

NON-ABELIAN $D = 11$ SUPERMEMBRANE

M. P. García del Moral^{a, 1}, *A. Restuccia*^{b, 2}

^a Departamento de Física, Universidad de Oviedo, Oviedo, Spain

^b Departamento de Física, Universidad Simón Bolívar, Caracas, Venezuela

We obtain a $U(M)$ action for supermembranes with central charges in the Light-Cone Gauge (LCG). The theory realizes all of the symmetries and constraints of the supermembrane together with the invariance under a $U(M)$ gauge group with M arbitrary. The world-volume action has (LCG) $\mathcal{N} = 8$ supersymmetry and it corresponds to M parallel supermembranes minimally immersed in the target $M_9 \times T^2$ (MIM2). In order to ensure the invariance under the symmetries and to close the corresponding algebra, a star-product determined by the central charge condition is introduced. It is constructed with a nonconstant symplectic two-form where curvature terms are also present. The theory is in the strongly coupled gauge-gravity regime. At low energies, the theory enters in a decoupling limit and it is described by an ordinary $\mathcal{N} = 8$ SYM in the IR phase for any number of M2-branes.

PACS: 11.25.Yb; 11.25.Uv; 11.30.Pb; 11.25.Mj; 11.30.Er

INTRODUCTION

An ultimate goal of String Theory is to find its nonperturbative quantization. M-theory has been elusive to this point although significative advances have been realized. To get contact with four dimensions, at the end of the day a phenomenological model that will take into account this nonperturbative effects will be required. From that point of view, obtaining a non-Abelian gauge formulation directly from M-theory — not just in its effective action — but in the full-fledged formulation, is an important goal. In this letter we summarize the results obtained in [1] where we have been able to find a non-Abelian extension of the supermembrane minimally immersed in $M_9 \times T^2$.

The supermembrane with a topological restriction associated with an irreducible winding has been shown to have very interesting properties: discreteness of the supersymmetric spectrum [2–4], spontaneous breaking of supersymmetry, stabilization of most of the moduli [5], a spectrum containing dyonic strings plus pure supermembrane excitations [6], formulation on a G2 manifold [7]. This restriction can be seen at algebraic level as a central charge condition on the 11D supersymmetric algebra and geometrically as a condition of being minimally immersed into the target space [8], so from now on, we will denote it as MIM2. The semiclassical supermembrane subject to an irreducible wrapping was first analyzed in [9].

¹E-mail: garciamormaria@uniovi.es

²E-mail: arestu@usb.ve

Recently, the low energy conformal description of multiple M2-branes has received a lot of attention from the scientific community. The original motivation has been to realize Maldacena's Conjecture for M-theory [10] according to which M-theory/ $AdS_4 \times S^7$ should be dual to a CFT_3 generated by the action of multiple M2-branes in the decoupling limit, that is, for a large number M of M2s. Authors of [11,12] and independently [13] were the first to obtain a realization of this algebra by imposing fields to be evaluated on a three-algebra with positive inner metric. In terms of a unique finite-dimensional gauge group $SO(4)$ with a twisted Chern–Simons terms. In order to generalize it for general $SU(M)$ gauge groups, a Chern–Simons-matter theory with $\mathcal{N} = 6$ was found by [14] (ABJM) in which they are able to generalize the theory to an arbitrary $SU(N)$ and recover also BLG theory for the case of $N = 2$. The ABJM, or at least a sector of it, can be also recovered from the three-algebra formulation by relaxing the condition of total antisymmetry of the structure constants [15]. A more complete list of relevant references for interested reader can be found in [1].

In this letter we summarize the results obtained in [1]. The approach is very different to the ABJM one. In [1] it consistently extended the action of a single MIM2 to a theory of interacting parallel M2-branes minimally immersed (MIM2s) preserving all of the symmetries of the theory: supersymmetry and invariance under area preserving diffeomorphisms. The theory is not conformal invariant. In the extension, the gauge and gravity sectors are strongly correlated. It corresponds to have an M-theory dual of the non-Abelian Born–Infeld action describing a bundle of multiple D2-D0 branes, so we work in the high energy approximation. When the energy scale is low, the theory decouples and it is effectively described by a $\mathcal{N} = 8$ SYM in the IR phase. As the energy scale raises, the YM coupling constant becomes weaker and at some point oscillations modes of the pure supermembrane appear and the theory enters in the strong correlated gauge-gravity sector.

