GAUGE FIELDS, NONLINEAR REALIZATIONS, SUPERSYMMETRY

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This is a brief survey of the all-years research activity in the Sector "Supersymmetry" (the former Markov Group) at the Bogoliubov Laboratory of Theoretical Physics. The focus is on the issues related to gauge fields, spontaneously broken symmetries in the nonlinear realizations approach, and diverse aspects of supersymmetry.

Представлен краткий обзор исследований в секторе «Суперсимметрия» (бывшая группа Маркова) Лаборатории теоретической физики им. Н. Н. Боголюбова за все годы. Упор сделан на вопросы, относящиеся к калибровочным полям, спонтанно нарушенным симметриям в подходе нелинейных реализаций и различным аспектам суперсимметрии.

*To the memory of V. I. Ogievetsky and
I. V. Polubarinov*

INTRODUCTION

The concepts which composed the title of this paper lie in the ground of modern mathematical theoretical physics. From their very invention [1–7], they are constantly among the most priority directions of research in Sector 3 of the Laboratory of Theoretical Physics. This Sector was originally named "Markov Group", after its first head, Academician Moisei Alexandrovich Markov (1908–1994). Later on, for more than 20 years, it was headed by Professor Victor Isaakovich Ogievetsky (1928–1996) and, since the early 1990s, by the present author. The aim of the overview is to focus on the milestones of this long-lasting research activity, with short explanations of their meaning and significance for further worldwide developments of the relevant subjects. Besides the studies concentrated around the title issues, for the years passed since 1956 there were many considerable contributions of Sector 3 members to other areas of theoretical physics, including the phenomenology of elementary particles, the conceptual and mathematical basics of quantum mechanics, the renowned Ising model, etc. The choice of

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the topics of this overview was determined by the preferences of the author and the fact that his scientific interests always bore upon just these lines of investigations.

The structure of the paper is as follows. In Sec. 1, we deal with the period before invention of supersymmetry. Section 2 describes the most sound results obtained in the domain of supersymmetry before the advent of Harmonic Superspace. The latter and related issues are the subject of Sec. 3. In Sec. 4, we give a brief account of some other contributions of the Dubna group to the directions related to the title.

Since many significant achievements go back to the pre-internet era, we describe them in some detail, with the hope that they could be of interest for the modern generation of theorists. This concerns the spin principle (Subsec. 1.1), the "notoph" and "inverse Higgs phenomenon" (Subsecs. 1.2 and 1.6), an interpretation of gravity and Yang–Mills theories as nonlinear realizations (Subsecs. 1.5 and 1.7), the complex superfield geometry of $\mathcal{N} = 1$ supergravity (Subsec. 2.3), the relation between the linear (superfield) and nonlinear (Volkov–Akulov) realizations of supersymmetry (Subsec. 2.4), and the whole Sec. 3.

The present review partly overlaps with the review [8] which was devoted mainly to the supersymmetry issues and so had a more narrow scope. Like in [8], the author apologizes for the inevitable incompleteness of the reference list and a possible involuntary bias in exposition of the investigations parallel to those performed in Dubna.

1. GAUGE FIELDS, GRAVITY AND NONLINEAR REALIZATIONS

The first studies in the directions claimed in the title are dated the early 1960s, and they were inspired by the invention of non-Abelian gauge fields by Yang and Mills in 1954 [1]. During a long time since its discovery, the Yang–Mills theory was apprehended merely as a kind of elegant mathematical toy, since no sign of non-Abelian counterparts of the $U(1)$ gauge field, photon, was observed and nobody knew to which class of physical phenomena such a theory could be applied. The situation changed in the early 1960s after detection of strongly interacting massive vector bosons. It was suggested that they can be analogs of photon for strong interactions and can be described by a mass-deformed Yang–Mills theory, with the minimally broken gauge invariance*.

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*Now we know that the genuine theory of strong interactions is the quantum chromodynamics which is the Yang–Mills theory for exact gauged "color" $SU(3)$ symmetry, with massless gluons as the relevant gauge fields. The second cornerstone of the Standard Model is the electroweak theory which is the Yang–Mills theory for the gauge group $U(2) = SU(2) \times U(1)$, with the photon and the triplet of intermediate vector bosons as the gauge fields. The intermediate bosons are massive on account of the Brout–Englert–Higgs effect within a linear realization of the spontaneous breaking of $U(2)$ symmetry. This mechanism of appearance of mass of the gauge fields does not break gauge invariance and preserves the remarkable property of Yang–Mills theory to be renormalizable, like quantum electrodynamics.

1.1. Spin Principle. The sharp growth of interest in non-Abelian gauge theories motivated Victor Isaakovich Ogievetsky and Igor Vasil'evich Polubarinov (1929–1998) (under the approval and support of M. A. Markov) to carefully elaborate on the nature and role of gauge fields. In the brilliant papers [9–11] they put forward the so-called "spin principle" as the basis of gauge theories. Namely, they showed that requiring one or another nonzero spin of field to be preserved in the interacting theory uniquely fixes the latter as a gauge theory and the field with the preserved spin as the relevant gauge field. The requirement of preservation of the spin 1 by the massless vector field uniquely reproduces Maxwell theory in the Abelian $U(1)$ case and Yang–Mills theory in the case of few vector bosons [9, 10]. Analogously, the theory of self-interacting massless spin 2 field proved to be just the Einstein gravity (treated as a field theory in Minkowski space-time) [12]. It is the relevant gauge invariances that ensure the neutralization of superfluous spins which the gauge field can carry (spin 0 in the vector field, spins 0 and 1 in the tensor field, etc.). Moreover, the spin principle applied to the theories with the gauge invariance broken by the mass terms fixes the latter in such a way that the "would-be" gauge field proves to be coupled to a conserved current, and this condition ensures the preservation of the given spin in the massive case as well.

The spin of interacting fields was the pioneer concept introduced by Ogievetsky and Polubarinov. Before their papers, it was a common belief that the quantum numbers of mass and spin characterizing the irreducible representations of the Poincaré group are applicable only to the free particles and on shell (with the evident substitution of the notion of helicity for that of spin for the massless particles). Ogievetsky and Polubarinov were the first to realize that the spin square Casimir operator of the Poincaré group $C_{(2)}$ can equally be defined for the interacting fields, as opposed to the mass square operator* $C_{(1)} = P^m P_m$, which cannot take any definite value on the interacting fields.

As an instructive example, we consider the case of vector field. The Casimir $C_{(2)}$ obtained as the square of the Pauli–Lubanski vector (divided by $P^2 \neq 0$ for further convenience) is in general expressed as

$$C_{(2)} = \frac{1}{2} S^{mn} S_{mn} - \frac{1}{P^2} S^{mn} S^q{}_n P_m P_q, \qquad (1.1)$$

where $S^{mn} = -S^{nm}$ is the matrix (spin) part of the full Lorentz generator $J^{mn} = S^{mn} + L^{mn}$, with $L^{mn} = i(x^m \partial^n - x^n \partial^m)$, and $P_m = (1/i)\partial_m$. The operator $P^2 = -\Box$ does not take any definite value in the theory with interaction, $P^2 \neq 0$. For the vector field b_m^i, where i is an index of some internal symmetry,

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*Hereafter, our conventions are as in the book [13]: $\eta^{mn} = \mathrm{diag}\,(1, -1, -1, -1)$, $\varepsilon_{0123} = 1$, $m, n = 0, 1, 2, 3$.

we have $(S^{pq})_m{}^n = i(\delta^p_m \eta^{qn} - \delta^q_m \eta^{pn})$ and

$$C_{(2)}\, b^i_m = C_{(2)m}{}^n b^i_n = 2\left[b^i_m - \frac{1}{\Box}\partial_m(\partial^n b^i_n)\right], \qquad (1.2)$$

i.e., we obtain that the field b^i_m is not an eigenfunction of $C_{(2)}$. Let us decompose b^i_m as

$$b^i_m = \mathbf{b}^i_m + \partial_m \phi^i, \quad \mathbf{b}^i_m := \left(\delta^n_m - \frac{1}{\Box}\partial_m \partial^n\right) b^i_n, \quad \phi^i := \frac{1}{\Box}(\partial^n b^i_n). \quad (1.3)$$

It is easy to see that

$$C_{(2)}\, \mathbf{b}^i_m = s(s+1)\,\mathbf{b}^i_m, \quad s = 1, \quad C_{(2)}\,\partial_m \phi^i = 0. \qquad (1.4)$$

Thus, (1.3) is the decomposition of the vector field b^i_m into the transverse (spin 1) part $\mathbf{b}^i_m$ and the longitudinal (spin 0) part $\partial_m \phi$. The question was how to arrange a theory in such a way that the vector field carries only spin 1 in the case of nontrivial interaction.

Ogievetsky and Polubarinov started from a general Lagrangian for the massive vector fields interacting with themselves and some matter fields Ψ^A, where $A = 1, 2, \ldots$ is an index of the same internal symmetry as for b^i_m,

$$L(b, \Psi) = -\frac{1}{4}F^{mn\,i}F^i_{mn} + \frac{1}{2}m^2 b^{mi}b^i_m + L_{\text{int}}(b, \Psi) + L_{\text{free}}(\Psi),$$
$$F^i_{mn} = \partial_m b^i_n - \partial_n b^i_m. \qquad (1.5)$$

The equations of motion for b^i_m read

$$\partial^m F^i_{mn} + J^i_n + m^2 b^i_n = 0, \quad J^i_n := \frac{\partial L_{\text{int}}}{\partial b^i_n} - \partial_p \frac{\partial L_{\text{int}}}{\partial(\partial_p b^i_n)}, \qquad (1.6)$$

where it is assumed that $L_{\text{int}}(b, \Psi)$ does not include higher-order derivatives of b^i_m. As the necessary and sufficient condition for b^i_m to possess only spin 1 in the interacting theory, Ogievetsky and Polubarinov rigorously proved that the equations of motion (including those for Ψ^A) should imply

$$m^2 \,\partial^n b^i_n = 0. \qquad (1.7)$$

This condition works for both the massive and the massless cases. If $m^2 \neq 0$, one has $\partial^n b^i_n = 0$, which just means that $C_{(2)} b^i_n = 2b^i_n$, i.e., b^i_n carries only spin 1. At $m = 0$, (1.7) is satisfied at any $\partial^m b^i_m$, which means that the latter quantity is arbitrary and so is not physical. Its arbitrariness is ensured by the gauge invariance which is thus the device to make b^i_m carry only spin 1 in the

massless case. One can always choose the gauge $\partial^m b_m^i = 0$, which implies that $b_n^i = \mathbf{b}_n^i$, $\partial^n \mathbf{b}_n^i = 0$, and so only spin 1 is really carried by the interacting b_n^i (this is true, of course, in any gauge).

The condition (1.7) amounts to the conservation of the current J_n^i defined in (1.6),

$$\partial^n J_n^i = 0, \tag{1.8}$$

which means that the spin 1 fields b_m^i couple to the conserved current. Using only this property, Ogievetsky and Polubarinov were able to uniquely restore the interaction Lagrangian L_{int} in (1.5). Together with the free F Lagrangian in (1.5) and modulo the extra fields Ψ^A, this Lagrangian is reduced in the general case to a sum of Yang–Mills Lagrangian for a semisimple gauge group and a number of the Abelian $U(1)$ Lagrangians, such that the dimension $d_{\{i\}}$ of the variety, where the indices i take their values, equals to the dimension of the adjoint representation of the Yang–Mills gauge group plus the dimension of the Abelian factors. For instance, if $d_{\{i\}} = 2$, only $U(1) \times U(1)$ gauge group is possible, if $d_{\{i\}} = 3$, the gauge group is either $SU(2)$ or $[U(1)]^3$, etc[*]. For the fields Ψ^A, there naturally arise minimal gauge-invariant couplings to the fields b_m^i. These structures are also uniquely fixed from the requirement that b_m^i are coupled to the conserved current.

To be more precise, the solution obtained by Ogievetsky and Polubarinov (ignoring trivial Abelian factors and the matter fields Ψ) is

$$L(b) = -\frac{1}{4g^2} G^{mn\,i} G_{mn}^i + \frac{1}{2} m^2 b^{mi} b_m^i,$$
$$G_{mn}^i = \partial_m b_n^i - \partial_n b_m^i - c^{ilt} b_m^l b_n^t, \tag{1.9}$$

where g is a coupling constant; c^{ilt} are the structure constants of some semi-simple gauge group, with the Hermitian generators T^i satisfying the algebra $[T^i, T^l] = ic^{ilt} T^t$. Ignoring the mass term, this Lagrangian is invariant under the gauge transformations with an arbitrary parameter $\lambda^i(x)$: $\delta b_n^i = -\partial_n \lambda^i + c^{ikl} b_n^k \lambda^l$. So in the limit $m = 0$ it becomes the standard massless gauge-invariant Yang–Mills Lagrangian. The mass term breaks the gauge invariance, but still retains the most important property of b_n^i to be coupled to the conserved current.

It is interesting that the approach based on the requirement of preservation of spins 1 in the interaction does not assume in advance any gauge group, the latter naturally arises, when revealing the structure of L_{int} from this requirement, as the invariance group of the full Lagrangian constructed in this way, modulo

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[*]It is assumed that the gauge groups contain no solvable factors.

the mass term. The systematic use of the condition of coupling of b^i_m to the conserved vector currents plays the crucial role in this derivation of the Yang–Mills Lagrangians (both massless and massive) from the spin principle.

The same machinery was used in [12] to derive the Einstein theory as a theory of symmetric tensor field carrying the spin 2 in interaction*. Ogievetsky and Polubarinov showed that the Einstein–Hilbert Lagrangian can be consistently derived by requiring the spin 2 field to be coupled to the conserved tensor current. Once again, not only the massless Lagrangian with the exact Diff R^4 gauge symmetry can be restored in this way, but also the appropriate mass deformations thereof. In both cases, the crucial role was played by the requirement of coupling to the conserved current. It is worth noting that [12] was one of the first papers where the gravity theory was treated on equal footing with other gauge theories as a field theory in the flat background space-time. It is distinguished by the property of preservation of the definite spin 2 in interaction, quite analogously to the treatment of the Yang–Mills theory as a field theory of definite spin 1 in interaction. Now such a treatment of gravity theories, as well as supergravities, is of common use.

The spin principle-inspired view of gauge invariance as just a way to ensure a definite spin of the interacting field proved to be very fruitful for further developments of gauge theories, including supergravity which is the unique self-consistent theory of interacting gauge fields of the spin 2 (graviton) and spin 3/2 (gravitino). Actually, in the lectures [11] Ogievetsky and Polubarinov have posed the question as to what could be the gauge theory in which the Rarita–Schwinger field carries spin 3/2 in the interacting case. They made serious efforts to find an answer [14], but failed because nobody was aware of supersymmetry that time.

The careful analysis of how the spin principle is obeyed in the course of quantization, on the examples of quantum electrodynamics and the theory of massive neutral gauge field, was accomplished by I. V. Polubarinov in the remarkable review [15]**. There, also a comparative detailed description of various approaches to quantizing the electrodynamics, including the historically first ones, was presented.

1.2. Notoph. While thinking on the group-theoretical grounds of gauge theories, Ogievetsky and Polubarinov discovered a new gauge theory, the gauge field of which is an antisymmetric rank two tensor field still propagating spin 1 off shell and describing on shell a massless particle with zero helicity, the "notoph" [16]. Later on, the notoph was rediscovered by Kalb and Ramond [17]. Now such gauge fields yielding an alternative off-shell description of zero spin,

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*To be more exact, in this case there is an admixture of spin 0. The pure spin 2 in interaction corresponds to the conformal gravity.

**This review was originally published in Russian as the JINR Preprint P-2421 (1965).

as well as their higher-rank p-form generalizations, are necessary ingredients of diverse superstring and supergravity theories.

It is instructive to dwell on the notoph theory in some detail. It is described by the following Lagrangian:

$$L = -\frac{1}{2} A^m A_m + L_{\text{int}}(f_{mn}, \ldots),$$
$$A^m := \frac{1}{2} \varepsilon^{mnpq} \partial_n f_{pq} \Longleftrightarrow \partial_m A^m = 0. \tag{1.10}$$

The antisymmetric tensor field f_{mn} is the notoph gauge potential, it possesses the following gauge transformation law:

$$\delta f_{mn} = \partial_m \lambda_n - \partial_n \lambda_m, \tag{1.11}$$

where $\lambda_m(x)$ is an arbitrary vector gauge parameter. The vector $A^m = (1/2) \times \varepsilon^{mnpq} \partial_n f_{pq}$ is the relevant gauge invariant field strength and the condition $\partial_m A^m = 0$ is the corresponding Bianchi identity. The equation of motion for f_{mn} reads

$$\frac{1}{2} \varepsilon^{mnst} \partial_s A_t = -J^{mn} \quad \text{or} \quad \Box f^{mn} - \partial^m \partial_p f^{pn} + \partial^n \partial_p f^{pm} = 2J^{mn}, \tag{1.12}$$

$$J^{mn} := \frac{\partial L_{\text{int}}}{\partial f_{mn}}. \tag{1.13}$$

For the compatibility of the left-handed and right-handed parts of (1.12), the tensor current J_{mn} should be conserved,

$$\partial^m J_{mn} = 0. \tag{1.14}$$

To see how many on-shell degrees of freedom the gauge field f_{mn} carries, one should take into account that the gauge freedom (1.11) actually involves three independent gauge parameters because of the additional freedom $\lambda_m \to \lambda_m + \partial_m \lambda$. So the field f_{mn} involves three independent off-shell degrees of freedom, like an Abelian gauge field, and represents spin 1 off shell. On shell, two additional degrees of freedom are eliminated by two analogs of the Gauss law in electrodynamics:

$$\Delta f^{0b} + \partial_0 (\partial_a f^{ab}) + \partial^b (\partial_a f^{0a}) = -2J^{0b},$$
$$\Delta = -\partial_a \partial^a = \partial_a \partial_a \quad (a, b = 1, 2, 3). \tag{1.15}$$

To be convinced that this relation indeed amounts to the two independent equations, one can check that, in virtue of the conservation law (1.14), $\partial_a J^{0a} = 0$,

only the transverse part f_{tr}^{0b} of f^{0b}, $\partial_b f_{\text{tr}}^{0b} = 0$, gives contribution to (1.15). As a result, we conclude that f^{mn} indeed comprises only one degree of freedom on shell.

