \tag{26}$$

where $b_{1z} \equiv \partial b_1/\partial z$, $b_{1\rho} \equiv \partial b_1/\partial \rho, \ldots$ Ground energy state has $s_z = -1/2$, $l_z = 0$. Taking $M = -1/2$, we should look for solution of (26) with $c_1 = b_1 = 0$:

$$\begin{aligned}
b_{2\rho} + \frac{eB\rho}{2}b_2 = 0, \\
c_{2\rho} + \frac{eB\rho}{2}c_2 = 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
(\varepsilon - m - e\varphi)c_2 + ib_{2z} = 0, \\
(\varepsilon + m - e\varphi)b_2 + ic_{2z} = 0.
\end{aligned} \tag{28}$$

The dependence on ρ is determined by (27)

$$\begin{aligned}
b_2(\rho, z) = \mathrm{e}^{-eB\rho^2/4}(-i)f(z), \\
c_2(\rho, z) = \mathrm{e}^{-eB\rho^2/4}g(z).
\end{aligned} \tag{29}$$

Substituting the last expressions in (28) and averaging over fast motion in transverse to the magnetic field plane we obtain two first-order differential equations which describe electron motion along magnetic field [7]:

$$\begin{aligned}
g_z - (\varepsilon + m_e - \bar{V})f = 0, \\
f_z + (\varepsilon - m_e - \bar{V})g = 0,
\end{aligned} \tag{30}$$

where $g_z \equiv dg/dz$, $f_z \equiv df/dz$. They describe the electron motion in the effective potential $\bar{V}(z)$:

$$\bar{V}(z) = -\frac{Ze^2}{a_H^2}\left[1 - \mathrm{e}^{-\sqrt{6m_e^2}|z|} + \mathrm{e}^{-\sqrt{(2/\pi)e^3B + 6m_e^2}|z|}\right] \int\limits_0^\infty \frac{\mathrm{e}^{-\rho^2/2a_H^2}}{\sqrt{\rho^2 + z^2}}\rho\, d\rho. \tag{31}$$

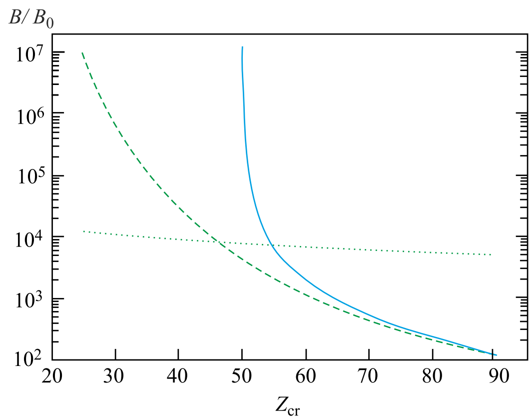

Fig. 2. The values of $B_{\rm cr}^{Z}$: without screening according to [7], dashed line; numerical results with screening, solid line. The dotted line corresponds to the field at which a_H becomes smaller than the size of the nucleus

Intergrating (30) numerically we find the dependence of $Z_{\rm cr}$ on the magnetic field with the account of screening. The results are shown in Fig. 2. For the given nucleus to become critical, larger magnetic fields are needed and the nuclei with $Z < 52$ do not become critical.

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