1. A SUPERMEMBRANE WITH DISCRETE SPECTRUM: THE MIM2

In this section we will make a self-contained summary of the construction of the minimally immersed M2-brane (MIM2). The Hamiltonian of the $D = 11$ supermembrane [16] may be defined in terms of maps X^M , $M = 0, \dots, 10$, from a base manifold $R \times \Sigma$, where Σ is a Riemann surface of genus g onto a target manifold which we will assume to be 11D Minkowski. The canonical reduced Hamiltonian to the Light-Cone Gauge has the expression [17]

$$\mathcal{H} = \int_{\Sigma} d\sigma^2 \sqrt{W} \left(\frac{1}{2} \left(\frac{P_M}{\sqrt{W}} \right)^2 + \frac{1}{4} \{X^M, X^N\}^2 - \bar{\Psi} \Gamma_- \Gamma_M \{X^M, \Psi\} \right) \quad (1)$$

subject to the constraints

$$\phi_1 := d \left(\frac{P_M}{\sqrt{W}} dX^M - \bar{\Psi} \Gamma_- d\Psi \right) = 0 \quad (2)$$

and

$$\phi_2 := \oint_{C_s} \left(\frac{P_M}{\sqrt{W}} dX^M - \bar{\Psi} \Gamma_- d\Psi \right) = 0, \quad (3)$$

where the range of M is now $M = 1, \dots, 9$ corresponding to the transverse coordinates in the Light-Cone Gauge, C_s , $s = 1, \dots, 2g$ is a basis of one-dimensional homology on Σ ,

$$\{X^M, X^N\} = \frac{\epsilon^{ab}}{\sqrt{W(\sigma)}} \partial_a X^M \partial_b X^N, \tag{4}$$

$a, b = 1, 2$ and σ^a are local coordinates over Σ . $W(\sigma)$ is a scalar density introduced in the Light-Cone Gauge fixing procedure. ϕ_1 and ϕ_2 are generators of area preserving diffeomorphisms, see [18]. That is

$$\sigma \rightarrow \sigma' \rightarrow W'(\sigma) = W(\sigma).$$

When the target manifold is simply connected dX^M are exact one-forms.

The spectral properties of (1) were obtained in the context of a $SU(N)$ regularized model [17] and it was shown to have continuous spectrum from $[0, \infty)$.

This property of the theory relies on two basic facts: supersymmetry and the presence of classical singular configurations, string-like spikes, which may appear or disappear without changing the energy of the model but may change the topology of the world-volume. Under compactification of the target manifold generically the same basic properties are also present and consequently the spectrum should be also continuous [19]. In what follows we will impose a topological restriction on the configuration space. It characterizes a $D = 11$ supermembrane with nontrivial central charges generated by the wrapping on the compact sector of the target space [2,4]. We will consider in this paper the case $g = 1$ Riemann surface as a base manifold Σ on a $M_9 \times T^2$ target space. The configuration maps satisfy

$$\oint_{C_s} dX^r = 2\pi L_s^r R^r, \quad r, s = 1, 2, \tag{5}$$

$$\oint_{C_s} dX^m = 0, \quad m = 3, \dots, 9, \tag{6}$$

where L_s^r are integers and R^r , $r = 1, 2$ is the radius of T^2 . These conditions ensure that we are mapping Σ onto a T^2 sector of the target manifold.

We now impose the central charge condition

$$I^{rs} \equiv \int_{\Sigma} dX^r \wedge dX^s = (2\pi R_1 R_2) n \omega^{rs}, \tag{7}$$

where ω^{rs} is a symplectic matrix on the T^2 sector of the target and $n = \det L_i^r$ represents the irreducible winding.

The topological condition (7) does not change the field equations of the Hamiltonian (1). In fact, any variation of I^{rs} under a change δX^r , single-valued over Σ , is identically zero. In addition to the field equations obtained from (1), the classical configurations must satisfy condition (7). It is only a topological restriction on the original set of classical solutions of the field equations. In the quantum theory the space of physical configurations is also restricted by condition (7). The geometrical interpretation of this condition has been discussed in previous

works [20,21]. We noticed that (7) only restricts the values of S_s^r , which are already integral numbers from (5).