An alternative way to demonstrate that the notoph presents just another description of massless particle with zero helicity is to perform the duality transformation relating the notoph theory to the theory of a single scalar field. For simplicity, we limit ourselves to the free theory, $L_{\text{int}} = 0$, and modify the free Lagrangian in (1.10) by adding to it, with the Lagrange multiplier φ, the 4-divergence $\partial_m A^m$ going to become the Bianchi identity:

$$L_0 = -\frac{1}{2}A^m A_m \Longrightarrow L_{\text{dual}} = -\frac{1}{2}A^m A_m + \varphi \partial_m A^m. \qquad (1.16)$$

When varying L_{dual} with respect to φ, we obtain the Bianchi identity $\partial^m A_m = 0$, after solving which through f^{mn} as in (1.10), the free Lagrangian of notoph is recovered. On the other hand, eliminating A^m from (1.16) by its algebraic equation of motion, $A_m = -\partial_m \varphi$, we obtain the free kinetic Lagrangian of the scalar field φ

$$L_{\text{dual}} \Longrightarrow L_\varphi = \frac{1}{2}\partial^m \varphi \partial_m \varphi. \qquad (1.17)$$

Note that in the case of nontrivial L_{int} this duality holds in a local way only if L_{int} depends on f^{mn} through the covariant gauge field strength. So in general the descriptions of the massless spin-zero particle through the scalar field and through the notoph field result in physically nonequivalent theories. It is just the description by the antisymmetric gauge fields that naturally appears in superstring theory and some extended supergravities.

One more interesting property of the notoph theory is that its massive version is equivalent to the massive deformation of the Abelian $U(1)$ theory and so describes three independent degrees of freedom on shell. Once again, for simplicity we will consider the case without interaction and modify the free action of the notoph as

$$L_0 = -\frac{1}{2}A^m A_m \Longrightarrow L_{(m)} = -\frac{1}{2}A^m A_m - \frac{1}{4}m^2 f^{mn} f_{mn},$$

$$A^m = \frac{1}{2}\varepsilon^{mnpq} \partial_n f_{pq}. \qquad (1.18)$$

The equation of motion (1.12) is modified as

$$\Box f^{mn} - \partial^m \partial_p f^{pn} + \partial^n \partial_p f^{pm} + m^2 f^{mn} = 0. \qquad (1.19)$$

Now the notoph gauge invariance is broken. Instead, Eq. (1.19) implies the transversality condition $\partial_m f^{mn} = 0$. It is easy to see that this equation actually amounts to three independent conditions (because $\partial_m \partial_n f^{mn} = 0$ is satisfied

identically), thus demonstrating that the massive f^{mn} indeed propagates three independent degrees of freedom on shell. We can dualize (1.18) as

$$L_{(m)} = -\frac{1}{2}A^m A_m - \frac{1}{4}m^2 f^{mn} f_{mn} \Longrightarrow L_{(m)}^{\text{dual}} = \frac{1}{2}A^m A_m +$$
$$+ \frac{1}{2}A^m \varepsilon_{mnpq}\partial^n f^{pq} - \frac{1}{4}m^2 f^{mn} f_{mn}, \quad (1.20)$$

where now A_m is treated as an independent auxiliary field. Varying with respect to A_m, we come back to the theory (1.18). On the other hand, varying with respect to f^{mn}, we obtain

$$f_{mn} = -\frac{1}{m^2}\varepsilon_{mnpq}\partial^p A^q. \quad (1.21)$$

Substituting it into (1.20), we obtain, up to a rescaling

$$L_{(m)}^{\text{dual}} = -\frac{1}{4}F^{mn} F_{mn} + \frac{1}{2}m^2 A^m A_m, \quad F_{mn} = \partial_m A_n - \partial_n A_m. \quad (1.22)$$

Thus, both the theory of gauge Abelian vector field describing on shell two degrees of freedom (helicities ± 1) and the gauge theory of notoph describing on shell one degree of freedom (zero helicity) after the minimal mass deformation yield the same theory of massive spin 1 which propagates three degrees of freedom on shell. So these two gauge theories are complementary to each other in the sense that the full set of helicities of the relevant particles equals to the set of the projections of the massive spin 1. So they can be treated as two different massless limits of the Abelian massive spin 1 theory[*].

Though notoph is a necessary mathematical ingredient of supergravities and string theory, it is still an open question whether it could manifest itself in a more phenomenological context as an elementary particle, like the standard vector gauge fields (photon, intermediate vector bosons, etc.). The authors of [16] indicated a few possible processes where the notoph could be produced, but no sign of it was detected so far. Despite the fact that the Standard Model and its currently discussed generalizations have seemingly no direct need in such an entity, nevertheless, to our knowledge, no no-go theorem against such a possibility was adduced.

1.3. Spinors in the Gravitation Theory. One more important and far-reaching result of the Ogievetsky–Polubarinov collaboration concerned the description of spinors in general relativity. They showed [18] that there is no direct necessity to introduce the orthogonal repere (vierbein) in order to construct the

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[*]This complementarity does not generalize to the non-Abelian case, at least in a direct way.

invariant coupling of Dirac fields to the gravitons; this can be done in a minimal way by ascribing, to spinorial fields, the nonlinear in graviton transformation law under the space-time diffeomorphism group, without introducing any extra entities. Actually, [18] anticipated the nonlinear realizations method which was discovered and applied for description of spontaneously broken symmetries in the low-energy strong interactions ("chiral dynamics") by Schwinger, Weinberg, Volkov, and others in a few years. Also, it was the first step towards interpreting gravity as a theory of two spontaneously broken space-time symmetries, the affine and conformal ones, by Borisov and Ogievetsky [19].

1.4. Nonlinear Realizations and Chiral Dynamics. V. I. Ogievetsky, together with his PhD student Boris Zupnik (1945–2015), took an active participation in developing and applying the above-mentioned nonlinear realization and effective Lagrangian methods of describing various low-energy systems. In particular, they proposed a new general method of constructing nonlinear realizations of the groups $U(N)$ [20], before the appearance of the seminal papers on the general theory of nonlinear realizations [2, 3]. Also, a new effective Lagrangian was proposed to describe the (π, ρ, A_1) system, with the maximally smooth momentum behavior of the corresponding amplitudes [21]. This model found a few interesting and unexpected applications, in particular, it was used to calculate contributions of the so-called exchange currents in some nuclear reactions [22].

1.5. Einstein Gravity from Nonlinear Realizations. As a natural continuation of this research activity, Ogievetsky soon got interested in applying the nonlinear realizations method, developed in [2] basically for internal symmetries, to the space-time symmetry groups including the Poincaré group as a subgroup [3, 4]. Studying the structure of the diffeomorphism group in R^4, he discovered that this infinite-dimensional group can be nicely represented as a closure of its two finite-dimensional subgroups, affine and conformal ones, intersecting over the common Weyl subgroup (the semidirect product of the Poincaré group and dilatations) [23]. This remarkable observation is now known as the Ogievetsky theorem. It has many applications, in particular, in supergravity and M-theory (see, e.g., [24] and references therein).

To explain the meaning of this theorem, let us write the special conformal transformation in the Minkowski space,

$$\delta_\beta x^m = \beta^m x^2 - 2(\beta \cdot x) x^m, \tag{1.23}$$

where β^m is the transformation parameter of the mass dimension, and the general linear $gl(4, R)$ transformation

$$\delta_\lambda x^m = -\Lambda^m{}_n x^n, \tag{1.24}$$

where the constant dimensionless parameters $\Lambda^m{}_n$ form a real 4×4 matrix comprising 16 independent parameters. The transformations (1.24) involve 6-parameter

transformations with the antisymmetric matrix $\Lambda_{[mn]}$, which form the Lorentz subalgebra $so(1,3)$ in $gl(4,R)$, and the 10-parameter transformations with the symmetric matrix $\Lambda_{(mn)}$, which belong to the symmetric coset of the group $GL(4,R)$ over the Lorentz group $SO(1,3)$. One can also add the translations of x^m, which complete $gl(4,R)$, to the affine algebra $A(4)$. The translations, together with the Lorentz transformations and dilatations, also extend (1.23) to the 4-dimensional conformal algebra $so(4,2)$.

Let us now define the generators corresponding to (1.23) and symmetric part of (1.24),

$$K_n := -i\left(x^2\delta_n^m - 2x^m x_n\right)\partial_m, \quad R_{st} := -i(x_s\partial_t + x_t\partial_s), \tag{1.25}$$

and compute their commutator, $[K_n, R_{st}]$. In this commutator, besides the generator K_n, we find new generators of the second order in x^m:

$$\sim \eta_{ns}x_t(x^p\partial_p), \quad \sim x_s x_t\partial_n. \tag{1.26}$$

Commuting these new generators with K_n, R_{st} and themselves, we encounter new generators of the third order in x^m, etc. V. I. Ogievetsky has proved that this process does not terminate at any finite step and produces the whole set of generators

$$L_m^{n_1 n_2 n_3 n_4} = (x^0)^{n_1}(x^1)^{n_2}(x^2)^{n_3}(x^3)^{n_4}\partial_m, \tag{1.27}$$

constituting the infinite-dimensional diffeomorphism group* Diff R^4

$$\delta x^m = f^m(x) = \sum_{n_1,n_2,n_3,n_4} c^{\{n_1 n_2 n_3 n_4\}\, m}(x^0)^{n_1}(x^1)^{n_2}(x^2)^{n_3}(x^3)^{n_4}. \tag{1.28}$$

Here $n_1,\ldots,n_4$ are arbitrary non-negative integers, $n_i \geqslant 0$, and $c^{\{n_1 n_2 n_3 n_4\}\, m}$ are constant parameters.

Based on this theorem, Ogievetsky with his PhD student Alexander Borisov constructed the sigma-model-type theory invariant under the simultaneous nonlinear realizations of the affine and conformal groups and showed that this theory is nothing else as the Einstein gravitation theory [19]. Let us recall the basic details of their construction.

The starting point is two algebras, affine and conformal, involving, respectively, the generators (P_m, L_{mn}, R_{mn}) and (P_m, K_n, L_{mn}, D) with the following

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*To be more exact, the connected subgroup of Diff R^4 consisting of all transformations expandable in the Taylor series around the origin $x^m = 0$.

commutation relations[*]:

$$[L_{mn}, L_{pq}] = i\big(\eta_{np}L_{mq} - \eta_{mp}L_{nq} - (p \leftrightarrow q)\big),$$
$$[L_{mn}, R_{pq}] = i\big(\eta_{np}R_{mq} - \eta_{mp}R_{nq} + (p \leftrightarrow q)\big),$$
$$[R_{mn}, R_{pq}] = i\big(\eta_{mp}L_{nq} + \eta_{np}L_{mq} + (p \leftrightarrow q)\big),$$
$$[L_{mn}, P_q] = i\big(\eta_{nq}P_m - \eta_{mq}P_n\big),$$
$$[R_{mn}, P_q] = i\big(\eta_{nq}P_m + \eta_{mq}P_n\big),$$

(1.29)

$$[P_m, P_n] = [K_m, K_n] = 0, \quad [D, P_m] = iP_m, \quad [D, K_m] = -iK_m,$$
$$[L_{mn}, K_q] = i\big(\eta_{nq}K_m - \eta_{mq}K_n\big), \quad [P_m, K_n] = 2i\big(\eta_{mn}D + L_{mn}\big).$$

(1.30)

These two algebras intersect over the Weyl algebra $(P_s, L_{mn}, D = (1/2)R^m_m)$.

As the next step, the authors of [19] constructed a nonlinear realization of the affine group $\mathcal{A}(4)$ in the coset space over the Lorentz subgroup,

$$\frac{\mathcal{A}(4)}{SO(1,3)} \sim \frac{\{P_m, L_{mn}, R_{mn}\}}{\{L_{mn}\}},$$

(1.31)

with the following parameterization of the coset element:

$$G(x, h_{mn}) = \mathrm{e}^{ix^m P_m} \, \exp\left(\frac{i}{2}h^{mn}(x)R_{mn}\right),$$

(1.32)

where $h^{mn}(x)$ is the symmetric tensor Goldstone field. Under the left multiplications by the element g of $\mathcal{A}(4)$, the coset representative is transformed as

$$G'(x', h') = \mathrm{e}^{ix^{m'}P_m} \, \exp\left(\frac{i}{2}h^{mn'}(x')R_{mn}\right) =$$
$$= g\,G(x, h)\,\exp\left(-\frac{i}{2}u^{mn}(x, h, g)L_{mn}\right), \quad (1.33)$$

where $u^{mn}(x, h, g)$ is the induced Lorentz group parameter. In particular, left multiplications by $\exp\big((i/2)\lambda^{[mn]}L_{mn}\big)$ and $\exp\big((i/2)\lambda^{(mn)}R_{mn}\big)$ yield for x^m the transformations (1.24), with $\Lambda^{mn} = \lambda^{[mn]} + \lambda^{(mn)}$. According to the general prescriptions of nonlinear realizations, one can now construct the left-covariant Cartan one-forms

$$G^{-1}dG = i\omega^m_{(P)}P_m + \frac{i}{2}\omega^{mn}_{(R)}R_{mn} + \frac{i}{2}\omega^{mn}_{(L)}L_{mn}.$$

(1.34)

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[*]We use slightly different conventions as compared to [19]. They are the same as in [13].

The form $\omega_{(P)}^m$ is calculated to be

$$\omega_{(P)}^m = e_p^m dx^p,$$

$$e_p^m = \left(\exp\left(\frac{i}{2}h^{st}\hat{R}_{st}\right)\right)_p^m = (\mathrm{e}^h)_p^m = \delta_p^m + h_p^m + \frac{1}{2}h_p^n h_n^m + \ldots, \quad (1.35)$$

$$(\hat{R}_{st})_p^m = -i(\eta_{sp}\delta_t^m + \eta_{tp}\delta_s^m).$$

The external product of four forms $\omega_{(P)}^m$ defines the invariant R_4 volume element, $\mathrm{Vol}\, R^4 = \det e_p^m\, d^4x = \mathrm{e}^{h^m_m}\, d^4x$, the form $\omega_{(R)}^{mn}$ defines the covariant derivative of the tensor Goldstone field h^{mn}, $\omega_{(R)}^{mn} = \omega_{(P)}^s \nabla_s h^{mn}$, and the inhomogeneously transforming form $\omega_{(L)}^{mn}$ — the covariant differential and the covariant derivative of the "matter" field Ψ^A transforming by some irreducible representation of the Lorentz group with the matrix generators $(S_{mn})^A{}_B$:

$$\mathcal{D}\Psi^A = d\Psi^A + \frac{i}{2}\omega_{(L)}^{mn}(S_{mn})^A{}_B\Psi^B := \omega_{(P)}^s \mathcal{D}_s\Psi^A, \qquad (1.36)$$

$$\mathcal{D}_s\Psi^A = (e^{-1})_s^p \partial_p\Psi^A + \frac{i}{2}\mathcal{V}_s^{mn}(S_{mn})^A{}_B\Psi^B, \quad \omega_{(L)}^{mn} := \omega_{(P)}^s \mathcal{V}_s^{mn}. \qquad (1.37)$$

All these objects were explicitly computed in [19]. An important observation was that the Lorentz connection in (1.37) can be generalized, without affecting its transformation properties, by adding three independent combinations of the covariant derivatives $\nabla_p h^{mn}$

$$\mathcal{V}_{mn,s} \Rightarrow \mathcal{V}_{mn,s}^{\mathrm{gen}} = \mathcal{V}_{mn,s} + \alpha_1 \nabla_{[m}h_{n]s} + \alpha_2\, \eta_{s[m}\nabla_{n]}h^p{}_p +$$
$$+ \alpha_3\, \eta_{s[m}\nabla_p h^p{}_{n]}. \quad (1.38)$$

The next step was the analogous construction of nonlinear realizations of the conformal group in the coset with the same stability subgroup

$$\frac{SO(2,4)}{SO(1,3)} \sim \frac{\{P_m, L_{mn}, K_n, D\}}{\{L_{mn}\}},$$
$$\tilde{G}(x,\sigma,\varphi) = \mathrm{e}^{ix^m P_m}\, \mathrm{e}^{i\varphi^m(x)K_m}\, \mathrm{e}^{i\sigma(x)D}. \qquad (1.39)$$

The conformal group is realized by left shifts on the coset element $\tilde{G}(x,\sigma,A)$. In particular, the left multiplication by $\mathrm{e}^{i\beta^m K_m}$ generates for x^m just the transformation (1.23). The left-covariant Cartan forms are defined by

$$\tilde{G}^{-1}d\tilde{G} = i\tilde{\omega}_{(P)}^m P_m + i\omega_{(D)}D + i\omega_{(K)}^m K_m + \frac{i}{2}\tilde{\omega}_{(L)}^{mn}L_n. \qquad (1.40)$$

They can be easily computed. In particular,

$$\tilde{\omega}_{(P)}^m = \mathrm{e}^\sigma dx^m, \quad \omega_{(D)} = d\sigma - 2\varphi_m dx^m, \quad \tilde{\omega}_{(L)}^{mn} = 2(\varphi^m dx^n - \varphi^n dx^m). \quad (1.41)$$

Taking into account that $\sigma = (1/4)h^m_m$ (because of the identification $D = (1/2)R^m_m$), we observe that the invariant volume $\mathrm{Vol}\, R^4$ is the same in both nonlinear realizations

$$\mathrm{Vol}\, R^4 = \mathrm{e}^{4\sigma} d^4 x = \mathrm{e}^{h^m_m} d^4 x.$$

The vector Goldstone field $\varphi_m(x)$ is unessential, as it can be covariantly eliminated by equating to zero the Cartan form $\omega_{(D)}$:

$$\omega_{(D)} = 0 \Rightarrow \varphi_m = \frac{1}{2}\partial_m \sigma \tag{1.42}$$

(this is a particular case of the "inverse Higgs phenomenon", see the next subsection). The covariant derivative of the "matter" fields is given by the expression

$$\tilde{\mathcal{D}}_m \Psi^A = \mathrm{e}^{-\sigma} \left(\partial_m \Psi^A - i\partial_n \sigma (S^n_m)^A_B \Psi^B \right). \tag{1.43}$$

As the last step in deriving the Einstein gravity, the authors of [19] analyzed the issue of simultaneous covariance under both nonlinear realizations constructed. As a consequence of the Ogievetsky theorem, the theory exhibiting such a covariance should be invariant under the full Diff R^4 group.