We consider now the most general map satisfying condition (7). Closed one-forms dX^r may be decomposed into the harmonic plus exact parts:

$$dX^r = M_s^r d\widehat{X}^s + dA^r, \tag{8}$$

where $d\widehat{X}^s$, $s = 1, 2$ is a basis of harmonic one-forms over Σ and dA^r are exact one-forms. We may normalize it by choosing a canonical basis of homology and imposing

$$\oint_{c_s} d\widehat{X}^r = \delta_s^r. \tag{9}$$

We have now considered a Riemann surface with a class of equivalent canonical basis. Condition (5) determines

$$M_s^r = 2\pi R^r L_s^r, \tag{10}$$

we rewrite $L_s^r = l_r S_s^r$ and $l_1 \cdot l_2 = n$. We now impose condition (7) and obtain

$$S_t^r \omega^{tu} S_u^s = \omega^{rs}, \tag{11}$$

that is, $S \in \text{Sp}(2, Z)$. This is the most general map satisfying (7). See [6] for details, in particular for $n > 1$.

The natural choice for $\sqrt{W(\sigma)}$ in this geometrical setting is to consider it as the density obtained from the pull-back of the Kähler two-form on T^2 . We then define

$$\sqrt{W(\sigma)} = \frac{1}{2} \partial_a \widehat{X}^r \partial_b \widehat{X}^s \omega_{rs}. \tag{12}$$

$\sqrt{W(\sigma)}$ is then invariant under the change

$$d\widehat{X}^r \rightarrow S_s^r d\widehat{X}^s, \quad S \in \text{Sp}(2, Z). \tag{13}$$

But this is just the change on the canonical basis of harmonics one-forms when a bi-holomorphic map in Σ is performed changing the canonical basis of homology. That is, the bi-holomorphic (and hence diffeomorphic) map associated with the modular transformation on a Teichmüller space. We thus conclude that the theory is invariant not only under the diffeomorphisms generated by ϕ_1 and ϕ_2 , homotopic to the identity, but also under the diffeomorphisms, bi-holomorphic maps, changing the canonical basis of homology by a modular transformation.

Having identified the modular invariance of the theory, we may go back to the general expression of dX^r , we may always consider a canonical basis such that

$$dX^r = 2\pi l^r R^r d\widehat{X}^r + dA^r \tag{14}$$

the corresponding degrees of freedom are described exactly by the single-valued fields A^r . After replacing this expression in the Hamiltonian (1) we obtain

$$\begin{aligned}
 H = \int_{\Sigma} \sqrt{W} d\sigma^1 \wedge d\sigma^2 & \left[\frac{1}{2} \left(\frac{P_m}{\sqrt{W}} \right)^2 + \frac{1}{2} \left(\frac{\Pi_r}{\sqrt{W}} \right)^2 + \right. \\
 & + \frac{1}{4} \{X^m, X^n\}^2 + \frac{1}{2} (\mathcal{D}_r X^m)^2 + \frac{1}{4} (\mathcal{F}_{rs})^2 + \\
 & \left. + (n^2 \text{Area}_{T^2}^2) + \int_{\Sigma} \sqrt{W} \Lambda \left(\mathcal{D}_r \left(\frac{\Pi_r}{\sqrt{W}} \right) + \left\{ X^m, \frac{P_m}{\sqrt{W}} \right\} \right) \right] + \\
 & + \int_{\Sigma} \sqrt{W} [-\bar{\Psi} \Gamma_- \Gamma_r \mathcal{D}_r \Psi - \bar{\Psi} \Gamma_- \Gamma_m \{X^m, \Psi\} - \Lambda \{\bar{\Psi} \Gamma_-, \Psi\}],
 \end{aligned}$$

where

$$\mathcal{D}_r X^m = D_r X^m + \{A_r, X^m\}, \quad \mathcal{F}_{rs} = D_r A_s - D_s A_r + \{A_r, A_s\}, \quad D_r = 2\pi l_r R_r \frac{\epsilon^{ab}}{\sqrt{W}} \partial_a \hat{X}^r \partial_b,$$

and P_m and Π_r are the conjugate momenta to X^m and A_r , respectively. \mathcal{D}_r and \mathcal{F}_{rs} are the covariant derivative and curvature of a symplectic noncommutative theory [20], constructed from the symplectic structure ϵ^{ab}/\sqrt{W} introduced by the central charge. The last term represents its supersymmetric extension in terms of Majorana spinors. The physical degrees of the theory are X^m, A_r, Ψ_α , they are single-valued fields on Σ .

Quantum Supersymmetric Analysis of a Single MIM2. We are going to summarize the spectral properties of the above Hamiltonian. The bosonic potential of (15) satisfies the following inequality [4] (in a particular gauge condition):

$$\begin{aligned}
 \int_{\Sigma} \sqrt{W} d\sigma^1 \wedge d\sigma^2 & \left[\frac{1}{4} \{X^m, X^n\}^2 + \frac{1}{2} (\mathcal{D}_r X^m)^2 + \frac{1}{4} (\mathcal{F}_{rs})^2 \right] \geq \\
 & \geq \int_{\Sigma} \sqrt{W} d\sigma^1 \wedge d\sigma^2 \left[\frac{1}{2} (\mathcal{D}_r X^m)^2 + (\mathcal{D}_r A_s)^2 \right].
 \end{aligned}$$

The right-hand member under regularization describes a harmonic oscillator potential. In particular, any finite-dimensional truncation of the original infinite-dimensional theory satisfies the above inequality. We consider regularizations satisfying the above inequality. We denote the regularized Hamiltonian of the supermembrane with the topological restriction by H , its bosonic part H_b and its fermionic potential V_f , then

$$H = H_b + V_f. \tag{15}$$

We can define rigorously the domain of H_b by means of Friederichs extension techniques. In this domain H_b is self-adjoint and it has a complete set of eigenfunctions with eigenvalues accumulating at infinity. The operator multiplication by V_f is relatively bounded with respect to H_b . Consequently, using Kato perturbation theory it can be shown that H is self-adjoint if we choose

$$\text{Dom } H = \text{Dom } H_b. \tag{16}$$

In [2] it was shown that H possesses a complete set of eigenfunctions and its spectrum is discrete, with finite multiplicity and with only an accumulation point at infinity. An independent proof was obtained in [3] using the spectral theorem and theorem 2 of that paper. In Sec.5 of [3] a rigorous proof of the Feynman formula for the Hamiltonian of the supermembrane was obtained. In distinction, the Hamiltonian of the supermembrane, without the topological restriction, although it is positive, its fermionic potential is not bounded from below and it is not a relative perturbation of the bosonic Hamiltonian. The use of the Lie product theorem in order to obtain the Feynman path integral is then not justified. It is not known and completely unclear whether a Feynman path integral formula exists for this case. In [4] it was proved that the theory of the supermembrane with central charges corresponds to a nonperturbative quantization of a symplectic Super Yang–Mills in a confined phase and the theory possesses a mass gap.

In [7] we constructed the supermembrane with the topological restriction on an orbifold with G_2 structure that can be ultimately deformed to lead to a true G_2 manifold. All the discussion of the symmetries on the Hamiltonian was performed directly in the Feynman path integral, at the quantum level, then valid by virtue of our previous proofs.

2. A $U(N)$ EXTENSION OF THE MIM2 FOR ARBITRARY RANK

In this section we extend the algebraic symplectic structure of the supermembrane with central charges in the LCG in terms of a noncommutative product and a $U(M)$ gauge group. The main point is to show that in such an extension the original area preserving constraint preserves the property of being first class. It is not enough to have the symplectic structure tensor $U(M)$ in order to close the algebra of the first-class constraint. The complete expansion related to a noncommutative associative product is needed. The noncommutative product we may introduce is constructed with the symplectic two-form already defined on the base manifold Σ :

$$\omega_{ab} = \sqrt{W} \epsilon_{ab}, \quad (17)$$

where $\sqrt{W} = \frac{n}{2} \text{Area}_{T^2}(\epsilon_{rs} \epsilon^{ab} \partial_a \hat{X}^r \partial_b \hat{X}^s)$. In this section, in order to get a better insight on the star-product, we use, without loosing generality, coordinates on the base manifold with length dimension +1 and define the dimensionless \sqrt{W} with the area factor. The two-form ω defines the area element which is preserved by the diffeomorphisms generated by the first-class constraint of the supermembrane theory in the Light-Cone Gauge, which are homotopic to the identity, and by the $SL(2, Z)$ group of large diffeomorphisms discussed in Sec.1. The two-form is closed and nondegenerate over Σ . By Darboux theorem one can choose coordinates on an open set \mathfrak{N} in Σ in a way that \sqrt{W} becomes constant on \mathfrak{N} . However, this property cannot be extended to the whole compact manifold Σ . The noncommutative theory must be globally constructed from a nonconstant symplectic ω . The construction of such noncommutative theories for symplectic manifolds was performed in [22, 23]. The general construction for Poisson manifolds was obtained in [24].