The $\mathcal{A}(4)$ Goldstone field h^{mn} can be divided as $h^{mn} = \hat{h}^{mn} + (1/4)\eta^{mn}h^p_p = \hat{h}^{mn} + \eta^{mn}\sigma$, where the traceless tensor field $\hat{h}^{mn}$ can be treated as a "matter" field with respect to the nonlinear realization of the conformal group. Then one requires that the $\mathcal{A}(4)$ covariant derivative (1.37) with the generalized Lorentz connection (1.38) involves the dilaton field σ and its derivatives only through the conformal covariant derivative (1.43). This requirement uniquely fixes the coefficients in (1.38) as $\alpha_1 = -2$, $\alpha_2 = \alpha_3 = 0$. The resulting covariant derivative is simultaneously covariant under the nonlinear realizations of both affine and conformal groups and hence under their closure Diff R^4. Borisov and Ogievetsky also showed that no combinations of the $\mathcal{A}(4)$ covariant derivatives $\nabla_s h^{mn}$ exist, such that they are covariant under the conformal group. The latter property corresponds to the well-known fact that in the Riemannian geometry no tensors involving the first derivative of the metric tensor can be constructed. The first nontrivial tensor contains two derivatives and in the formulation of [19] it is constructed as the covariant curl of the Lorentz connection $\mathcal{V}^{\mathrm{gen}}_{mn,s}$:

$$(\mathcal{D}_m \mathcal{D}_n - \mathcal{D}_m \mathcal{D}_n)\Psi^A = \frac{i}{2} R^{pq}_{mn} (S_{pq})^A_B \Psi^B. \tag{1.44}$$

Since R^{pq}_{mn} undergoes the induced Lorentz rotation with respect to all of its indices, the object $R := R^{mn}_{mn}$ is invariant under the simultaneous nonlinear realizations of the affine and conformal groups and hence under the group Diff R^4. It can be represented as the standard Riemann scalar curvature with the metric

$$g_{mn} = e^p_m e_{np} = (\mathrm{e}^{ih^{sq}\hat{R}_{sq}})^p_m \eta_{np} = \eta_{mn} + 2h_{mn} + \ldots, \tag{1.45}$$

having the standard transformation properties under the coordinate transformations, $\delta g_{mn} = -\partial_m \delta x^p g_{pn} - \partial_n \delta x^p g_{mp}$. The minimal invariant action coincides with the Einstein–Hilbert action

$$-\frac{1}{16\pi G} \int d^4 x \sqrt{-g}\, R, \tag{1.46}$$

where $G = (1/4\pi) f^2$ and the constant f, $[f] = 1$, arises as the result of the standard rescaling of the dimensionless Goldstone field h_{mn}, $h_{mn} \to f h_{mn}$. The couplings to matter fields are constructed using the covariant derivative (1.37) with the connection $\mathcal{V}_{mn,s}^{\mathrm{gen}}$ (with the fixed coefficients ensuring the conformal group covariance). The connection $\mathcal{V}_{mn,s}^{\mathrm{gen}}$ can be related to the Christoffel symbols. The nonlinear transformation law for spinors deduced in [18] immediately follows from the general transformation law of matter fields in the realization (1.33), with the induced Lorentz parameter,

$$\delta \Psi^A = \frac{i}{2} u^{mn}(h)(S_{mn})^A{}_B \Psi^B, \tag{1.47}$$

upon specializing, e.g., to the $(1/2, 0)$ spinor representation, with $(1/2)(\sigma_{mn})^\alpha_\beta$ as the spin part of the Lorentz generators. In fact, the theory obtained in [19] can be reproduced from the Einstein theory formulated in terms of vierbeins e_m^a by gauge-fixing the local Lorentz rotations in the tangent space in such a way that the antisymmetric part of e_m^a vanishes.

The work [19] turned out very important in the conceptual sense, because it exposed, for the first time, the double nature of the spin 2 graviton field. On the one hand, it is a gauge field for the diffeomorphism group (modulo some subtle issues, see Subsec. 1.8) and, on the other hand, as follows from [19], it is the tensorial Goldstone field accompanying the spontaneous breakdown of the finite-dimensional affine and conformal groups. Later on, it was shown that any gauge field can be interpreted as a Goldstone field associated with some infinite-dimensional global symmetry [25] (see Subsec. 1.7). The space-time diffeomorphisms are distinguished in that they can be represented as a closure of two finite-dimensional groups, while there is no analogous theorem for the Yang–Mills type gauge groups. As was shown in [26, 27], the super Yang–Mills and supergravity theories (at least, the simple $\mathcal{N} = 1, 4D$ ones) can also be reproduced from the nonlinear realizations of some supergroups. The interpretation of the Yang–Mills and graviton fields, as well as their super Yang–Mills and supergravity counterparts, as the Goldstone (super)fields, and the associate (super)gauge and (super)gravity theories as nonlinear realizations, posed a natural question as to what could be *linear* realizations of these groups and which generalizations of *linear* sigma models of the underlying spontaneously broken symmetries could correspond to such realizations. Until now, there is no answer

to this question. Since in such hypothetical theories the Yang–Mills and/or gravity fields should appear inside some linear multiplets, while the symmetry generators carry the Lorentz indices, these multiplets should be infinite-dimensional and include fields with higher spins. So these hypothetical theories should be a sort of higher-spin or string-like theories (M-theory?). Note that in the nonlinear realization formulation the space-time coordinate x^m itself appears as a coset parameter. In the conjectured linear multiplets it should be present on equal footing with those components which are going to become Goldstone fields after spontaneous breaking.

1.6. Inverse Higgs Phenomenon. The classical Nambu–Goldstone theorem claims that, to any generator of spontaneously broken symmetry in the quantum field theory, there should correspond a massless Goldstone field with the inhomogeneous transformation law under this generator, such that it starts with the relevant transformation parameter. The basic result of [28] was the observation that in nonlinear realizations of space-time symmetries certain Goldstone fields can be covariantly traded for space-time derivatives of some minimal set of such fields, and there were established the general criteria under which this becomes possible. The condition under which the given Goldstone field admits an elimination is that the commutator of the space-time translation generator with the corresponding spontaneously broken generator again yields a spontaneously broken generator. This phenomenon was called "inverse Higgs phenomenon" or "inverse Higgs effect". It proved to work with an equal efficiency in the superfield theories as well. Now it is of indispensable use in theories with nonlinear realizations of space-time (super)symmetries.

As an example of inverse Higgs effect, let us reproduce the massive particle (0-brane) in the flat 2D space-time by the nonlinear realization method applied to the 2D Poincaré group $\mathcal{P}_{(2)}$ [29]. The latter involves two translation generators P_0, P_1 and the $SO(1,1)$ Lorentz generator L, with the only two nonvanishing commutators

$$[L, P_0] = iP_1, \quad [L, P_1] = iP_0. \tag{1.48}$$

Then we construct a nonlinear realization of $\mathcal{P}_{(2)}$, with the one-dimensional "Poincaré" generator P_0 as the only one to which a coordinate (time) is associated as the coset parameter. Two other generators pick up as the relevant parameters the "Goldstone" fields $X(t)$ and $\Lambda(t)$, giving rise to the following coset element:

$$G = \mathrm{e}^{itP_0}\, \mathrm{e}^{iX(t)P_1}\, \mathrm{e}^{i\Lambda(t)L}. \tag{1.49}$$

The group $\mathcal{P}_{(2)}$ acts as left shifts of G, $G \rightarrow G' = \mathrm{e}^{iaP_0}\, \mathrm{e}^{ia_1 P_1}\, \mathrm{e}^{i\sigma L} G$. The Cartan forms

$$G^{-1}\, dG = i\omega_0 P_0 + i\omega_1 P_1 + i\omega_L L, \tag{1.50}$$

$$\omega_0 = \sqrt{1 + \Sigma^2}\, dt + \Sigma\, dX, \quad \omega_1 = \sqrt{1 + \Sigma^2}\, dX + \Sigma\, dt,$$

$$\omega_L = \frac{1}{\sqrt{1 + \Sigma^2}}\, d\Sigma, \qquad \Sigma \equiv \operatorname{sh}\Lambda \tag{1.51}$$

by construction are invariant under this left action. We observe that the Lorentz Goldstone field $\Sigma(t)$ can be traded for $\dot{X}(t)$ by the inverse Higgs constraint

$$\omega_1 = 0 \Rightarrow \Sigma = -\frac{\dot{X}}{\sqrt{1 - \dot{X}^2}}. \tag{1.52}$$

This constraint is covariant, since ω_1 is the group invariant (in the generic case, the coset Cartan forms undergo homogeneous rotation in their stability subgroup indices). Thus, the obtained expression for Σ possesses correct transformation properties[*]. Substituting it into the remaining Cartan forms, we find

$$\omega_0 = \sqrt{1 - \dot{X}^2}\, dt, \quad \omega_L = \sqrt{1 - \dot{X}^2}\, \frac{d}{dt}\left(\frac{\dot{X}}{\sqrt{1 - \dot{X}^2}}\right) dt. \tag{1.53}$$

The simplest invariant action, the covariant length

$$S = \int \omega_0 = \int dt\, \sqrt{1 - \dot{X}^2}, \tag{1.54}$$

is recognized, up to a renormalization factor of the dimension of mass, as the action of 2D massive particle in the static gauge $X^0(t) = t$.

Another text-book example of how the inverse Higgs phenomenon works is the derivation of the Alfaro–Fubini–Furlan conformal mechanics from the non-linear realization of the $d = 1$ conformal group $SO(2,1) \sim SU(1,1)$ [30]. Its important application in constructing nonlinear realizations of 4D conformal group was already discussed in the previous subsection. Its use is crucial for deducing the superfield actions of branes in the approach based on the concept of partial spontaneous breaking of global supersymmetry (PBGS) (see, e.g., [31] and references therein). In the PBGS models, this effect has also dynamical manifestations, giving rise, in some cases, to the equations of motion as the result of equating to zero some appropriate Cartan forms (see, e.g., [32]). Such an extended inverse Higgs effect was also applied for deducing some two-dimensional integrable systems from nonlinear realizations of (super)groups (see, e.g., [33,34]) and deriving new kinds of superconformal mechanics in the superfield approach [35]. Recently, it was applied for construction of the Galilean conformal mechanics in [36].

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[*]The possibility to eliminate the field Σ (or Λ) follows from the criterion of the elimination mentioned earlier. Indeed, the commutator of the time-translation operator P_0 with the broken generator L yields the broken generator P_1 (see (1.48)).

The inverse Higgs phenomenon plays the key role in proving that any gauge theory, like the gravitation theory, admits an alternative interpretation as a theory of spontaneous breakdown, with the gauge fields as the corresponding unremovable Goldstone fields [25].

1.7. Yang–Mills Theory as a Nonlinear Realization. The basic idea of [25] was to represent the Yang–Mills gauge group* as a group with constant parameters and an infinite number of tensorial generators.

One starts with some internal symmetry group with the generators T^i,

$$[T^i, T^k] = ic^{ikl}T^l, \qquad (1.55)$$

and decomposes $\lambda^i(x)T^i$ as

$$\lambda^i(x)T^i = \lambda^i T^i + \sum_{n \geqslant 1} \frac{1}{n!} \lambda^i_{m_1 \cdots m_n} x^{m_1} \cdots x^{m_n} T^i. \qquad (1.56)$$

Denoting $T^{(m_1 \cdots m_n)i} := x^{m_1} \cdots x^{m_n} T^i$, we indeed can rewrite the gauge parameter with values in the Lie algebra (1.55) as

$$\lambda^i(x)T^i = \lambda^i T^i + \sum_{n \geqslant 1} \frac{1}{n!} \lambda^i_{m_1 \cdots m_n} T^{(m_1 \cdots m_n)i}, \qquad (1.57)$$

i.e., as a particular representation of the infinite-dimensional algebra generated by $T^{(m_1 \cdots m_n)i}$. Viewed as the abstract algebra, this set of generators, together with the 4-translation generator $P_m = (1/i)\partial_m$, is closed under the commutation relations

$$[T^{(m_1 \cdots m_n)i}, T^{(p_1 \cdots p_s)k}] = ic^{ikl}T^{(m_1 \cdots m_n p_1 \cdots p_s)l}, \quad n \geqslant 1, \quad s \geqslant 1,$$
$$[T^i, T^{(m_1 \cdots m_n)k}] = ic^{ikl}T^{(m_1 \cdots m_n)l}, \qquad (1.58)$$
$$[P_n, T^{m\,i}] = -i\delta_n^m T^i,$$

$$[P_n, T^{(m_1 \cdots m_k)i}] = -i\Big(\delta_n^{m_1} T^{(m_2 \cdots m_k)i} + \ldots + \delta_n^{m_k} T^{(m_1 \cdots m_{k-1})i}\Big), \quad k \geqslant 2, \qquad (1.59)$$

to which one should add (1.55). With respect to the Lorentz group, the generators $T^{(m_1 \cdots m_n)i}, n \geqslant$, form symmetric tensors of the rank n. In fact, the Lorentz group can be considered as external automorphisms of the algebra of the remaining generators and so it decouples.

The full infinite-dimensional group involving the abstract generators $\big(P_s, T^i, T^{(m_1 \cdots m_n)k}\big)$ can be called $\mathcal{K}$. By analogy with the interpretation of

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*To be more exact, the connected component of the full gauge group, spanned by the gauge functions admitting a decomposition into the Taylor series in a vicinity of $x^m = 0$.

the Einstein gravity as a nonlinear realization of the group Diff R^4, it was suggested in [25] that the Yang–Mills theory can be interpreted as the appropriate nonlinear realization of $\mathcal{K}$. An essential difference from the case of gravity is that $\mathcal{K}$ cannot be obtained as a closure of any two finite-dimensional subgroups. Actually, the only closed nontrivial subgroup of $\mathcal{K}$ is the original internal symmetry group G_0 generated by T^i subjected to (1.55). So the appearance of an infinite number of Goldstone fields in any nonlinear realization of $\mathcal{K}$ seems inevitable. Fortunately, most of such fields are eliminated by the inverse Higgs effect.

Thus, let us consider the realization of $\mathcal{K}$ by the left shifts on the coset manifold $\mathcal{K}/G_0$. The coset element can be written as

$$G(x,b) = \mathrm{e}^{ix^m P_m} \, \exp\left(i \sum_{n \geqslant 1} \frac{1}{n!} \, b^i_{m_1 \cdots m_n}(x) T^{(m_1 \cdots m_n)i} \right) \tag{1.60}$$

and under the left $\mathcal{K}$ multiplications is transformed as

$$G(x,b') = g G(x,b) \, \mathrm{e}^{-iu^i(x,b,g)T^i},$$

$$g = \mathrm{e}^{ia^k T^k} \, \exp\left(i \sum_{n \geqslant 1} \frac{1}{n!} \, a^k_{m_1 \cdots m_n} T^{(m_1 \cdots m_n)k} \right). \tag{1.61}$$

The "matter" fields Ψ^α, in accord with the general rules of nonlinear realizations [2], are transformed as

$$\Psi^{\alpha\prime} = (\mathrm{e}^{iu^i(x,b,g)\hat{T}^i})^\alpha_\beta \Psi^\beta, \tag{1.62}$$

where $\hat{T}^i$ is the matrix realization of the generators T^i in the representation of G_0 by which Ψ^α is transformed.

The first factor in g (1.61) just homogeneously rotates all fields with respect to the adjoint representation index i, so $u^i(x,b,g) = a^i$ in this case and (1.62) yields global G_0 transformation of Ψ^α. The parameters of the second factor generate some nonlinear inhomogeneous transformations of the coset fields like $\delta b^i_{m_1 \cdots m_n}(x) = a^i_{m_1 \cdots m_n} + O(b)$. Using the commutation relations (1.58) and (1.59), it is rather direct to establish that $u^k(x,b,g) = \sum_{n \geqslant 1} \frac{1}{n!} \, a^k_{m_1 \cdots m_n} x^{m_1} \cdots x^{m_n} := \lambda^k(x)$ in this case. In other words, the induced G_0 transformation is just the standard G_0 gauge transformation. Where is then the gauge field? To answer this question, we need to construct the corresponding Cartan forms and the covariant derivative of Ψ^α

$$G^{-1}dG = idx^m P_m + i\mathcal{V}^k_m dx^m T^k + i \sum_{n \geqslant 1} \frac{1}{n!} \, \nabla_s b^k_{(m_1 \cdots m_n)} dx^s T^{(m_1 \cdots m_n)k}, \tag{1.63}$$

$$\mathcal{D}_m \Psi^\alpha = \partial_m \Psi^\alpha + i\mathcal{V}_m^k (\hat{T}^k)^\alpha_{\ \beta} \Psi^\beta. \tag{1.64}$$

It is easy to compute

$$\mathcal{V}_m^k = b_m^k, \quad \nabla_m b_n^i = \partial_m b_n^i + b_{(mn)}^i - \frac{1}{2} c^{ikl} b_m^k b_n^l, \tag{1.65}$$

$$\nabla_s b_{(m_1 \cdots m_n)}^i = \partial_s b_{(m_1 \cdots m_n)}^i + \dots, \tag{1.66}$$

$$\delta b_m^i = -\partial_m \lambda^i + c^{ikl} b_m^k \lambda^l, \quad \delta \nabla_m b_n^i = c^{ikl} (\nabla_m b_n^k) \lambda^l, \text{ etc.} \tag{1.67}$$

From (1.64), (1.65), and (1.67) we observe that b_m^i possesses the standard transformation properties of the Yang–Mills field and enters the covariant derivative of Ψ^α in the right way. Further, we observe that the skew-symmetric and symmetric parts of the covariant derivative $\nabla_m b_n^i$,

$$2\nabla_{[m} b_{n]}^i = \partial_m b_n^i - \partial_n b_m^i - c^{ikl} b_m^k b_n^l,$$
$$2\nabla_{(m} b_{n)}^i = \partial_m b_n^i + \partial_n b_m^i + 2 b_{(mn)}^i, \tag{1.68}$$

are covariant separately. The skew-symmetric part is just the covariant field strength of b_m^i, while the symmetric part can be put equal to zero, yielding the inverse Higgs expression for b_{mn}^i

$$\nabla_{(m} b_{n)}^i = 0 \Rightarrow b_{(mn)}^i = -\partial_{(m} b_{n)}^i. \tag{1.69}$$

As follows from (1.58) and (1.59), the commutators of P_m with all spontaneously broken generators $T^{(m_1 \cdots m_n)i}$, besides $T^{m\,i}$, contain the generators of the same (spontaneously broken) type, so all the related Goldstone fields can be eliminated by the inverse Higgs effect. This can be accomplished, like in (1.69), by equating to zero the totally symmetric parts of the appropriate covariant derivatives

$$\nabla_{(s} b_{m_1 \cdots m_n)}^i = 0. \tag{1.70}$$

The commutator (1.58) yields the generator of the stability subgroup, so the Goldstone (gauge) field b_m^i cannot be eliminated, and it is the only "true" Goldstone field in the considered case.

In [37], a general solution of (1.70) was found. The abstract algebra $\mathcal{K}$ was realized as*

$$P_m = \frac{1}{i} \frac{\partial}{\partial y^m}, \quad T^{(m_1 \cdots m_n)i} = y^{m_1} \cdots y^{m_n} T^i, \tag{1.71}$$

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*Our conventions here are slightly different from those in [37].

where y^m is some new 4-vector coordinate. Then the coset element (1.60) can be rewritten in the concise form as

$$G(x,y) = e^{ix^m P_m} e^{ib^k(x,y)T^k},$$

$$b^k(x,y) = \sum_{n \geqslant 1} \frac{1}{n!} b^k_{(m_1 \cdots m_n)}(x) \, y^{m_1} \cdots y^{m_n}, \quad b^k(x,0) = 0, \tag{1.72}$$

while the covariant derivatives of the Goldstone fields corresponding to the Cartan forms, (1.64) as

$$e^{-ib^k(x,y)T^k}(\partial^x_m + \partial^y_m) e^{-ib^k(x,y)T^k} = i\omega^k_m(x,y)T^k,$$

$$\omega^k_m(x,y) = b^k_m(x) + \sum_{n \geqslant 1} \frac{1}{n!} \nabla_m b^k_{(m_1 \cdots m_n)}(x) \, y^{m_1} \cdots y^{m_n}. \tag{1.73}$$

The inverse Higgs constraints (1.70) in this formalism are rewritten as

$$y^m(\partial^x_m + \partial^y_m) e^{-ib^k(x,y)T^k} = -iy^m b^k_m(x)T^k e^{-ib^k(x,y)T^k} \tag{1.74}$$

and are solved by

$$e^{-ib^k(x,y)T^k} = P \exp \left\{ i \int_{x-y}^{x} b^k_m(\xi)T^k \, d\xi^m \right\}, \tag{1.75}$$

where P denotes ordering in the matrices T^i along the straight line connecting the points $(x-y)^m$ and x^m. This representation could be suggestive for identifying the hypothetical *linear* sigma model for gauge fields corresponding to the nonlinear realization constructed above.

To summarize, all gauge theories including gravity and Yang–Mills theory, correspond to the spontaneous breaking of some underlying symmetry, finite- or infinite-dimensional, and can be consistently derived by applying the general non-linear realizations machinery to these symmetries. The inverse Higgs phenomenon plays a crucial role in this derivation.

1.8. Gravitation Theories as Gauge Theories. Finally, it is worth mentioning here the papers [38,39] closely related to the circle of problems discussed in this section.

Actually, for a long time there were certain difficulties with the treatment of gravitation theories as gauge theories. The most direct analogy with the Yang–Mills theories is achieved, when treating gravity as a gauge theory associated with local symmetries in the tangent space, rather than with Diff R^4 obtained by gauging rigid space-time symmetries, like x^m-translations and Lorentz rotations.

The basic objects in such formulations are the direct and inverse vierbeins e_m^a and e^{ma} treated as gauge fields for some translation-like generators in the tangent space. The basic problem with such formulations was how to covariantly eliminate other gauge fields associated with these tangent-space groups, which include some other generators in parallel with the translation-like ones. In our papers [38, 39] with Jiri Niederle (1939–2010), the correct way of deriving various versions of gravitation theories by gauging the groups in the tangent space was formulated. It was shown there how to construct the correct formulations which make manifest the deep analogies of the gravitation theories with the standard Yang–Mills theories and ensure the covariance of the conditions eliminating the redundant gauge fields (actually, these constraints are quite similar to the inverse Higgs conditions). One should treat the tangent space gauge groups as spontaneously broken ones, with some additional Goldstone fields associated to the translation-like generators*. Then the vierbeins are to be identified with the covariant derivatives of such fields, rather than directly with the gauge fields (e.g., in the Einstein gravity, $e_m^a = \partial_m \phi^a + \ldots$, where $\phi^a(x)$ is the Goldstone field parameterizing the spontaneously broken tangent space 4-translations). The standard gravity theories naturally come out after choosing the "soldering" gauge, in which these extra Goldstone fields are identified with the space-time coordinates. It was also explicitly shown that the diverse gravity theories which differ in the maximally symmetric classical backgrounds (e.g., Weyl, Poincaré, de Sitter and anti-de Sitter gravities, etc.) correspond to gauging different tangent space groups having as the common important feature the presence of some spontaneously broken translation-like generators (in general, corresponding to curved "translations").

2. SUPERSYMMETRY: EARLY YEARS

The invention of supersymmetry in the early 1970s [5–7] sharply influenced the further fate of the mainstream research activity in the Markov Group. V. I. Ogievetsky rapidly realized the potential importance of this new concept for the particle and mathematical theoretical physics. One of the discoverers of supersymmetry was Dmitry Vasil'evich Volkov (1925–1996) from the Kharkov Institute of Physics and Technology. For a long time he and his group had close scientific and human contacts with Ogievetsky and his collaborators. So it is not surprising that the study of supersymmetric theories became the basic research direction in the group of young researchers formed and headed by Ogievetsky. The active members of this team were Luca Mezincescu from Bucharest and Emery Sokatchev from Sofia. Boris Zupnik, who defended his PhD by that time and re-

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*A similar proposal was made in [40].

ceived a position at the Institute of Nuclear Physics in Ulugbek near Tashkent, has penetrated deeply into this new area despite a few thousand kilometers separating Dubna and Ulugbek (Boris succeeded to come back to Dubna and join our group only in 1994). The author, after defending his PhD in 1976 under supervision of V. I. Ogievetsky, has also focused on this line of investigations. Later on, this research team constituted the main staff of the Sector "Supersymmetry" at BLTP headed by V. I. Ogievetsky. In the early 1980s it was enriched by the talented and enthusiastic PhD student Sasha Galperin from Tashkent. Also, I would like to mention Stilian Kalitzin from Sofia. Like A. Galperin, he was a representative of the second generation of the supersymmetry adepts. To the same category I would attribute Alexander Sorin, my first PhD student. The permanent contacts with the Volkov group, as well as with the Fradkin group from Lebedev Institute, were certainly very helpful and conducive for the successful development of investigations on supersymmetry in Dubna. In particular, an essential contribution to this development was brought by Volkov's scholar Anatoly Pashnev (1948–2004) who was employed on the contract at BLTP from Kharkov in 1991 and worked in Dubna until his untimely and tragic death. Frequent visits to Dubna by Sasha Kapustnikov (1945–2003) from Dniepropetrovsk, my friend and co-author for many years, have also played an invaluable role. More recently, in the 1990s, there were established fruitful and firm contacts with the research groups of the theorists from Tomsk, Joseph Buchbinder and Anton Galajinsky.

Since the start of the supersymmetry epoch, the basic interests of Ogievetsky and his surrounding were concentrated on the superspace approach to supersymmetric theories.

2.1. Invariant Actions in Superspace. The natural arena for supersymmetry is superspace, an extension of some bosonic space by anticommuting fermionic (Grassmann) coordinates. For the $\mathcal{N} = 1$ Poincaré supersymmetry, it was introduced in [6] as a coset of the $\mathcal{N} = 1$ Poincaré supergroup over its bosonic Lorentz subgroup. However, the fermionic coset parameters were treated in [6] as Nambu–Goldstone fields "living" on Minkowski space and supporting a nonlinear realization of $\mathcal{N} = 1$ Poincaré supersymmetry. It was suggested by Salam and Strathdee [41] to treat the fermionic coordinates $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}, (\alpha, \dot{\alpha} = 1, 2)$, on equal footing with x^m as *independent* coordinates. The fields on such an extended space (superspace) were christened superfields. They naturally encompass the irreducible $\mathcal{N} = 1$ supermultiplets, the fields of which appear as coefficients in the expansions of superfields over Grassmann coordinates. The remarkable property of superfields is that these expansions terminate at a finite step due to the nilpotency of the Grassmann coordinates [41, 42]. Another advantage of superfields is the simple rule of constructing component actions invariant under supersymmetry. In any products of superfields and their ordinary and/or covariant spinor derivatives the highest components in the expansions of these products over Grassmann coordinates (the so-called D component, if the product is a general superfield,

and the F component, if the product is a chiral superfield) is transformed to a total x-derivative and so is a candidate for the supersymmetric action in the Minkowski space-time.

In [43], Ogievetsky and Mezincescu proposed an elegant way of writing down the invariant superfield actions directly in superspace. As just mentioned, the invariant actions can be constructed as the x-integrals of the coefficients of the highest-degree θ monomials in the appropriate products of the involved superfields. The question was how to extract these components in a manifestly supersymmetric way. Ogievetsky and Mezincescu proposed to use the important notion of Berezin integral [44] for this purpose. In fact, Berezin integration is equivalent to the Grassmann differentiation and, in the case of $\mathcal{N} = 1$ superspace, is defined by the rules

$$\int d\theta_\alpha \, \theta^\beta = \delta_\alpha^\beta, \quad \int d\theta^\alpha \, 1 = 0, \quad \{d\theta_\alpha, d\theta_\beta\} = \{\theta_\alpha, d\theta_\beta\} = 0, \qquad (2.1)$$

and analogous ones for the conjugated coordinates $\bar{\theta}^{\dot\alpha}$. It is easy to see that, up to the appropriate normalization,

$$\int d^2\theta \, (\theta)^2 = 1, \quad \int d^2\bar{\theta} \, (\bar{\theta})^2 = 1, \quad \int d^2\theta d^2\bar{\theta} \, (\theta)^4 = 1, \qquad (2.2)$$

and hence Berezin integration provides the manifestly supersymmetric way of extracting the coefficients of the highest-order θ monomials. For example, the simplest invariant action of chiral superfields can be written as

$$S \sim \int d^4x \, d^4\theta \, \varphi(x_L, \theta)\bar{\varphi}(x_R, \bar{\theta}), \quad x_L^m = x^m + i\theta\sigma^m\bar{\theta}, \quad x_R^m = \overline{(x_L^m)}, \quad (2.3)$$

where the superfields satisfy the chirality and antichirality conditions

$$\bar{D}_{\dot\alpha}\varphi(x_L, \theta) = 0, \quad D_\alpha\bar{\varphi}(x_R, \bar{\theta}) = 0, \qquad (2.4)$$

with

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^m\bar{\theta})_\alpha\partial_m, \quad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} - i(\theta\sigma^m)_{\dot\alpha}\partial_m,$$
$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i(\sigma^m)_{\alpha\dot\alpha}\partial_m. \qquad (2.5)$$

Using the θ expansion of the chiral superfield $\varphi(x_L, \theta)$,

$$\varphi(x_L, \theta) = \varphi(x_L) + \theta^\alpha\psi_\alpha(x_L) + (\theta)^2 F(x_L), \qquad (2.6)$$

and its conjugate $\bar{\varphi}(x_R, \bar{\theta})$, it is easy to integrate over $\theta, \bar{\theta}$ in (2.3) and, discarding total x-derivatives, to obtain the component form of the action

$$S \sim \int d^4x \left(\partial^m\bar{\varphi}\partial_m\varphi - \frac{i}{2}\psi\sigma^m\partial_m\bar{\psi} + F\bar{F} \right). \qquad (2.7)$$

It is just the free action of the massless scalar $\mathcal{N} = 1$ multiplet. It can be generalized to the case with interaction, choosing the Lagrangian as an arbitrary function $K(\bar{\varphi}, \varphi)$ and adding independent potential terms

$$\sim \int d^4 x_L \, d^2\theta \, P(\varphi) + \text{c. c.} \tag{2.8}$$

The sum of (2.3) and the superpotential term (2.8) with $P(\varphi) \sim g\varphi^3 + m\varphi^2$ yields the Wess–Zumino model [45], which was the first example of nontrivial $\mathcal{N} = 1$ supersymmetric model and the only renormalizable model of scalar $\mathcal{N} = 1$ multiplet. Ogievetsky and Mezincescu argued that the representation of the action of the Wess–Zumino model in terms of Berezin integral is very useful and suggestive while developing the superfield perturbation theory for it. All quantum corrections have the form of the integral over the whole $\mathcal{N} = 1$ superspace, so the superpotential term (and hence the parameters g and m) is not renormalized. This was the first example of the nonrenormalization theorems, which nowadays are the powerful ingredients of the quantum superfield approach.

In 1975, Ogievetsky and Mezincescu wrote a comprehensive review on the basics of supersymmetry and superspace techniques [46]. Until now it is still one of the best introductory reviews in this area.

2.2. Superfields with Definite Superspins and Supercurrents. The dream of Ogievetsky was to generalize the spin principle formulated by him and Polubarinov to the superfield approach. Indeed, the notion of the Poincaré spin of fields naturally extends to the case of supersymmetry as the notion of superspin, the eigenvalue of one of the Casimir operators of the Poincaré supersymmetry algebra. The irreducible $\mathcal{N} = 1$ supermultiplets are characterized by definite superspins, and the latter can be well defined for the interacting superfields, like spin in the Poincaré invariant theories. The means to ensure the definite spin of superfields are either the appropriate irreducibility constraints (in the massive case) or the appropriate gauge invariance (in the massless case). The concept of conserved current also admits "supersymmetrization", and the appropriate supercurrents were already known for a number of simple models. The requirement of preserving definite superspins by interacting superfields was expected to fully determine the structure of the corresponding actions, as well as the gauge group intended to make harmless extra superspins carried by the given off-shell superfield. However, because of the existence of new differential operators in the superfield case, the covariant spinor derivatives (2.5), along with the standard x-derivative, yet defining the irreducibility conditions for the most cases of interest was a very difficult technical problem. In the pioneering paper [41], the decomposition into the superspin-irreducible parts was discussed only for a scalar $\mathcal{N} = 1$ superfield.

The general classification of $\mathcal{N} = 1$ superfields by superspin was given by Sokatchev in [47], where the corresponding irreducibility superfield constraints,

together with the relevant projection operators on definite superspins, were found. In fulfilling the program of generalizing the spin principle to supersymmetry, the formalism of the projection operators of [47] proved to be of key significance.

The main efforts of Ogievetsky and Sokatchev were soon concentrated on seeking a self-consistent theory of massless axial-vector superfield (carrying superspins 3/2 and 1/2). This superfield $H^n(x, \theta, \bar{\theta})$ was of special interest because its component field expansion involved the massless tensor field e_a^n and the spin-vector field ψ_α^n,

$$H^n = \theta \sigma^a \bar{\theta} e_a^n + (\bar{\theta})^2 \theta^\alpha \psi_\alpha^n + (\theta)^2 \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha} n} + \ldots$$

These fields could naturally be identified with those of graviton and gravitino of $\mathcal{N} = 1$ supergravity (SG) known to that time in the component form [48]. In [49], Ogievetsky and Sokatchev have put forward the hypothesis that the correct "minimal" $\mathcal{N} = 1$ superfield SG should be a theory of gauge axial-vector superfield $H^m(x, \theta, \bar{\theta})$ generated by the conserved supercurrent. The latter unifies into an irreducible $\mathcal{N} = 1$ supermultiplet the energy–momentum tensor and spin-vector current associated with the supertranslations [50] (see also [51, 52] and references therein). Ogievetsky and Sokatchev relied upon the clear analogy with the Einstein gravity which can be viewed as a theory of massless tensor field generated by the conserved energy–momentum tensor. As was already mentioned in Sec. 1, the whole Einstein action and its non-Abelian 4D diffeomorphism gauge symmetry can be uniquely restored step-by-step, starting with a free action of symmetric tensor field and requiring its source (constructed from this field and its derivatives, as well as from matter fields) to be conserved [12]. In [49], this Noether procedure was applied to the free action of $H^m(x, \theta, \bar{\theta})$. The first-order coupling of H^m to the conserved supercurrent of the matter chiral superfield was restored, and the superfield gauge symmetry generalizing bosonic diffeomorphism symmetry was identified at the linearized level. The geometric meaning of this supergauge symmetry and its full non-Abelian form were revealed by Ogievetsky and Sokatchev later, in the remarkable papers [53, 54].

2.3. Complex Superfield Geometry of $\mathcal{N} = 1$ Supergravity. After the discovery of the component $\mathcal{N} = 1$ supergravity in [48]*, it was an urgent problem to find its complete off-shell formulation, i.e., to extend the set of physical fields of graviton and gravitino to an off-shell multiplet by adding the appropriate auxiliary fields and/or to formulate $\mathcal{N} = 1$ supergravity in superspace, making all its symmetries manifest.

One of the approaches to the superspace formulations of $\mathcal{N} = 1$ supergravity was to start from the most general differential geometry in $\mathcal{N} = 1$ super-

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*A version of supergravity with the spontaneously broken supersymmetry (based on the Higgs effect for the Goldstone fields with the spin 1/2) was worked out in [55] (see also recent paper [56].

space. One defines supervielbeins, supercurvatures, and supertorsions, which are covariant under arbitrary $\mathcal{N} = 1$ superdiffeomorphisms, and then imposes the appropriate covariant constraints, so as to single out the minimal set of off-shell $\mathcal{N} = 1$ superfields carrying the irreducible field content of supergravity [57]. An alternative approach would consist of revealing the fundamental minimal gauge group of supergravity and defining the basic unconstrained prepotential, an analog of $\mathcal{N} = 1$ super Yang–Mills (SYM) prepotential [58]. This was just the strategy which Ogievetsky and Sokatchev follow in [53] to construct a beautiful geometric formulation of the conformal and "minimal" Einstein $\mathcal{N} = 1$ SG*.

It is based on a generalization of the notion of $\mathcal{N} = 1$ chirality to the curved case. The flat chiral $\mathcal{N} = 1$ superspace (x_L^m, θ_L^μ) possesses the complex dimension $(4|2)$ and contains the $\mathcal{N} = 1$ superspace $(x^m, \theta^\mu, \bar\theta^{\dot\mu})$ as a real $(4|4)$ dimensional hypersurface defined by the following embedding conditions:

$$\text{(a)}\ x_L^m + x_R^m = 2x^m, \quad \text{(b)}\ x_L^m - x_R^m = 2i\theta\sigma^m\bar\theta, \quad \theta_L^\mu = \theta^\mu, \quad \bar\theta_R^{\dot\mu} = \bar\theta^{\dot\mu}, \quad (2.9)$$

and $x_R^m = \overline{(x_L^m)}, \bar\theta_R^{\dot\mu} = \overline{(\theta_L^\mu)}$. The underlying gauge group of conformal $\mathcal{N} = 1$ SG proved to be the group of general diffeomorphisms of the chiral superspace:

$$\delta x_L^m = \lambda^m(x_L, \theta_L), \quad \delta\theta_L^\mu = \lambda^\mu(x_L, \theta_L), \quad (2.10)$$

where λ^m, λ^μ are arbitrary complex functions of their arguments. The fermionic part of the embedding conditions (2.9) does not change, while the bosonic part is generalized as

$$\text{(a)}\ x_L^m + x_R^m = 2x^m, \quad \text{(b)}\ x_L^m - x_R^m = 2iH^m(x, \theta, \bar\theta). \quad (2.11)$$

The basic gauge prepotential of conformal $\mathcal{N} = 1$ SG is just the axial-vector superfield $H^m(x, \theta, \bar\theta)$ in (2.11). It specifies the superembedding of real $\mathcal{N} = 1$ superspace as a hypersurface into the complex chiral $\mathcal{N} = 1$ superspace (x_L^m, θ_L^μ). Through relations (2.11), the transformations (2.10) generate field-dependent nonlinear transformations of the $\mathcal{N} = 1$ superspace coordinates $(x^m, \theta^\mu, \bar\theta^{\dot\mu})$ and of the superfield $H^m(x, \theta, \bar\theta)$. The irreducible field content of H^m is revealed in the WZ gauge which requires knowing only the linearized form of the transformations:

$$\delta_{\text{lin}} H^m = \frac{1}{2i} \left[\lambda^m(x + i\theta\sigma\bar\theta, \theta) - \bar\lambda^m(x - i\theta\sigma\bar\theta, \bar\theta) \right] -$$
$$- \lambda(x + i\theta\sigma\bar\theta, \theta)\sigma^m\bar\theta - \theta\sigma^m\bar\lambda(x - i\theta\sigma\bar\theta, \bar\theta). \quad (2.12)$$

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*A closely similar formulation was worked out in the parallel investigations by Siegel and Gates [59].