The Hamiltonian we propose in this section is not related to a Seiberg–Witten limit of String Theory [25] in which one obtains a noncommutative theory with constant B -field.

2.1. The Non-Abelian Hamiltonian. We now extend the above construction and consider the tensor product of the Weyl-algebra bundle times the enveloping algebra of $U(M)$. It may

be constructed in terms of the Weyl-algebra generators T_A introduced in the previous section, with the inclusion of the identity associated with $A = (0, 0)$. This complete set of generators determines an associative algebra under matrix multiplication. The inclusion of the identity allows one to realize the generators of $U(M)$ in terms of T_A matrices, with $A = (a_1, a_2)$ and $a_1, a_2 = -(M-1), \dots, 0, \dots, M-1$. All the properties of the Fedosov construction remain valid, in particular the associativity of the star-product. The trace property is also valid. In order to construct the Hamiltonian of the theory, we consider the following connection on the Weyl bundle [26]:

$$\mathcal{D} \diamond = \frac{i}{\hbar} [G_r e^r, \diamond]_{\circ} + \frac{i}{\hbar} [\mathcal{A}_r e^r, \diamond]_{\circ}, \quad (18)$$

where $G_r, \mathcal{A}_r \in C^\infty(W_{\text{Abel}})$, $\sigma G_r = \delta_{rs} X_h^s$ and $X_h^s = 2\pi R_s l_s \widehat{X}^s$. It corresponds to the harmonic sector of the map to the compact sector of the target space. $\sigma \mathcal{A}_r = A_r$ using the notation of Sec. 1, $e^r = \partial_a \widehat{X}^r d\sigma^a$. Its curvature is given by

$$\Omega = \frac{i}{2\hbar} [G, G]_{\circ} + \frac{i}{\hbar} [G, \gamma]_{\circ} + \frac{i}{2\hbar} [\gamma, \gamma]_{\circ}, \quad \gamma = \mathcal{A}_r e^r. \quad (19)$$

We now consider $(X^m, P_m), (A_r, \Pi^r)$ the canonical conjugate pairs as well as the spinor fields Ψ lifted to the quantum algebra $W_{\text{Abel}} \in C^\infty(W)$. The constraint is then defined as

$$\phi(\sigma, \xi, \hbar) \equiv \mathcal{D}_r \frac{\Pi^r A}{\sqrt{W}} T_A + \frac{i}{\hbar} \left(X^{mB} \circ \frac{P_m^C}{\sqrt{W}} - \frac{P_m^B}{\sqrt{W}} \circ X_m^C \right) T_B T_C + \frac{i}{\hbar} [\overline{\Psi} \Gamma_-, \Psi]_{\circ}. \quad (20)$$

with

$$\mathcal{D}_r \frac{\Pi^r A}{\sqrt{W}} T_A = \frac{i}{\hbar} \left[G_r, \frac{\Pi^r A}{\sqrt{W}} \right]_{\circ} T_A + \frac{i}{\hbar} \left(A_r^B \circ \frac{\Pi^r C}{\sqrt{W}} - \frac{\Pi^r B}{\sqrt{W}} \circ A_r^C \right) T_B T_C. \quad (21)$$

We notice that the first two terms of the commutator

$$\left[X^m, \frac{P_m}{\sqrt{W}} \right]_{\circ} = X^{mB} \frac{P_m^C}{\sqrt{W}} f_{BC}^E T_E + \left(-i \frac{\hbar}{2} \right) \left\{ X^{mB}, \frac{P_m^C}{\sqrt{W}} \right\} d_{BC}^E T_E + O((\hbar\omega)^2) \quad (22)$$

are the terms which we considered in the previous section as extensions of the algebraic structure of the supermembrane in the Light-Cone Gauge. The additional terms arising from the noncommutative product, ensuring an associative product, are relevant in order to close the constraint algebra. In fact, using the trace properties $\phi \in W_{\text{Abel}}$ is a first-class constraint generating a gauge transformation which is a deformation of the original and preserving diffeomorphisms.