Here we assumed the presence of the "flat" part $\theta \sigma^m \bar{\theta}$ in $H^m = \theta \sigma^a \bar{\theta} (\delta_a^m + \kappa h_a^m) + \ldots$ The WZ gauge form of H^m is then as follows:

$$H_{\mathrm{WZ}}^m = \theta \sigma^a \bar{\theta} \, e_a^m + (\bar{\theta})^2 \theta^\mu \psi_\mu^m + (\theta)^2 \bar{\theta}_{\dot{\mu}} \bar{\psi}^{m\dot{\mu}} + (\theta)^2 (\bar{\theta})^2 A^m. \tag{2.13}$$

Here, one finds the inverse vierbein e_a^m presenting the conformal graviton (gauge-independent spin 2 off-shell), the gravitino ψ_μ^m (spin $(3/2)^2$), and the gauge field A^m (spin 1) of the local γ_5 R-symmetry. They constitute just $(8+8)$ off-shell degrees of freedom of the superspin 3/2 $\mathcal{N} = 1$ Weyl multiplet.

The Einstein $\mathcal{N} = 1$ SG can now be deduced in the two basically equivalent ways. The first one was used in the original paper [53], and it consists in restricting the group (2.10) by the constraint

$$\partial_m \lambda^m - \partial_\mu \lambda^\mu = 0, \tag{2.14}$$

which is the infinitesimal form of the requirement that the integration measure of chiral superspace (x_L, θ^μ) is invariant. One can show that, with this constraint, the WZ form of H^m collects two extra scalar auxiliary fields, while A^m ceases to be gauge and also becomes an auxiliary field. On top of this, there disappears one fermionic gauge invariance (corresponding to conformal supersymmetry) and, as a result, spin-vector field starts to comprise 12 independent components. So, one ends up with the $(12 + 12)$ off-shell multiplet of the so-called "minimal" Einstein SG [60].

Another, more suggestive way to come to the same off-shell content is to use the compensator techniques which can be traced back to the interpretation of Einstein gravity as conformal gravity with the compensating (Goldstone) scalar field [61]. Since the group (2.10) preserves the chiral superspace, in the local case one can still define a chiral superfield $\Phi(x_L, \theta)$ as an unconstrained function on this superspace and ascribe to it the following transformation law:

$$\delta \Phi = -\frac{1}{3} \left(\partial_m \lambda^m - \partial_\mu \lambda^\mu \right) \Phi, \tag{2.15}$$

where the specific choice $(-1/3)$ of the conformal weight of Φ is needed for constructing the invariant SG action. One can show that such a compensating chiral superfield together with the prepotential H^m yield, in the appropriate WZ gauges, just the required off-shell $(12 + 12)$ representation.

The basic advantage of the compensating method is the possibility to easily write the action of the minimal Einstein SG as an invariant action of the compensator Φ in the background of the Weyl multiplet carried by H^m:

$$S_{\mathrm{SG}} = -\frac{1}{\kappa^2} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, E \Phi(x_L, \theta) \, \bar{\Phi}(x_R, \bar{\theta}) +$$
$$+ \xi \left(\int d^4 x_L \, d^2\theta \, \Phi^3(x_L, \theta) + \mathrm{c.\,c.} \right). \tag{2.16}$$

Here E is a density constructed from H^m and its derivatives [54], such that its transformation cancels the total weight transformation of the integration measure $d^4x\, d^2\theta\, d^2\bar{\theta}$ and the product of chiral compensators. In components, the first term in (2.16) yields the minimal Einstein $\mathcal{N} = 1$ SG action without cosmological term, while the second term in (2.16) is the superfield form of the cosmological term $\sim \xi$.

Later on, many other off-shell component and superfield versions of $\mathcal{N} = 1$ SG were constructed. They mainly differ in the choice of the compensating supermultiplet. The uncertainty in choosing compensating superfields is related to the fact that the same on-shell scalar $\mathcal{N} = 1$ multiplet admits variant off-shell representations.

The Ogievetsky–Sokatchev formulation of $\mathcal{N} = 1$ SG was one of the main indications that the notion of chiral superfields and chiral superspace play the pivotal role in $\mathcal{N} = 1$ supersymmetry. Later it was found that the superfield constraints of $\mathcal{N} = 1$ SG have the nice geometric meaning: they guarantee the existence of chiral $\mathcal{N} = 1$ superfields in the curved case, once again pointing out the fundamental role of chirality in $\mathcal{N} = 1$ theories. The constraints defining the $\mathcal{N} = 1$ SYM theory can also be derived from requiring chiral representations to exist in the full interaction case. The parameters of the $\mathcal{N} = 1$ gauge group are chiral superfields, so this group manifestly preserves the chirality. The geometric meaning of $\mathcal{N} = 1$ SYM prepotential $V(x, \theta, \bar{\theta})$ was discovered in [26]. By analogy with $H^n(x, \theta, \bar{\theta})$, the superfield V specifies a real $(4|4)$ dimensional hypersurface, this time in the product of $\mathcal{N} = 1$ chiral superspace and the internal coset space G^c/G, where G^c is the complexification of the gauge group G. At last, chiral superfields provide the most general description of $\mathcal{N} = 1$ matter, since any variant off-shell representation of $\mathcal{N} = 1$ scalar multiplet is related to chiral multiplet via duality transformation.

In parallel with these investigations, in the second half of 1970s – the beginning of 1980s two other important themes related to supersymmetry were worked out in Sector 3, which exerted a sound influence on further developments in this area.

2.4. Relation between Linear and Nonlinear Realizations of Supersymmetry. One of the first known realizations of $\mathcal{N} = 1$ supersymmetry was its nonlinear (Volkov–Akulov) realization [6]

$$y^{m\,\prime} = y^m + i[\lambda(y)\sigma^m\bar{\epsilon} - \epsilon\sigma^m\bar{\lambda}(y)],$$
$$\lambda^{\alpha\,\prime}(y\,\prime) = \lambda^\alpha(y) + \epsilon^\alpha, \quad \bar{\lambda}^{\dot{\alpha}\,\prime}(y\,\prime) = \bar{\lambda}^{\dot{\alpha}}(y) + \bar{\epsilon}^{\dot{\alpha}}, \tag{2.17}$$

where the corresponding Minkowski space coordinate is denoted by y^m to distinguish it from x^m corresponding to the superspace realization

$$\theta^{\alpha\prime} = \theta^\alpha + \epsilon^\alpha, \quad \bar{\theta}^{\dot{\alpha}\prime} = \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}, \quad x^{m\prime} = x^m + i(\theta\sigma^m\bar{\epsilon} - \epsilon\sigma^m\bar{\theta}). \tag{2.18}$$

In (2.17), (2.18), ϵ^α and $\bar\epsilon^{\dot\alpha}$ are the mutually conjugated Grassmann transformation parameters associated with the $\mathcal{N}=1$ supertranslation generators Q_α and $\bar Q_{\dot\alpha}$.

The main difference between (2.17) and (2.18) is that (2.17) involves the Volkov–Akulov $\mathcal{N}=1$ Goldstone fermion (goldstino) $\lambda(y)$, the characteristic feature of which is the inhomogeneous transformation law under supertranslations corresponding to the spontaneously broken supersymmetry. It is a field given on Minkowski space, while θ^α in (2.18) is an independent Grassmann coordinate, and $\mathcal{N}=1$ superfields support a linear realization of supersymmetry. The invariant action of $\lambda, \bar\lambda$ is [6]

$$S_{(\lambda)} = \frac{1}{f^2} \int d^4y \, \det E_m^a, \quad E_m^a = \delta_m^a + i \left(\lambda \sigma^a \partial_m \bar\lambda - \partial_m \lambda \sigma^a \bar\lambda \right), \quad (2.19)$$

where f is a coupling constant ($[f] = -2$).

The natural question was as to what is the precise relation between the nonlinear and superfield (linear) realizations of the same $\mathcal{N}=1$ Poinacaré supersymmetry. The explicit answer was for the first time presented in [62–64]. There we showed that, given the Goldstone fermion $\lambda(y)$ with the transformation properties (2.17), the relation between two types of the supersymmetry realizations, (2.18) and (2.17), is accomplished through the following invertible change of the superspace coordinates:

$$x^m = y^m + i \left[\theta \sigma^m \bar\lambda(y) - \lambda(y) \sigma^m \bar\theta \right], \quad \theta^\alpha = \tilde\theta^\alpha + \lambda^\alpha(y), \quad \bar\theta^{\dot\alpha} = \tilde{\bar\theta}^{\dot\alpha} + \bar\lambda^{\dot\alpha}(y), \tag{2.20}$$

where

$$\tilde\theta^{\alpha\,\prime} = \tilde\theta^\alpha. \tag{2.21}$$

Then the transformations (2.17) imply for $(x^m, \theta^\alpha, \bar\theta^{\dot\alpha})$ just the transformations (2.18), and vice versa (2.18) imply (2.17). Using (2.20), any linearly transforming superfield can be put in the new "splitting" basis

$$\Phi(x, \theta, \bar\theta) = \tilde\Phi(y, \tilde\theta, \tilde{\bar\theta}). \tag{2.22}$$

Since the new spinor coordinate $\tilde\theta^\alpha$ is "inert" under $\mathcal{N}=1$ supersymmetry, Eq. (2.21), the components of $\tilde\Phi$ transform as "sigma-fields"*,

$$\delta^\star \phi(y) = -i[\lambda(y)\sigma^m \bar\epsilon - \epsilon\sigma^m \bar\lambda(y)]\partial_m \phi(y), \quad \text{etc.}, \tag{2.23}$$

independently of each other, that explains the adjective "splitting" for the basis $(y^m, \tilde\theta^\alpha, \tilde{\bar\theta}^{\dot\alpha})$.

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*Below, $\delta^\star$ stands for the "active" variation, as distinct from other group variations in this section which are "passive".

As demonstrated in [64], irrespective of the precise mechanism of generating goldstino in a theory with the linear realization of spontaneously broken $\mathcal{N} = 1$ supersymmetry, the corresponding superfield action can be rewritten in the splitting basis (after performing integration over the inert Grassmann variables) as

$$S_{\text{lin}} \sim \int d^4 y \det E_m^a \left[1 + \mathcal{L}(\sigma, \nabla_a \sigma, \ldots) \right]. \qquad (2.24)$$

Here $\mathcal{L}$ is a function of the "sigma" fields and their covariant derivatives $\nabla_a = E_a^m \partial_m$ only, while $\lambda^\alpha(y)$ is related to the goldstino of the linear realization through a field redefinition. Thus, the Goldstone fermion is always described by the universal action (2.19), independently of details of the given dynamical theory with the spontaneous breaking of $\mathcal{N} = 1$ supersymmetry, in the spirit of the general theory of nonlinear realizations.

The transformations (2.20) and (2.22) can be easily generalized to chiral superfields and to higher $\mathcal{N}$. It proved very useful for exhibiting the low-energy structure of theories with spontaneously broken supersymmetry [65], as well as in some other problems (see, e.g., [66] and references therein). It was generalized to the case of local $\mathcal{N} = 1$ supersymmetry in [67, 68]. At present, in connection with some cosmological problems, great attention is paid to models in which $\mathcal{N} = 1$ supergravity interacts with the matter superfields constructed solely from the Goldstone fermions [69]*. The approach based on (2.20) and (2.22) (and their generalizations to local supersymmetry) is very appropriate for constructing such multiplets. Indeed, as follows from the transformation law (2.21), the quantities $\tilde{\theta}^\alpha(x, \theta, \bar{\theta})$, $\bar{\tilde{\theta}}^{\dot{\alpha}}(x, \theta, \bar{\theta})$ are $\mathcal{N} = 1$ superfields properly constrained because their dependence on the superspace coordinates basically appear through the dependence on y^m. Using the definitions in (2.20), it is easy to deduce the corresponding superspace constraints [63]:

$$D_\beta \tilde{\theta}^\alpha = \delta_\beta^\alpha + i(\sigma^m)_{\beta\dot{\beta}} \bar{\tilde{\theta}}^{\dot{\beta}} \partial_m \tilde{\theta}^\alpha, \quad \bar{D}_{\dot{\beta}} \tilde{\theta}^\alpha = -i(\sigma^m)_{\gamma\dot{\beta}} \tilde{\theta}^\gamma \partial_m \tilde{\theta}^\alpha, \qquad (2.25)$$

where $D_\alpha, \bar{D}_{\dot{\alpha}}$ are defined in (2.5), and the analogous ones for $\bar{\tilde{\theta}}^{\dot{\beta}}$. Thus, $\tilde{\theta}^\alpha$ and $\bar{\tilde{\theta}}^{\dot{\beta}}$ can be considered as bricks from which more complicated $\mathcal{N} = 1$ superfields as functions of the goldstino field (and its x-derivative) can be assembled.

The constraints (2.25) look similar to those derived by Samuel and Wess in [67]. The latter are in fact equivalent to (2.25) and can be readily derived using a modification of the variable change (2.20). They follow by starting from the realization of $\mathcal{N} = 1$ supersymmetry in the right-handed chiral superspace

$$\delta x_R^m = -2i\epsilon\sigma^m\bar{\theta}, \quad \delta\theta^\alpha = \epsilon^\alpha, \quad \delta\bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}}, \qquad (2.26)$$

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*See also recent [56].

and defining the new complex coordinate

$$z_+^m = x_R^m + 2i\chi(z_+)\sigma^m\bar\theta, \quad \chi^\alpha(z_+) = \lambda^\alpha(\tilde z_+),$$
$$z_+^m = \tilde z_+^m + i\lambda(\tilde z_+)\sigma^m\bar\lambda(\tilde z_+), \tag{2.27}$$

$$\delta z_+^m = 2i\chi(z_+)\sigma^m\bar\epsilon,$$
$$\delta\tilde z_+^m = i[\lambda(\tilde z_+)\sigma^m\bar\epsilon - \epsilon\sigma^m\bar\lambda(\tilde z_+)], \tag{2.28}$$
$$\delta\chi^\alpha(z_+) = \epsilon^\alpha, \quad \delta\bar\chi^{\dot\alpha}(z_+) = \bar\epsilon^{\dot\alpha}.$$

Next, we define

$$\tilde\theta_+^\alpha(x,\theta,\bar\theta) = \theta^\alpha - \chi^\alpha(z_+) \tag{2.29}$$

and find the following constraints for this complex spinor $\mathcal{N}=1$ superfield:

$$D_\beta\tilde\theta_+^\alpha = \delta_\beta^\alpha, \quad \bar D_{\dot\beta}\tilde\theta_+^\alpha = -2i(\sigma^m)_{\gamma\dot\beta}\tilde\theta_+^\gamma\partial_m\tilde\theta_+^\alpha. \tag{2.30}$$

These constraints are just those given in [67] (with $\tilde\theta_+^\alpha$ denoted there as Λ^α). One can establish the explicit equivalency relation between $\tilde\theta^\alpha$ and $\tilde\theta_+^\alpha$.

Another possibility, which is also related to the original transformations through an equivalency change of the goldstino field, is to start from the left-chiral realization

$$\delta x_L^m = 2i\theta\sigma^m\bar\epsilon, \quad \delta\theta^\alpha = \epsilon^\alpha, \quad \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha} \tag{2.31}$$

and define

$$z_- = x_L^m - 2i\theta\sigma^m\bar\omega(z_-), \quad \omega^\alpha(z_-) = \lambda^\alpha(\tilde z_-),$$
$$z_-^m = \tilde z_-^m - i\lambda(\tilde z_-)\sigma^m\bar\lambda(\tilde z_-), \tag{2.32}$$

$$\delta z_-^m = -2i\epsilon\sigma^m\bar\omega(z_-),$$
$$\delta\tilde z_-^m = i[\lambda(\tilde z_-)\sigma^m\bar\epsilon - \epsilon\sigma^m\bar\lambda(\tilde z_-)], \tag{2.33}$$
$$\delta\omega^\alpha(z_-) = \epsilon^\alpha, \quad \delta\bar\omega^{\dot\alpha}(z_-) = \bar\epsilon^{\dot\alpha}.$$

The corresponding composite $\mathcal{N}=1$ superfield,

$$\tilde\theta_-^\alpha(x,\theta,\bar\theta) = \theta^\alpha - \omega^\alpha(z_-), \tag{2.34}$$

satisfies the constraints

$$D_\beta\tilde\theta_-^\alpha = \delta_\beta^\alpha + 2i(\sigma^m)_{\beta\dot\beta}\bar{\tilde\theta}_-^{\dot\beta}\partial_m\tilde\theta_-^\alpha, \quad \bar D_{\dot\beta}\tilde\theta_-^\alpha = 0. \tag{2.35}$$

So the superfield $\tilde{\theta}_-^\alpha(x, \theta, \bar{\theta})$ is chiral, and one can construct the nilpotent chiral scalar superfield as the bilinear of these "bricks"

$$\varphi = \tilde{\theta}_-^\alpha \tilde{\theta}_{-\alpha}, \quad \bar{D}_{\dot{\beta}}\varphi = 0, \quad \varphi^2 = 0. \tag{2.36}$$

It is worth pointing out that all components of such a nilpotent superfield are model-independent functions of the goldstino field $\lambda^\alpha(x), \bar{\lambda}^{\dot{\alpha}}(x)$ and its x-derivatives only. In the standard description of $\mathcal{N} = 1$ goldstino through the nilpotent scalar chiral superfield [66, 70], the latter still includes a scalar auxiliary field as the independent one. It is eliminated either through its equations of motion or by imposing additional differential constraints.

2.5. Anti-de Sitter Supersymmetry. Soon after the discovery of the $\mathcal{N} = 1$ Poincaré supersymmetry as a symmetry of theories in the flat Minkowski space treated as a coset of the Poincaré group $\mathcal{P}^4$ over its Lorentz subgroup, i.e., $\mathcal{P}^4/SO(1,3)$, there arose an interest in analogous supersymmetries preserving nonflat background solutions of the Einstein equations. The renowned manifolds of this kind are de Sitter and anti-de Sitter spaces $dS_4 \sim SO(1,4)/SO(1,3)$ and $AdS_4 \sim SO(2,3)/SO(1,3)$. These are solutions of the Einstein equations with a nonzero cosmological constant, positive and negative, respectively, so the study of the relevant supersymmetries was expected to give some hints why this constant is so small (if nonzero). One more source of interest in these "curved" supersymmetries was related to the important role of the superconformal group $SU(2,2|4)$ involving such supersymmetries as subgroups, along with the flat $\mathcal{N} = 1$ Poincaré supersymmetry. As was already mentioned, various 4D supergravities follow from the conformal supergravity through the compensator mechanism.

The anti-de Sitter supersymmetry is the easiest one to analyze because it is very similar to $\mathcal{N} = 1$ Poincaré supersymmetry and goes over to it in the limit of infinite anti-de Sitter radius. While the dS_4 spinor comprises eight independent components, no such doubling as compared to the Minkowski space occurs for AdS_4: the AdS_4 spinor is the Weyl one with two complex components, i.e., the number of supercharges in the AdS supersymmetry is the same as in the $\mathcal{N} = 1$ Poincaré one. A self-consistent superfield formalism for AdS_4 supersymmetry was constructed in [71, 72].

$\mathcal{N} = 1$ AdS_4 superalgebra is $osp(1|4) \subset su(2,2|1)$, and it is defined by the following (anti)commutation relations:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^m)_{\alpha\dot{\alpha}} P_m, \quad \{Q_\alpha, Q_\beta\} = \mu(\sigma^{mn})_{\alpha\beta} L_{mn},$$

$$[Q_\alpha, P_m] = \frac{\mu}{2}(\sigma_m)_{\alpha\dot{\alpha}} \bar{Q}^{\dot{\alpha}}, \quad [P_m, P_n] = -i\mu^2 L_{mn}. \tag{2.37}$$

Here $\mu \sim r^{-1}$ is the inverse radius of AdS_4 and L_{mn} are generators of the Lorentz $SO(1,3)$ subgroup of $SO(2,3) \propto (P_m, L_{mn})$. To Eqs. (2.37) one should

add complex-conjugate relations and evident commutators with L_{mn}. In the limit $\mu \rightarrow 0$ ($r \rightarrow \infty$), Eqs. (2.37) go over into the algebra of $\mathcal{N} = 1$ Poincaré supersymmetry.

In [71, 72], we defined the true AdS_4 analogs of the general and chiral $\mathcal{N} = 1$ superfields, as well as the vector and spinor covariant derivatives, invariant superspace integration measures, etc. Having developed the AdS_4 superfield techniques, we constructed the $OSp(1|4)$ invariant actions generalizing the actions of the Wess–Zumino model and $\mathcal{N} = 1$ super Yang–Mills theory. For instance, an analog of the free massless action (2.7) of $\mathcal{N} = 1$ scalar multiplet, with the auxiliary fields eliminated by their equations of motion, reads

$$S \sim \int d^4x \, a^4(x) \left(\nabla^m \bar{\varphi} \nabla_m \varphi - \frac{i}{4} \psi \sigma^m \nabla_m \bar{\psi} + \frac{i}{4} \nabla_m \psi \sigma^m \bar{\psi} + 2\mu^2 \, \varphi \bar{\varphi} \right). \tag{2.38}$$

Here, $a(x) = \dfrac{2}{1 + \mu^2 x^2}$ is a scalar factor specifying the AdS_4 metric in a conformally flat parameterization, $ds^2 = a^2(x)\eta_{mn} \, dx^m \, dx^n$, and $\nabla_m = a^{-1}\partial_m$ *. Taking into account that $\mu^2 = (-1/12)R$, where R is the scalar curvature of AdS_4, this action matches the standard form of the massless scalar field action in a curved background.

In [72], the vacuum structure of the general massive AdS_4 Wess–Zumino model was studied. This structure proved to be much richer as compared to the standard "flat" Wess–Zumino model due to the presence of the intrinsic mass parameter μ. It was also shown that both the AdS_4 massless Wess–Zumino model and super Yang–Mills theory can be reduced to their flat $\mathcal{N} = 1$ super Minkowski analogs via some superfield transformation generalizing the Weyl transformation

$$\varphi(x) = a^{-1}(x) \, \tilde{\varphi}(x), \quad \psi^\alpha(x) = a^{-3/2}(x) \, \tilde{\psi}^\alpha(x), \tag{2.39}$$

which reduces (2.38) to (2.7). The existence of the superfield Weyl transformation was an indication of the superconformal flatness of the AdS_4 superspace (although this property has been proven much later, in [73]).

The simplest supermultiplets of $OSp(1|4)$ derived for the first time in [71] in the superfield approach and the corresponding projection operators were used in [74] to give a nice algebraic interpretation of the superfield constraints of $\mathcal{N} = 1$ supergravity. The interest in $OSp(1|4)$ supersymmetry has especially grown up in recent years in connection with the famous AdS/CFT correspondence. For instance, the theories invariant under rigid supersymmetries in various curved manifolds are now under intensive study (see, e.g., [75, 76]), and they are just generalizations of the AdS supersymmetric models the analysis of which was initiated in [71, 72].

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*In general, ∇_m contains a spin connection, but it drops out from (2.38).

3. HARMONIC SUPERSPACE AND ALL THAT

After creating the minimal geometric formulation of $\mathcal{N} = 1$ SG described in Sec. 2, there was posed a natural question as to how it can be generalized to the most interesting case of extended supergravities and, as a first step, to $\mathcal{N} = 2$ supergravity. To answer this question, it proved necessary to realize what the correct generalization of $\mathcal{N} = 1$ chirality to $\mathcal{N} \geqslant 2$ supersymmetry is and to invent a new type of superspaces, the harmonic ones.

It was even unclear how to define, in the suggestive geometric way, the appropriate $\mathcal{N} = 2$ analog of the $\mathcal{N} = 1$ SYM prepotential $V(x, \theta, \bar{\theta})$, $\delta V = (i/2)(\Lambda(x_L, \theta) - \bar{\Lambda}(x_R, \bar{\theta})) + \mathcal{O}(V)$. While the $\mathcal{N} = 1$ SYM constraints are just the integrability conditions for preserving covariant chirality,

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0, \quad \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 0, \tag{3.1}$$

their $\mathcal{N} = 2$ counterparts read [79]

$$\{\mathcal{D}_\alpha^{(i}, \mathcal{D}_\beta^{k)}\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}^{(k}, \bar{\mathcal{D}}_{\dot{\beta}}^{i)}\} = \{\mathcal{D}_\alpha^{(i}, \bar{\mathcal{D}}_{\dot{\beta}}^{k)}\} = 0. \tag{3.2}$$

Here, $\mathcal{D}_\alpha^i = D_\alpha^i + i\mathcal{A}_\alpha^i(x, \theta^i, \bar{\theta}^k)$ and $i, k = 1, 2$ are the doublet indices of the automorphism group $SU(2)_A$ of $\mathcal{N} = 2$ Poincaré superalgebra. Obviously, these constraints cannot be interpreted as the conditions for preserving $\mathcal{N} = 2$ chirality. Luca Mezincescu solved these constraints in the Abelian case through an unconstrained prepotential [77]. However, the latter has a nonstandard dimension –2, and the corresponding gauge freedom does not admit a geometric interpretation (equally as a reasonable generalization to the non-Abelian case).

There also existed difficulties with an off-shell description of $\mathcal{N} = 2$ hypermultiplet, the direct analog of $\mathcal{N} = 1$ chiral multiplet. The natural irreducibility constraints on the relevant superfield $q^i(x, \theta^k, \bar{\theta}^k)$,

$$D_\alpha^{(i} q^{k)} = \bar{D}_{\dot{\alpha}}^{(i} q^{k)} = 0, \tag{3.3}$$

are solved by $q^i = f^i + \theta^{i\alpha}\psi_\alpha + \bar{\theta}_{\dot{\alpha}}^i \bar{\chi}^{\dot{\alpha}} + \ldots$, but simultaneously put the involved fields on their free mass shell. This is a reflection of the "no-go" theorem [78] which states that no off-shell representation for hypermultiplet in its "complex form" (i.e., with bosonic fields arranged into $SU(2)$ doublet) can be achieved with any *finite* number of auxiliary fields. No reasonable way to relax (3.3) was known.

3.1. Way Out: Grassmann Harmonic Analyticity. In [80], it was observed that extended supersymmetries, besides the standard chiral superspaces generalizing the $\mathcal{N} = 1$ one, also admit some other types of the invariant subspaces which were called "Grassmann-analytic". Like chiral superspaces, these analytic subspaces are revealed by passing to some new basis in the original general

superspace, such that spinor covariant derivatives with respect to some subset of Grassmann variables become "short" in this basis. Then one can impose *Grassmann* Cauchy–Riemann conditions with respect to these variables. They preserve the full original supersymmetry, but force the relevant *analytic* super-fields to depend on a smaller number of Grassmann coordinates (in a deep analogy with the chirality conditions (2.4)). As a nontrivial example of such Grassmann analyticity in extended supersymmetries, in [80] the existence of a complex "$O(2)$ analytic subspace" in $\mathcal{N} = 2$, 4D superspace was found. Unfortunately, it can be defined only provided that the full automorphism $SU(2)$ symmetry is broken down to $O(2)$. Despite this, it was natural to assume that the Grassmann analyticity of the similar type could play the fundamental role in extended supersymmetry and provide the correct generalization of $\mathcal{N} = 1$ chirality. In [81], the hyper-multiplet constraints (3.3) were shown to imply that different components of the $\mathcal{N} = 2$ superfield q^i "live" on different $O(2)$-analytic subspaces. Since (3.3) is $SU(2)$ covariant, it remained to "$SU(2)$- covariantize" the $O(2)$ analyticity.

All these problems were solved in the framework of the harmonic super-space [13, 82, 83].

$\mathcal{N} = 2$ harmonic superspace (HSS) is defined as the product

$$(x^m, \theta_{\alpha\,i}, \bar{\theta}^k_{\dot{\beta}}) \otimes S^2. \tag{3.4}$$

Here, the internal two-sphere $S^2 \sim SU(2)_A/U(1)$ is represented, in a parameteri-zation-independent way, by the lowest (isospinor) $SU(2)_A$ harmonics

$$S^2 \in (u^+_i, u^-_k), \quad u^{+i} u^-_i = 1, \quad u^\pm_i \to \mathrm{e}^{\pm i\lambda}\, u^\pm_i. \tag{3.5}$$

It is required that nothing depends on the $U(1)$ phase $\mathrm{e}^{i\lambda}$, so one effectively deals with the 2-sphere $S^2 \sim SU(2)_A/U(1)$. The superfields given on (3.4) (harmonic $\mathcal{N} = 2$ superfields) are assumed to admit the harmonic expansions on S^2, with the set of all symmetrized products of u^+_i, u^-_i as the basis. Such an expansion is fully specified by the harmonic $U(1)$ charge of the given superfield[*].

The main advantage of HSS is that it contains an invariant subspace, the $\mathcal{N} = 2$ *analytic* HSS, involving only half of the original Grassmann coordinates

$$\left(x^m_A, \theta^+_\alpha, \bar{\theta}^+_{\dot{\alpha}}, u^\pm_i\right) \equiv \left(\zeta^M, u^\pm_i\right),$$
$$x^m_A = x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{k)}u^+_i u^-_k, \quad \theta^+_\alpha = \theta^i_\alpha u^+_i, \quad \bar{\theta}^+_{\dot{\alpha}} = \bar{\theta}^i_{\dot{\alpha}} u^+_i. \tag{3.6}$$

It is just $SU(2)$ covariantization of the $O(2)$ analytic superspace of [80]. It is closed under $\mathcal{N} = 2$ supersymmetry transformations and is real with respect to

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[*] Another off-shell approach to $\mathcal{N} = 2$ supersymmetric theories is based on the concept of projective superspace [84], an extension of the ordinary $\mathcal{N} = 2$ superspace by a complex $\mathbb{CP}^1$ coordinate.

the special involution defined as the product of the ordinary complex conjugation and the antipodal map (Weyl reflection) of S^2.

All $\mathcal{N} = 2$ supersymmetric theories have off-shell formulations in terms of unconstrained superfields defined on (3.6), the *Grassmann analytic* $\mathcal{N} = 2$ superfields. An analytic superfield $\varphi_{\mathrm{an}}^{+n}$ with the harmonic $U(1)$ charge $+n$ satisfies the Grassmann harmonic analyticity constraints

$$D_\alpha^+ \varphi_{\mathrm{an}}^{+n} = \bar{D}_{\dot\alpha}^+ \varphi_{\mathrm{an}}^{+n} = 0 \Rightarrow \varphi_{\mathrm{an}}^{+n} = \varphi_{\mathrm{an}}^{+n}(\zeta, u), \qquad (3.7)$$

$$D_\alpha^\pm = D_\alpha^i u_i^\pm, \quad \bar{D}_{\dot\alpha}^\pm = \bar{D}_{\dot\alpha}^i u_i^\pm. \qquad (3.8)$$

These constraints are self-consistent just due to the conditions

$$\{D_\alpha^+, D_\beta^+\} = \{\bar{D}_{\dot\alpha}^+, D_{\dot\beta}^+\} = \{D_\alpha^+, \bar{D}_{\dot\beta}^+\} = 0, \qquad (3.9)$$

which are equivalent to the "flat" version of (3.2) (these are their projections on u_i^+). The solution (3.7) is obtained in the analytic basis, where D_α^+ and $\bar{D}_{\dot\alpha}^+$ are reduced to the partial derivatives with respect to $\theta^{-\alpha}$ and $\bar\theta^{-\dot\alpha}$. The opportunity to choose such a basis is just ensured by the integrability conditions (3.9).

3.2. $\mathcal{N} = 2$ Matter. In general case, the $\mathcal{N} = 2$ matter is described by $2n$ hypermultiplet analytic superfields $q_a^+(\zeta, u)$ ($\overline{(q_a^+)} = \Omega^{ab} q_b^+$, $\Omega^{ab} = -\Omega^{ba}$; $a, b = 1, \ldots, 2n$) with the following off-shell action [85]:

$$S_q = \int du\, d\zeta^{(-4)} \left\{ q_a^+ D^{++} q^{+a} + L^{+4}(q^+, u^+, u^-) \right\}. \qquad (3.10)$$

Here, $du\, d\zeta^{(-4)}$ is the charged measure of integration over the analytic superspace (3.6), $D^{++} = u^{+\,i} \dfrac{\partial}{\partial u^{-i}} - 2i\theta^+ \sigma^m \bar\theta^+ \dfrac{\partial}{\partial x^m}$ is the analytic basis form of one of three harmonic derivatives one can define on S^2 (it preserves the harmonic Grassmann analyticity), and the indices are raised and lowered by the $Sp(n)$ totally skew-symmetric tensors $\Omega^{ab}, \Omega_{ab}, \Omega^{ab}\Omega_{bc} = \delta_c^a$. The crucial feature of the general q^+ action (3.10) is an infinite number of auxiliary fields coming from the harmonic expansion on S^2. Just this fundamental property made it possible to evade the no-go theorem about the nonexistence of off-shell formulations of the $\mathcal{N} = 2$ hypermultiplet in the complex form. The on-shell constraints (3.3) (and their nonlinear generalizations) amount to *both* the harmonic analyticity of q^{+a} (which is a kinematic property like $\mathcal{N} = 1$ chirality) and the dynamical equations of motion following from the action (3.10). After eliminating infinite sets of auxiliary fields by their algebraic equations, one ends up with the most general self-interaction of n hypermultiplets. In the bosonic sector it yields the generic sigma model with $4n$-dimensional hyper-Kähler (HK) target manifold, in accord with the theorem of Alvarez–Gaumé and Freedman about the one-to-one

correspondence between $\mathcal{N} = 2$ supersymmetric sigma models and HK manifolds [86]. In general, the action (3.10) and the corresponding HK sigma model possess no isometries. The object L^{+4} is the HK potential [87], an analog of the Kähler potential of $\mathcal{N} = 1$ supersymmetric sigma models [88]. Choosing one or another specific L^{+4}, one gets the explicit form of the relevant HK metric by eliminating the auxiliary fields from (3.10). So the general hypermultiplet action (3.10) provides an efficient universal tool of the *explicit* construction of the HK metrics [85, 89].

The appearance of the HK geometry prepotential as the most general hypermultiplet interaction superfield Lagrangian is quite similar to the way how the Kähler geometry potential appears as the most general sigma-model super Lagrangian for $\mathcal{N} = 1$ chiral superfields [88]. In many other cases, the superfield Lagrangians describing the sigma-model type interactions of the matter multiplets of diverse supersymmetries prove also to coincide with the fundamental objects (prepotentials) of the relevant target complex geometries (see, e.g., [90] and references therein).

3.3. $\mathcal{N} = 2$ Super Yang–Mills Theory. The HSS approach makes manifest that the $\mathcal{N} = 2$ SYM constraints (3.2) are the integrability conditions for the existence of the harmonic analytic superfields in such an interacting theory, like in the flat case*. They are solved in terms of the fundamental geometric object of $\mathcal{N} = 2$ SYM theory, the analytic harmonic connection $V^{++}(\zeta, u)$, which covariantizes the analyticity-preserving harmonic derivative

$$D^{++} \to \mathcal{D}^{++} = D^{++} + iV^{++}, \quad (V^{++})' = \frac{1}{i}\, \mathrm{e}^{i\omega}\left(D^{++} + iV^{++}\right)\mathrm{e}^{-i\omega}, \quad (3.11)$$

where $\omega(\zeta, u)$ is an arbitrary analytic gauge parameter containing infinitely many component gauge parameters in its combined θ, u-expansion. The harmonic connection V^{++} contains infinitely many component fields, however almost all of them can be gauged away by $\omega(\zeta, u)$. The rest of the $(8 + 8)$ components is just the off-shell $\mathcal{N} = 2$ vector multiplet. More precisely, in the WZ gauge V^{++} has the following form:

$$\begin{aligned}
V^{++}_{\mathrm{WZ}} = {} & (\theta^+)^2 w(x_A) + (\bar{\theta}^+)^2 \bar{w}(x_A) + i\theta^+ \sigma^m \bar{\theta}^+ V_m(x_A) + \\
& + (\bar{\theta}^+)^2 \theta^{+\alpha} \psi^i_\alpha(x_A) u^-_i + (\theta^+)^2 \bar{\theta}^+_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}i}(x_A) u^-_i + \\
& + (\theta^+)^2 (\bar{\theta}^+)^2 D^{(ij)}(x_A) u^-_i u^-_j. \quad (3.12)
\end{aligned}$$

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*An interpretation of the constraints of $\mathcal{N} = 2, 3, 4$ SYM theories as the integrability conditions along some directions in the (complexified) automorphism group manifolds was given by A. Rosly [91].

Here, V_m, w, $\bar{w}$, ψ_i^α, $\bar{\psi}^{\dot\alpha i}$, $D^{(ij)}$ are the vector gauge field, complex physical scalar field, doublet of gaugini, and the triplet of auxiliary fields, respectively. All the geometric quantities of $\mathcal{N} = 2$ SYM theory (spinor and vector connections, covariant superfield strengths, etc.), as well as the invariant action, can be expressed in terms of $V^{++}(\zeta, u)$. The closed V^{++} form of the $\mathcal{N} = 2$ SYM action was found by Boris Zupnik [92]:

$$S_{\text{SYM}}^{(\mathcal{N}=2)} = \frac{1}{2g^2} \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \operatorname{Tr} \int d^4x \, d^8\theta \, du_1 \cdots du_n \times$$
$$\times \frac{V^{++}(x, \theta, u_1) \cdots V^{++}(x, \theta, u_n)}{(u_1^+ u_2^+) \cdots (u_n^+ u_1^+)}, \quad (3.13)$$

where $(u_1^+ u_2^+), \ldots, (u_n^+ u_1^+)$ are the harmonic distributions defined in [83]. An important role is played by the second, nonanalytic gauge connection V^{--}, which covariantizes the second harmonic derivative D^{--} on the harmonic sphere S^2 and is related to V^{++} by the harmonic flatness condition

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0. \quad (3.14)$$

Most of the objects of the $\mathcal{N} = 2$ SYM differential geometry have a concise representation just in terms of V^{--}.