The projection of Ω in (19) has the expression [26]

$$\begin{aligned} \sigma \Omega = -\omega + \mathcal{F} - \frac{\hbar^2}{96} \left(R_{bcda} \left(D_{\widehat{b}} D_{\widehat{c}} D_{\widehat{d}} \right) A_m - \frac{1}{4} R_{\widehat{b}\widehat{c}\widehat{d}p} \epsilon^{pq} D_q A_m \right) \epsilon^{\widehat{b}\widehat{b}} \epsilon^{\widehat{c}\widehat{c}} \epsilon^{\widehat{d}\widehat{d}} e^a \wedge e^m - \\ - \frac{\hbar^2}{96 \cdot 8} R_{bcda} R_{\widehat{b}\widehat{c}\widehat{d}m} \epsilon^{\widehat{b}\widehat{b}} \epsilon^{\widehat{c}\widehat{c}} \epsilon^{\widehat{d}\widehat{d}} e^a \wedge e^m + O(\hbar^3) \dots \end{aligned}$$

and

$$\mathcal{F} = \frac{1}{2} e^r \wedge e^s \left(D_r A_s - D_s A_r + \frac{i}{\hbar} \{A_r, A_s\}_* \right), \quad (23)$$

with $\omega = 1/2h\sqrt{w}\epsilon_{ab}d\sigma^a \wedge d\sigma^b$, D_r, D_s are the ones defined in Sec. 1. We remark that G_r is the lifting to the Weyl algebra of the harmonic \widehat{X}_r of Sec. 1, and \mathcal{A}_r is the lifting of A_r . The $O(\hbar^2)$ depends explicitly on the Riemann tensor of the symplectic connection, which itself depends on the symplectic two-form introduced by the central charge. The $O(\hbar)$ terms are necessary in order to close the constraint algebra. The star-product formula involves the covariant derivatives constructed from the symplectic connection as well as terms involving the Riemann tensor of symplectic connection, which are absent in the Moyal product.

The Hamiltonian of the theory for M multiple parallel M2-branes with $U(N)$ gauge group is then

$$\begin{aligned} \text{Tr} \int_{\Sigma} \mathcal{H} = \text{Tr} \int_{\Sigma} \sqrt{W} & \left[\frac{1}{2} \left(\frac{P^m}{\sqrt{W}} \right)^2 + \frac{1}{2} \left(\frac{\Pi^r}{\sqrt{W}} \right)^2 + \frac{1}{2\hbar^2} (\{X_h^r, X^m\}_* + \{A^r, X^m\}_*)^2 + \right. \\ & + \frac{1}{4\hbar^2} \{X^m, X^n\}_*^2 + \frac{1}{2} (\mathcal{F}_{rs} - \omega_{rs})(\mathcal{F}^{rs} - \omega^{rs}) - \\ & \left. - \frac{i}{\hbar} \overline{\Psi} \Gamma_- \Gamma_r (\{X_h^r, \Psi\}_* + \{A^r, \Psi\}_*) - \frac{i}{\hbar} \overline{\Psi} \Gamma_- \Gamma_m \{X^m, \Psi\}_* \right], \end{aligned}$$

where the term $\{X_h^r, X^m\}_* + \{A^r, X^m\}_* = \delta^{rs} \mathcal{D}_s X^m + O(\hbar)$ in the notation of Sec. 1. The Hamiltonian is subject to the first-class constraint

$$\phi \equiv \left\{ X_h^r, \frac{\Pi^r}{\sqrt{W}} \right\}_* + \left\{ A_r, \frac{\Pi^r}{\sqrt{W}} \right\}_* + \left\{ X^m, \frac{\Pi^r}{\sqrt{W}} \right\}_* - \left\{ \overline{\Psi} \Gamma_-, \Psi \right\}_* = 0. \quad (24)$$