3.4. $\mathcal{N} = 2$ Conformal Supergravity. The $\mathcal{N} = 2$ Weyl multiplet is represented in HSS by the analytic vielbeins covariantizing D^{++} with respect to the analyticity-preserving diffeomorphisms of the superspace $(\zeta^M, u^{\pm i})$ [93, 94]:

$$D^{++} \to \mathcal{D}^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} + H^{++M}(\zeta, u) \frac{\partial}{\partial \zeta^M} + H^{++++}(\zeta, u) u^{-i} \frac{\partial}{\partial u^{+i}},$$
$$\delta \zeta^M = \lambda^M(\zeta, u), \quad \delta u_i^+ = \lambda^{++}(\zeta, u) u_i^-,$$
$$\delta H^{++M} = \mathcal{D}^{++} \lambda^M - \delta_{\mu+}^M \theta^{\mu+} \lambda^{++}, \quad \delta H^{++++} = \mathcal{D}^{++} \lambda^{++}, \quad \mu \equiv (\alpha, \dot\alpha), \quad (3.15)$$
$$\delta \mathcal{D}^{++} = -\lambda^{++} D^0, \quad D^0 \equiv u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} + \theta^{\mu+} \frac{\partial}{\partial \theta^{\mu+}}.$$

The vielbein coefficients H^{++M}, H^{++++} are unconstrained analytic superfields involving an infinite number of the component fields which come from the harmonic expansions. Most of these fields, like in V^{++}, can be gauged away by the analytic parameters λ^M, λ^{++}, leaving in the WZ gauge just the $(24 + 24)$ component fields of $\mathcal{N} = 2$ Weyl multiplet. The invariant actions of various versions of $\mathcal{N} = 2$ Einstein SG are given by a sum of the action of $\mathcal{N} = 2$ vector compensating superfield $H^{++5}(\zeta, u), \delta H^{++5} = \mathcal{D}^{++} \lambda^5(\zeta, u)$, and that of matter compensator superfields, both in the background of $\mathcal{N} = 2$ conformal SG. The superfield $H^{++5}(\zeta, u)$ and extra gauge parameter $\lambda^5(\zeta, u)$ have, respectively, the

geometric meaning of the vielbein coefficient associated with an extra coordinate x^5 (central charge coordinate) and the shift along this coordinate [95]. It is assumed that nothing depends on x^5. The most general off-shell version of $\mathcal{N} = 2$ Einstein SG is obtained by choosing the superfield $q^{+a}(\zeta, u)$ as the conformal compensator. It involves an infinite number of auxiliary fields and yields all the previously known off-shell versions with finite sets of auxiliary fields via the appropriate superfield duality transformations. Only this version allows for the most general SG-matter coupling. The latter gives rise to a generic quaternion-Kähler (QK) sigma model in the bosonic sector, in accordance with the theorem of Bagger and Witten [96]. The general superfield Lagrangian of hypermultiplets in the background of $\mathcal{N} = 2$ Weyl multiplet is a generalization of (3.10) to the SG case [94], and it is the fundamental prepotential of the quaternion-Kähler target geometry [97]. It can be used for the explicit computation of the QK metrics, e.g., through the appropriate quotient construction in HSS [98, 99].

More references related to the basics of HSS can be found in the monograph [13].

3.5. $\mathcal{N} = 3$ Harmonic Superspace. The HSS method can be generalized to $\mathcal{N} > 2$. It was used to construct, for the first time, an unconstrained off-shell formulation of $\mathcal{N} = 3$ SYM theory (that is equivalent to $\mathcal{N} = 4$ SYM on shell) in the harmonic $\mathcal{N} = 3$ superspace with the purely harmonic part $SU(3)/[U(1) \times U(1)]$, $SU(3)$ being the automorphism group of $\mathcal{N} = 3$, 4D supersymmetry [100]. The corresponding action is written in the analytic $\mathcal{N} = 3$ superspace and has a nice form of the superfield Chern–Simons term. This peculiarity supports the general statement that the structure and geometry of one or another gauge theory in superspace are radically different from those in the ordinary space-time.

Let us dwell on this formulation in some details. The $\mathcal{N} = 3$ SYM constraints in the standard $\mathcal{N} = 3$, 4D superspace read

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} = \varepsilon_{\alpha\beta} \bar{W}^{ij}, \quad \{\bar{\mathcal{D}}_{\dot\alpha i}, \bar{\mathcal{D}}_{\dot\beta j}\} = \varepsilon_{\dot\alpha\dot\beta} W_{ij},$$
$$\{\mathcal{D}_{\dot\alpha}^i, \bar{\mathcal{D}}_{\dot\beta j}\} = -2i\delta_j^i \mathcal{D}_{\alpha\dot\beta}, \tag{3.16}$$

where $i, j = 1, 2, 3$ are indices of the fundamental representations of $SU(3)$ and $\bar{W}^{ij} = -\bar{W}^{ji}$ (together with its conjugate) is the only independent covariant superfield strength of the theory. Unlike the $\mathcal{N} = 2$ SYM constraints, Eqs. (3.16) put the theory on shell.

The basic steps in [100] were the definition of the $\mathcal{N} = 3$ harmonic superspace with the harmonic part $SU(3)/[U(1) \times U(1)]$ parameterized by the mutually conjugated sets of harmonic variables possessing two independent harmonic $U(1)$ charges,

$$\left(u_i^{(1,0)}, u_i^{(0,-1)}, u_i^{(-1,1)}\right), \quad \left(u^{i(-1,0)}, u^{i(0,1)}, u^{i(1,-1)}\right), \quad u^{i(a,b)} u_i^{(c,d)} = \delta^{ac} \delta^{bd}, \tag{3.17}$$

and then the interpretation of the constraints (3.16) as the integrability conditions for the existence of an analytic subspace in such HSS:

$$\{\mathcal{D}_\alpha^{(1,0)}, \mathcal{D}_\beta^{(1,0)}\} = \{\mathcal{D}_\alpha^{(1,0)}, \bar{\mathcal{D}}_{\dot\beta}^{(0,1)}\} = \{\bar{\mathcal{D}}_{\dot\alpha}^{(0,1)}, \bar{\mathcal{D}}_{\dot\beta}^{(0,1)}\} = 0, \qquad (3.18)$$

where $\mathcal{D}_\alpha^{(1,0)} = u_i^{(1,0)} \mathcal{D}_\alpha^i$, $\bar{\mathcal{D}}_{\dot\beta}^{(0,1)} = u^{i(0,1)} \bar{\mathcal{D}}_{\dot\alpha i}$. The conditions (3.18) amount to the existence of a subclass of general $\mathcal{N} = 3$ harmonic superfields, the analytic superfields $\Phi^{(q_1,q_2)}(\zeta, u)$ living on the invariant analytic subspace with eight independent Grassmann coordinates (as compared with 12 such coordinates in the general $\mathcal{N} = 3$ superspace),

$$\{\zeta, u\} = \{x_{\mathrm{an}}^{\alpha\dot\alpha}, \theta_\alpha^{(1,-1)}, \theta_\alpha^{(0,1)}, \bar{\theta}_{\dot\alpha}^{(1,0)}, \bar{\theta}_{\dot\alpha}^{(-1,1)}, u\}. \qquad (3.19)$$

The corresponding $\mathcal{N} = 3$ Grassmann analyticity conditions are

$$\mathcal{D}_\alpha^{(1,0)} \Phi^{(q_1,q_2)} = \bar{\mathcal{D}}_{\dot\beta}^{(0,1)} \Phi^{(q_1,q_2)} = 0, \qquad (3.20)$$

and they are solved as $\Phi^{(q_1,q_2)} = \Phi^{(q_1,q_2)}(\zeta, u)$ in the basis and frame in which the covariant spinor derivatives $\mathcal{D}_\alpha^{(1,0)}$ and $\bar{\mathcal{D}}_{\dot\beta}^{(0,1)}$ simultaneously become "short". On the other hand, the triple of the harmonic derivatives $(D^{(2,-1)}, D^{(-1,2)}, D^{(1,1)})$, which commute with $\mathcal{D}_\alpha^{(1,0)}$, $\bar{\mathcal{D}}_{\dot\beta}^{(0,1)}$ and so preserve the $\mathcal{N} = 3$ analyticity, acquire the analytic harmonic connections which are analogs of the $\mathcal{N} = 2$ analytic gauge connection V^{++}:

$$(D^{(2,-1)}, D^{(-1,2)}, D^{(1,1)}) \Rightarrow (\mathcal{D}^{(2,-1)}, \mathcal{D}^{(-1,2)}, \mathcal{D}^{(1,1)}),$$
$$\mathcal{D}^{(a,b)} = D^{(a,b)} + i V^{(ab)}(\zeta, u). \qquad (3.21)$$

These harmonic derivatives satisfy, in both the original and the analytic bases, the commutation relations

$$[\mathcal{D}^{(2,-1)}, \mathcal{D}^{(-1,2)}] = \mathcal{D}^{(1,1)}, \quad [\mathcal{D}^{(1,1)}, \mathcal{D}^{(2,-1)}] = [\mathcal{D}^{(1,1)}, \mathcal{D}^{(-1,2)}] = 0. \quad (3.22)$$

As was already mentioned, the constraints (3.16) amount to the $\mathcal{N} = 3$ SYM equations of motion and the same is true for the equivalent form (3.18) of the same constraints. In the original basis the harmonic derivatives are short and their commutation relations with $\mathcal{D}_\alpha^{(1,0)}$ and $\bar{\mathcal{D}}_{\dot\beta}^{(0,1)}$,

$$[D^{(2,-1)}, \mathcal{D}_\alpha^{(1,0)}] = [D^{(-1,2)}, \mathcal{D}_\alpha^{(1,0)}] = [D^{(1,1)}, \mathcal{D}_\alpha^{(1,0)}] = 0,$$
$$[D^{(2,-1)}, \bar{\mathcal{D}}_{\dot\alpha}^{(0,1)}] = [D^{(-1,2)}, \bar{\mathcal{D}}_{\dot\alpha}^{(0,1)}] = [D^{(1,1)}, \bar{\mathcal{D}}_{\dot\alpha}^{(0,1)}] = 0, \qquad (3.23)$$

are satisfied for $\mathcal{D}_\alpha^{(1,0)}$ and $\bar{\mathcal{D}}_{\dot\beta}^{(0,1)}$ linearly depending on $SU(3)$ harmonics. Moreover, it can be shown that (3.23) are also the *necessary* conditions for

$\mathcal{D}_\alpha^{(1,0)}$ and $\bar{\mathcal{D}}_{\dot\beta}^{(0,1)}$ to be linear in $SU(3)$ harmonics. Thus, the constraints (3.16) are actually equivalent to the set of conditions (3.18), (3.23), and (3.22) (with $\mathcal{D}^{(a,b)} = D^{(a,b)}$).

On the other hand, after solving (3.18) by passing to the short $\mathcal{D}_\alpha^{(1,0)}$, $\bar{\mathcal{D}}_{\dot\beta}^{(0,1)}$, and making the appropriate similarity transformation of the remaining constraints, the relations (3.23) become the analyticity conditions for the three harmonic gauge connections $V^{(2,-1)}$, $V^{(-1,2)}$, $V^{(1,1)}$ appearing in the transformed harmonic derivatives. The whole dynamics proves to be concentrated in the purely harmonic constraints (3.22) which are just the equations of motion of the $\mathcal{N} = 3$ SYM theory in the analytic basis and frame. The final (and crucial) observation of [100] was that these equations can be reproduced by varying the following Chern–Simons-type off-shell analytic superfield action:

$$
\begin{aligned}
S_{\text{SYM}}^{(\mathcal{N}=3)} = \int du\, d\zeta^{(-2,-2)}\, \text{Tr}\Big\{ & V^{(2,-1)}(D^{(-1,2)}V^{(1,1)} - D^{(1,1)}V^{(-1,2)}) - \\
& - V^{(-1,2)}(D^{(2,-1)}V^{(1,1)} - D^{(1,1)}V^{(2,-1)}) + \\
& + V^{(1,1)}(D^{(2,-1)}V^{(-1,2)} - D^{(-1,2)}V^{(2,-1)}) - \\
& - (V^{(1,1)})^2 + 2iV^{(1,1)}[V^{(2,-1)}, V^{(-1,2)}]\Big\}, \quad (3.24)
\end{aligned}
$$

where $du\, d\zeta^{(-2,-2)}$ is the appropriate integration measure over the analytic $\mathcal{N} = 3$ superspace. Like in the ordinary 3D non-Abelian Chern–Simons action, varying (3.24) with respect to the unconstrained analytic gauge potentials yields the vanishing of three harmonic curvatures, which is equivalent to the relations (3.22). The off-shell invariance of the action (3.24) under the $\mathcal{N} = 3$ superconformal group $SU(2,2|3)$ has been shown in [101].

The presence of just three harmonic gauge connections with three equations for them is only one reason for the existence of an off-shell action for $\mathcal{N} = 3$ SYM theory. Two other reasons are the zero dimension of the integration measure of the $\mathcal{N} = 3$ analytic superspace and the charge assignment $(-2,-2)$ of this measure, which precisely matches the zero dimension and the charge assignment $(2,2)$ of the analytic Lagrangian. This threefold coincidence looks as a kind of "miracle". Unfortunately, it fails to hold in the maximally extended $\mathcal{N} = 4$ SYM theory. Though various harmonic superspace reformulations of this theory were proposed (see, e.g., [102], where the $\mathcal{N} = 4$ HSS with the harmonic part $SU(4)/[U(1) \times SU(2) \times SU(2)]$ was considered), no reasonable off-shell actions were constructed in their framework so far. They merely serve to provide some new geometric interpretations of the on-shell constraints of this theory.

Soon after its invention, the harmonic superspace approach was worldwide recognized as an adequate framework for exploring theories with extended supersymmetry in diverse dimensions. Some of its further developments and uses are briefly outlined below.

3.6. Quantum Harmonic Superspace. The quantization of $\mathcal{N} = 2$ theories in the harmonic formalism was fulfilled in [83]. The actual applications of these quantum techniques started with [103] (see also the review [104]) where there was computed, for the first time, the quantum one-loop effective action of the Coulomb phase of $\mathcal{N} = 2$ SYM theory interacting with the massless and massive matter hypermultiplets. The complete agreement with the Seiberg–Witten duality hypothesis [105] was found. The preservation of the manifest off-shell $\mathcal{N} = 2$ supersymmetry at all stages of computation was confirmed to be the basic advantage of the harmonic superspace quantum formalism. While in [103] the effective action was constructed in the sector of gauge fields, in [106] the analogous HSS-based one-loop computation was made in the hypermultiplet sector. It was shown there that some nontrivial induced hyper-Kähler metrics (e.g., the Taub-NUT one) surprisingly come out as a quantum effect.

In [107, 108], we studied the issue of finding the leading term of the low-energy quantum effective action of $\mathcal{N} = 4$ SYM theory in the Coulomb phase in the $\mathcal{N} = 2$ HSS formulation. In this formulation, the $\mathcal{N} = 4$ SYM action is represented as a sum of the $\mathcal{N} = 2$ SYM action and the action of the hypermultiplet in the adjoint representation minimally coupled to the $\mathcal{N} = 2$ gauge potential V^{++}:

$$S_{\mathrm{SYM}}^{(\mathcal{N}=4)} = S_{\mathrm{SYM}}^{(\mathcal{N}=2)} - \frac{1}{2}\mathrm{Tr}\int du\, d\zeta^{(-4)}\, q^{+a}(D^{++} + iV^{++})q_a^+. \qquad (3.25)$$

Here $S_{\mathrm{SYM}}^{(\mathcal{N}=2)}$ was defined in (3.13), and $a = 1, 2$ is an index of the so-called Pauli–Gürsey group $SU(2)_{\mathrm{PG}}$ which commutes with $\mathcal{N} = 2$ supersymmetry. This combined action is invariant under the extra hidden $\mathcal{N} = 2$ supersymmetry

$$\delta V^{++} = (\varepsilon^{\alpha a}\theta_a^+ + \bar{\varepsilon}_{\dot{\alpha}}^a\bar{\theta}^{+\dot{\alpha}})q_a^+,$$

$$\delta q_a^+ = -\frac{1}{2}(D^+)^4\big[(\varepsilon_a^\alpha\theta_\alpha^- + \bar{\varepsilon}_{\dot{\alpha}a}\bar{\theta}^{-\dot{\alpha}})V^{--}\big] \qquad (3.26)$$

(with $(D^+)^4 = (1/16)\, D^{+\alpha}D_\alpha^+ \bar{D}_{\dot{\alpha}}^+\bar{D}^{+\dot{\alpha}}$), which builds up the manifest $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 4^*$. The nonanalytic gauge potential V^{--} is related to V^{++} by the harmonic flatness condition (3.14). In [107], based purely on the transformations (3.26), we computed the leading term in the one-loop $\mathcal{N} = 4$ SYM effective action in the Coulomb phase (with the $SU(2)$ gauge group broken to $U(1)$) as

$$\Gamma(V, q) = \frac{1}{(4\pi)^2}\int d^{12}z\left\{\ln\frac{\mathcal{W}}{\Lambda}\ln\frac{\bar{\mathcal{W}}}{\bar{\Lambda}} + \mathrm{Li}_2(X) + \ln(1 - X) - \frac{1}{X}\ln(1 - X)\right\}, \qquad (3.27)$$

---

*Though (3.26) is the symmetry of the off-shell action (3.25), its correct closure with itself and with the manifest $\mathcal{N} = 2$ supersymmetry is achieved only on shell.

where Λ is an arbitrary scale, $X = \dfrac{-2q^{ai}q_{ai}}{\mathcal{W}\bar{\mathcal{W}}}$, and $\mathrm{Li}_2(X)$ is the Euler dilogarithm. In this formula, $\mathcal{W}$ is the chiral $U(1)$ superfield strength and q^{ia} is related to the on-shell $U(1)$ component of q^{+a} as $q^{+a} = q^{ia}u_i^+$. Before [107], only the $\mathcal{W}$ part of (3.27) was exactly known. The result (3.27) was reproduced from the quantum $\mathcal{N} = 2$ supergraph techniques in [108].

The quantum calculations in $\mathcal{N} = 4$ SYM theory with making use of the harmonic $\mathcal{N} = 2$ quantum supergraph techniques are widely performed by other groups, in particular, for checking the AdS/CFT correspondence (see, e.g., [109] and references therein).