The first terms in the star-product expansion are

$$\phi \equiv \mathcal{D}_r \frac{\Pi^r}{\sqrt{W}} + \left\{ X^m, \frac{\Pi^r}{\sqrt{W}} \right\}_* - \left\{ \overline{\Psi} \Gamma_-, \Psi \right\}_* + O(\hbar), \quad (25)$$

where \langle, \rangle_* has been normalized in a way to be a deformation of $\{, \}$ the symplectic bracket of the supermembrane in the LCG. In the notation of Sec. 1, the fields are now $u(M)$ valued. An explicit expression for $O(\hbar)$, the first terms with the explicit dependence were found in [26], for example, if we make manifest the dimensional dependence of the star-product, we can realize that the parameter $[h] = n \text{Area}_{T^2}$, n is the wrapping number. In fact, due to the minimal immersion map there is a local bijection between the coordinates in the base manifold and those in the compact part of the target space:

$$\int_{\Sigma} d^2\sigma \sqrt{W} = \int_{\Sigma} dX_h^1 dX_h^2 \quad (26)$$

with $X_h^r = 2\pi R^r l^r \widehat{X}^r$. The star-product is explicitly given by

$$\begin{aligned} \frac{i}{\hbar} \{f, g\}_*^a &= \frac{i}{\hbar} f^b g^c f_{bc}^a + \{f^b, g^c\} d_{bc}^a + O(\hbar) = \\ &= \frac{i}{n \text{Area}_{T^2}} f^b g^c f_{bc}^a + \{f^b, g^c\} d_{bc}^a + O(n \text{Area}_{T^2}), \end{aligned}$$

where $\{f^b, g^c\} = \epsilon^{rs} D_r f^b D_s g^c$. D_r was defined in Sec.1. The factor $1/h$ ensures that this formalism is a non-Abelian extension of the Abelian MIM2-brane, since for the Abelian case f_{bc}^a vanishes, $d_{bc}^a = 1$, and the algebra closes exactly with the ordinary symplectic bracket corresponding to a single M2 action without further contributions.

The associated action to this Hamiltonian [1] is invariant under the following supersymmetric transformations with parameter $\epsilon = \Gamma_- \Gamma_+ \epsilon$:

$$\begin{aligned}\delta A_M &= \delta A_M^B T_B = \bar{\epsilon} \Gamma_M \Psi^B T_B, \quad M = r, m, \\ \delta A_0 &= \delta A_0^B T_B = -\bar{\epsilon} \Psi^B T_B, \\ \delta \Psi &= \delta \Psi^B T_B = \frac{1}{4} \Gamma_+ \Omega_{MN}^B \Gamma^{MN} \epsilon T_B + \frac{1}{2} \Gamma_+ \Omega_{0M}^B \Gamma^M \epsilon T_B.\end{aligned}$$

These transformations are a $U(N)$ extension of the SUSY transformations for the supermembrane in the LCG found by [17,27] and they preserve $\mathcal{N} = 8$ supersymmetry. The invariance of the action arises in a similar way as it does for Super Yang–Mills.

2.2. Decoupling Limit. The mass square operator may be written as

$$\begin{aligned}-\text{mass}^2 &= \int \left(\frac{1}{2} d\hat{x}^r \wedge d\hat{x}^s \epsilon_{rs} \right) \left[\frac{1}{2} \left(\frac{P}{\sqrt{W}} \right)^2 + \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\Pi}{\sqrt{W}} \right)^2 + (T \text{Area}_{T^2}^2)(V_B + V_F) \right], \quad (27)\end{aligned}$$

where V_B and V_F are the bosonic and fermionic potentials of the Hamiltonian. The scale of the theory is then $T n \text{Area}_T^2$. The measure of integration reduces to the dimensionless $(1/2)d\hat{x}^r \wedge d\hat{x}^s \epsilon_{rs}$. The conjugate momenta have mass dimension $+1$, and the corresponding configuration variable mass dimension -1 . T has mass dimension $+3$. On the other hand, by considering the contribution to Yang–Mills arising from the first term in the above expansion of the star-product and by taking canonical dimensions for the conjugate pairs, we get for the coupling constant

$$g_{\text{YM}} = \frac{1}{T_{M2}^{1/2} n \text{Area}_{T^2}}. \quad (28)$$

It has dimension of mass^{1/2}. It represents the coupling constant of the first term in the star-product expansion. We assume that the compactification radii are $R_i \gg l_p$ but with the theory still defined at high energies. For a fixed tension and winding number n , the only relevant contribution in the star-product at low energies is the $U(M)$ commutator since the natural length is much larger than the effective radii $R_{\text{eff}} = n^{1/2} \sqrt{R_1 R_2}$. This is the decoupling limit of the theory since the Yang–Mills field strength becomes the coupling constant of the theory. The g_{YM} is very large in this phase and the theory is in the IR phase. It corresponds to have a description of M multiple MIM2-branes as point-like particles, representing M the number of supermembranes. As we raise the energy, the g_{YM} coupling constant gets weaker and for energies high enough, comparable with the natural scale of a MIM2-brane with an effective area of $(n \text{Area}_{T^2})$, the oscillation and vibrational modes containing the gauge but also gravity interactions between the supermembranes are