3.7. Harmonic Approach to the Target Geometries. The fact that the general harmonic analytic Lagrangians of the hypermultiplets in the rigid and local $\mathcal{N} = 2$ supersymmetries can be identified with the prepotentials of the target space hyper-Kähler (HK) and quaternion-Kähler (QK) geometries was proved in [87,97]. The general HK and QK constraints can be solved quite analogously to those of $\mathcal{N} = 2$ SYM or conformal SG theories, by passing to $SU(2)$ harmonic extensions of the HK and QK manifolds and revealing there the appropriate analytic subspaces the dimension of which is twice as less compared to that of the manifold one started with. The HK and QK constraints prove to admit a general solution in terms of unconstrained prepotentials defined on these analytic subspaces, and they are just the hypermultiplet Lagrangians mentioned above. The hypermultiplets q^{+n} are none other than the coordinates of these analytic subspaces. This deep affinity between the target and Grassmann harmonic analyticities in the $\mathcal{N} = 2$, 4D (or $\mathcal{N} = 4$, 2D) sigma models in the HSS approach looks very suggestive and surely deserves further study and understanding. The examples of such an interplay between the two types of the analyticity were also found for more complicated target geometries. For instance, in recent paper [90] the so-called HKT ("hyper-Kähler with torsion") geometries (both "weak" and "strong" HKT) [110] were shown to select, as their natural prepotentials, the objects appearing in the description of the most general 1D multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ by the $\mathcal{N} = 4$, 1D analytic harmonic superfields [111] constrained by the further harmonic conditions. One of the prepotentials arises as the superfield Lagrangian of the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ analytic superfields, while the other one, as a function defining the most general harmonic constraint for these superfields.

3.8. Harmonic Superspaces in Diverse Dimensions. In [112,113], the biharmonic superspace with two independent sets of $SU(2)$ harmonics was introduced and shown to provide an adequate off-shell description of $\mathcal{N} = (4, 4)$, 2D sigma models with torsion. The analogous biharmonic $\mathcal{N} = 4$, 1D superspace [114] secures the natural uniform description of the models of $\mathcal{N} = 4$ supersymmetric mechanics with the simultaneous presence of the "mutually mirror" worldline $\mathcal{N} = 4$ multiplets. The harmonic superspace approach to extended supersymmetries in three dimensions was the subject of the important

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papers [115–117]. As a recent contribution in this direction, the $\mathcal{N} = 3$, 3D harmonic superspace formulation of the conformally invariant ABJM (Aharony–Bergman–Jafferis–Maldacena) theories was given in [118, 119]. The harmonic superspace description of $\mathcal{N} = (1, 0)$, 6D gauge theories and hypermultiplets was worked out in [120, 122, 123] (see also [121]) and recently has received a further prospective development in [124]. Various applications of the harmonic superspace method in one-dimensional mechanics models and integrable systems are presented in [111] and [125–130], as well as in [131, 132]. In particular, $\mathcal{N} = 4$, 1D HSS was used in [125] to construct $\mathcal{N} = 4$ super KdV hierarchy. It was argued in [129] that the $\mathcal{N} = 4$, 1D harmonic superspace provides a unified description of all known off-shell multiplets of $\mathcal{N} = 4$ supersymmetric mechanics. The corresponding $\mathcal{N} = 4$, 1D superfields are related to each other via gauging the appropriate isometries of the superfield actions by nonpropagating "topological" $\mathcal{N} = 4$ gauge multiplets.

Some other important applications of the HSS approach involve classifying "short" and "long" representations of various superconformal groups in diverse dimensions in the context of the AdS/CFT correspondence [133], study of the domain-wall solutions in the hypermultiplet models [134], description of self-dual supergravities [135], construction of $\mathcal{N} = 3$ supersymmetric Born–Infeld theory [136], etc. The Euclidean version of $\mathcal{N} = 2$ HSS was used in [137–139] to construct string theory-motivated nonanticommutative (nilpotent) deformations of $\mathcal{N} = (1, 1)$ hypermultiplet and gauge theories.

By now, the HSS method has proved its power as the adequate approach to off-shell theories with extended supersymmetries. Without doubt, in the future it will remain the efficient and useful tool of dealing with such theories.

4. OTHER RELATED DOMAINS

Here we briefly outline some other results obtained in Sector 3 after the invention of supersymmetry.

4.1. 2D Integrable Systems with Extended Supersymmetry. In [33], there was constructed, for the first time, $\mathcal{N} = 2$ supersymmetric extension of the renowned 2D Liouville equation and the superfield Lax pair for it was found, as well as the general solution in a superfield form. There was established, independently of [140], the existence of the twisted chiral representation of $\mathcal{N} = 2$, 2D supersymmetry besides the standard chiral one. The method used in this construction was based on a nonlinear realization of infinite-dimensional $\mathcal{N} = 2$ superconformal group in two dimensions, augmented with the inverse Higgs effect. Later on, the $\mathcal{N} = 2$ Liouville equation appeared in many contexts, including the $\mathcal{N} = 2$, 2D quantum supergravity closely related to string theory.

This research activity was continued in [34], where the same nonlinear realization methods were applied to the "small" $\mathcal{N} = 4$, 2D superconformal group to construct the new integrable superfield system, $\mathcal{N} = 4$ supersymmetric Liouville equation. Both the Lax representation and general $\mathcal{N} = 4$ superfield solution of this system were found. The $\mathcal{N} = 4$ super Liouville equation is written as an equation for the superfield describing the $\mathcal{N} = 4$, 2D "twisted chiral" multiplet and encompasses in its bosonic sector, along with the Liouville equation, also the equations of Wess–Zumino–Novikov–Witten (WZNW) sigma model for the group $SU(2)$. So the system constructed simultaneously yielded the first example of $\mathcal{N} = 4$ supersymmetric extension of the WZNW sigma models playing the fundamental role in string theory and 2D conformal field theory*.

As a next development in the same direction, in [141] new $\mathcal{N} = 4$ super-extensions of WZNW sigma models were found, in particular those exhibiting invariance under the "large" $\mathcal{N} = 4$, 2D superconformal groups. The relevant superfield and component actions were presented and it was shown that these systems admit deformations which preserve the original $\mathcal{N} = 4$ superconformal symmetry and generate Liouville potential terms in the actions. In this way, new simultaneous superextensions of the Liouville equation and WZNW sigma models come out. The $\mathcal{N} = 4$, 2D WZNW sigma models at the quantum level were studied in [142].

A different sort of $\mathcal{N} = 4$ supersymmetric integrable system was discovered in [125]. It is an $\mathcal{N} = 4$ superextension of the KdV hierarchy. Before this paper, only $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric KdV systems were known. The second Hamiltonian structure of the new system was shown to be the small $\mathcal{N} = 4$ superconformal algebra with a central charge. The basic object is the doubly charged harmonic analytic superfield subjected to a simple harmonic constraint. Later on, there appeared a lot of papers devoted to further integrable extensions of this system and their applications in many mathematical and physical problems.

4.2. Supersymmetric and Superconformal Mechanics. The supersymmetric quantum mechanics [143] is the simplest (1D) supersymmetric theory. The first work on the extended superconformal mechanics in the nonlinear realization superfield approach was [35]. There, the $\mathcal{N} = 4$ superconformal mechanics associated with the multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})^{**}$ was reproduced and the new model with the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ was found. Also, a new kind of the on-shell $\mathcal{N}$ extended superconformal mechanics with the internal symmetry group $U(\mathcal{N})$ and $\mathcal{N}$ fermionic fields in the fundamental representation of this group was constructed. The methods used in [35] are based on the inverse Higgs phenomenon which in this

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*It was also the historically first example of system with the target "strong" HKT geometry [110].

**Such a notation for the off-shell multiplets of 1D supersymmetry was suggested by A. Pashnev and F. Toppan in [144]. For the $\mathcal{N} = 4$, 1D case $(\mathbf{n}, \mathbf{4}, \mathbf{4} - \mathbf{n})$ denotes a multiplet with 4 fermions, $\mathbf{n}$ physical bosons, and $(\mathbf{4} - \mathbf{n})$ auxiliary fields.

case has not only kinematic consequences, giving rise to the elimination of certain Goldstone superfields in terms of few basic ones, but also yields the dynamics, implying the equations of motion for the basic superfields. Results and methods developed in this pioneer paper are actively applied and developed in the studies related to the superconformal quantum mechanics, including the corresponding version of the AdS/CFT correspondence. The closely related paper is [145], where the phenomenon of partial breaking of $\mathcal{N} = 4$, 1D supersymmetry was studied for the first time, on the example of the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$.

As other benchmarks on the way of developing this line of research it is worth distinguishing [111] and [131].

In [111], the harmonic superspace method was adapted to 1D supersymmetric models, i.e., the models of supersymmetric quantum mechanics, and then applied for constructing the superfield actions of diverse $\mathcal{N} = 4$, 1D multiplets, including the sigma-model type actions, superpotentials and the superfield Wess–Zumino (or Chern–Simons) terms. The realization of the most general $\mathcal{N} = 4$, 1D superconformal group $D(2, 1; \alpha)$ in the 1D harmonic superspace was found and a wide class of new models of supersymmetric (and superconformal) $\mathcal{N} = 4$ mechanics was constructed. This paper triggered many subsequent papers of different authors on the related subjects.

In [131], new superconformal extensions of integrable 1D Calogero-type models were constructed by gauging the $U(n)$ isometries of matrix superfield models (with the use of methods of [129]). The cases of $\mathcal{N} = 1, 2$, and $\mathcal{N} = 4$ superconformal systems were considered. The $\mathcal{N} = 4$ extension of the so-called "$U(2)$ spin" Calogero system was deduced. The paper [131] was first to introduce the spin (or "isospin") superfield variables, with the WZ type action of the first order in the time derivative for their bosonic physical components. In subsequent studies, these variables proved to be a useful tool of constructing various new models of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric mechanics, including the models in which the $\mathcal{N} = 4$, 1D multiplets couple to the external non-Abelian gauge fields [132].

The further developments along these lines with the participation of the Dubna group, together with the relevant references, can be retrieved from the reviews [146, 147]. As a recent new direction of research, it is worth mentioning the deformed $\mathcal{N} = 4$ mechanics associated with the supergroup $SU(2|1)$ [148]. The relevant models involve the intrinsic mass parameter and go over to the standard $\mathcal{N} = 4$ mechanics models, when this parameter goes to zero.

A different approach was represented by the papers [149–151] and [152, 153], in which the target-space supersymmetrization of the quantum-mechanical Landau problem on a plane and two-sphere was treated, as well as the closely related issue of "fuzzy" supermanifolds. In some cases, the worldline supersymmetry arises as a hidden symmetry of such models. These studies look rather interesting and perspective, since, e.g., they are expected to give rise to a deeper understanding of

quantum Hall effect and its possible superextensions. The relationships of these models to superparticles and superbranes are also worthy to learn in more depth.

4.3. Superparticles, Branes, Born–Infeld, Chern–Simons, and Higher Spins. In the late nineties, there was growth of interest in the superfield description of superbranes as systems realizing the concept of Partial Breaking of Global Supersymmetry (PBGS) pioneered by Bagger and Wess [154] and Hughes and Polchinsky [155]. In this approach, the physical worldvolume superbrane degrees of freedom are represented by Goldstone superfields, on which the worldvolume supersymmetry acts by linear transformations. The rest of the full target supersymmetry is spontaneously broken and is realized nonlinearly. In components, the transverse coordinates of the superbrane (if they exist) are described by a gauge-fixed Nambu–Goto action. In the cases when the Goldstone supermultiplets are vector ones, the Goldstone superfield actions simultaneously provide supersymmetrization of the appropriate Born–Infeld-type actions. The relevant references can be found, e.g., in [156, 157].

Among the most important results obtained in this domain with the decisive participation of the Dubna group, it is worth mentioning the perturbative-theory construction of the $\mathcal{N} = 2$ superfield Born–Infeld action with the spontaneously broken $\mathcal{N} = 4$ supersymmetry [32, 158], as well as the interpretation of a hypermultiplet as a Goldstone multiplet supporting a partial breaking of $\mathcal{N} = 1$, 10D supersymmetry [159]. The peculiarities of the partial breaking $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ in 4D within the $\mathcal{N} = 2$ superfield formalism (including the interplay between the electric and magnetic Fayet–Ilopoulos terms) were discussed in [160]. The $\mathcal{N} = 3$ supersymmetric extension of the Born–Infeld theory was constructed in [136].

In [161], the so-called "AdS/CFT equivalence transformation" was proposed. It relates the standard realization of the spontaneously broken 4D conformal group $SO(2, 4)$ on the dilatonic field ("conformal basis") with its realization as the isometry group of the gauge-fixed AdS_5 brane ("AdS basis"). The 1D version of this transformation allowed us to show [162] that the standard one-dimensional conformal mechanics is in fact equivalent to the so-called "relativistic conformal mechanics" of [163] (alias AdS_2 particle). This correspondence can be extended to superconformal mechanics models in the Hamiltonian formalism [164] and is now widely applied in many domains (see, e.g., [165]).

Supersymmetric extensions of the Chern–Simons terms in three dimensions (as well as of their generalization, the so-called BF Lagrangians) were constructed and studied in [115–119] and [166–168]. In particular, in [166,167] the manifestly supersymmetric superfield form of the $\mathcal{N} = 2$ Chern–Simons action was given for the first time.

In [169], it was shown that the $AdS_3 \times S^3$ and $AdS_5 \times S^5$ superstring theories in the Pohlmeyer-reduced form [170] reveal hidden $\mathcal{N} = (4, 4)$ and $\mathcal{N} = (8, 8)$ worldsheet supersymmetries. The explicit form of the supersymmetry transformations was found for both the off-shell action and the superstring equations.

A new superfield approach to the higher-spin multiplets based on nonlinear realizations of the generalized 4D superconformal group $OSp(1|8)$ has been developed in [171]. It was argued that the higher-spin generalization of $\mathcal{N} = 1$ supergravity should be based, *a la* Ogievetsky and Sokatchev, on the preservation of the $OSp(1|8)$ analog of chirality. There were also given a few proposals of how to reproduce the higher-spin equations by quantizing various kinds of superparticles [172–174]. In particular, it was shown in [174] that a new kind of such equations can be obtained by quantizing a particle in the tensorial space associated with the so-called Maxwell extension of the Poincaré group. The BRST approach to Lagrangian formulation of higher-spin fields was successfully elaborated by A. Pashnev with co-authors (see [175] and references therein).

4.4. Last but not Least: Auxiliary Tensor Fields for Duality Invariant Theories. Nowadays, the duality invariant systems attract a lot of attention (see, e.g., [176] and references therein). The simplest example of duality in 4D is the covariance of the free Maxwell equation and Bianchi identity,

$$\partial^m F_{mn} = 0, \quad \partial^m \tilde{F}_{mn} = 0, \quad \tilde{F}_{mn} := \frac{1}{2}\varepsilon_{mnpq}F^{pq}, \tag{4.1}$$

under the $O(2)$ duality rotation

$$\delta F_{mn} = \omega \tilde{F}_{mn}, \quad \delta \tilde{F}_{mn} = -\omega F_{mn}, \tag{4.2}$$

where ω is a real transformation parameter. Another example of the duality-invariant system is supplied by the renowned nonlinear Born–Infeld theory. The duality invariant systems involving, besides gauge fields, also the coset scalar fields described by nonlinear sigma models naturally appear in various extended supergravities and are important ingredients of string/brane theory.

Even in the simplest $O(2)$ duality case it was not so easy to single out the most general set of duality-invariant nonlinear generalizations of the Maxwell theory. There were developed a few approaches based on solving some nonlinear equations. In [177, 178], a new purely algebraic approach to this problem was proposed. Namely, it was shown that the most general duality-invariant nonlinear extension of Maxwell theory is described by the Lagrangian

$$\mathcal{L}(V, F) = \mathcal{L}_2(V, F) + E(\nu, \bar{\nu}), \tag{4.3}$$

$$\mathcal{L}_2(V, F) = \frac{1}{2}(\varphi + \bar{\varphi}) + \nu + \bar{\nu} - 2\,(V \cdot F + \bar{V} \cdot \bar{F}), \tag{4.4}$$

where the auxiliary unconstrained fields $V_{\alpha\beta}$ and $\bar{V}_{\dot{\alpha}\dot{\beta}}$ were introduced, with $\nu = V^2$, $\bar{\nu} = \bar{V}^2$, $\varphi = F^{\alpha\beta}F_{\alpha\beta}$, $\bar{\varphi} = \bar{F}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}$ and $F^{\alpha\beta}$, $\bar{F}^{\dot{\alpha}\dot{\beta}}$ representing the Maxwell field strength in the spinorial notation. In (4.3), $\mathcal{L}_2(V, F)$ is the bilinear part only through which the Maxwell field strength enters the action, and $E(\nu, \bar{\nu})$

is the nonlinear interaction involving only auxiliary fields. The duality group acts on $V_{\alpha\beta}$ as

$$\delta V_{\alpha\beta} = -i\omega V_{\alpha\beta}, \quad \delta\nu = -2i\omega\nu,$$

and it was proved that the requirement of duality invariance of the full set of equations of motion following from (4.3) amounts to $O(2)$ invariance of the function $E(\nu, \bar{\nu})$,

$$E(\nu, \bar{\nu}) = \mathcal{E}(a), \quad a = \nu\bar{\nu}. \tag{4.5}$$

Eliminating the auxiliary fields from (4.3) with such $E(\nu, \bar{\nu})$ by their algebraic equations of motion, we obtain a nonlinear version of Maxwell action, such that the relevant equations of motion necessarily respect duality invariance. Thus, the variety of all possible duality invariant extensions of the Maxwell theory is parameterized by the single function $\mathcal{E}(a)$ which can be chosen at will.

Later on, this formalism was generalized to the cases of $U(N)$ duality [179] and $Sp(2, \mathbb{R})$ duality [180]. The $\mathcal{N} = 1, 2$ superfield extensions were built in [181–183].

Note that the tensorial auxiliary field representation was guessed from the construction of $\mathcal{N} = 3$ superfield Born–Infeld theory in [136]. These auxiliary fields naturally appear as the necessary components of the off-shell $\mathcal{N} = 3$ SYM multiplet in the HSS approach. Keeping this in mind, it seems probable that the tensor auxiliary fields formulation of the duality invariant systems could also enter as an element into the hypothetical harmonic superfield formulations of various extended supergravities.

Finally, it is worth mentioning that, besides the topics listed above, in the Sector "Supersymmetry" for the last decade the investigations on a few different important subjects were also accomplished. These include the twistor approach to strings and particles (see, e.g., [184, 185]), the studies related to the AGT (Alday–Gaiotto–Tachikawa) conjecture (see, e.g., [186]) and, more recently, the explicit construction of instantons and monopoles (see, e.g., [187]). In view of lacking of space, I will not dwell on these issues.

CONCLUSIONS

In this paper, the author reviewed the mainstream scientific activity of the Sector of Markov–Ogievetsky–Ivanov–... for more than fifty years. In retrospect, the most influential pioneering results and methods which have successfully passed the examination by time are, in my opinion, the following: I. Notoph; II. "Ogievetsky Theorem" and the view of the gravitation theory as a theory of spontaneous breaking, with the graviton as a Goldstone field; III. The inverse Higgs phenomenon; IV. The complex superfield geometry of $\mathcal{N} = 1$ supergravity; V. The general relationship between linear and nonlinear realizations of supersymmetry; VI. Grassmann analyticity and harmonic superspace.

As for the future directions of research, I think that in the nearest years they will be mainly concerned with exploring the geometry and quantum structure of supersymmetric gauge theories and supergravity in diverse dimensions in the superfield approach, as well as studying various aspects of supersymmetric and superconformal mechanics models in their intertwining relationships with the higher-dimensional field theories and string theory.

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