no longer negligible, so the full star-expansion has to be considered. All terms associated with the supermembrane symplectic structure of the star-bracket contribute, while the ordinary SYM contribution vanishes. The point-like particle picture is no longer valid, and it is substituted for that of an extended $(2 + 1)D$ object and the gauge and gravity contributions are strongly coupled. One can define formally effective physical coupling constant for the ordinary $F_{\mu\nu}$ field strength which would correspond to $\Lambda = Mg_{\text{YM}}$ with M representing the number of supermembranes and then one can try to obtain the 't Hooft coupling expansion in the large M . In this picture, however, one should take care of the limit. By keeping Λ fixed with M going to infinity, for a fixed tension and a fixed compactification radii, one has to consider the wrapping number n also going to infinity. But $n \text{Area}$ is the order parameter that would also go multiplied by M in the expansion, so one enters «faster» in the strong correlated limit where the rest of the terms of the star-product expansion cannot be neglected, moreover, from a physical point of view, $n \text{Area}_{T^2}$ is related to the size of the MIM2 as an extended object and it cannot be larger than the present energy bounds we have, otherwise, it would be in contradiction with our point-particle description at low scales. In order to perform a more accurate analysis, one should be working with the non-Abelian extension of the MIM2 for 4D noncompact, — it will be considered elsewhere — however, we believe that the qualitative arguments presented here should remain valid also in that case.

3. DISCUSSION AND CONCLUSIONS

We have obtained a $N = 8$ non-Abelian $U(M)$ formulation of the minimally immersed supermembrane for arbitrary number of colors M with all the symmetries of the supermembrane, in the LCG. This corresponds to the M-theory dual of the non-Abelian formulation of a *bundle* of Dirac–Born–Infeld representing a D2-branes-D0 bound state. It is the first time that a non-Abelian gauge theory can be directly obtained from a full-fledged sector of M-theory element, so far restricted to String Theory: Heterotics and Dp-branes in type II theories. This opens a new interesting window for models in phenomenology. At energies of the order of the compactification scale, the theory has the gauge and gravity sector strongly coupled. It describes all of the oscillations modes of the multiple parallel M2-branes minimally immersed. At low energies the theory enters in a decoupling regime and the physics is then described by a $\mathcal{N} = 8$ SYM theory of point-like particles in the IR phase. We then expect to describe correctly many aspects of phenomenology when realistic gauge groups will be considered. From the point of view of the target space, the theory has $N = 1$ SUSY in 9D flat dimensions. In [5] a $N = 1$ target space, $D = 4$ formulation of a single supermembrane minimally immersed together with a number of interesting phenomenological properties were found. Moreover, in [7] a formulation of the supermembrane minimally immersed in a G2 manifold was also obtained. Its quantum supersymmetric spectrum is also purely discrete. The analysis in 4D can be also extended to the non-Abelian case following the lines shown in this paper, allowing one to obtain models with reduced number of target and world-volume supersymmetries.

Acknowledgements. The work of M. P. G. M. has been partially supported by the grants MICINN-09/FPA2009-07122, FICYT-09/IB09-069 and the MEC-DGI Consolider CSD2007-00042. The work of A. R. is partially supported by PROSUL, under contract CNPq 490134/2006-08.

REFERENCES

1. *García del Moral M. P., Restuccia A.* A $N = 8$ Action for Multiple M2-Branes with an Arbitrary Number of Colors. hep-th/0903.5202.
2. *Boulton L., García del Moral M. P., Restuccia A.* Discreteness of the Spectrum of the Compactified $D = 11$ Supermembrane with Non-Trivial Winding // Nucl. Phys. B. 2003. V. 671. P. 343–358.
3. *Boulton L., Restuccia A.* The Heat Kernel of the Compactified $D = 11$ Supermembrane with Non-Trivial Winding // Nucl. Phys. B. 2005. V. 724. P. 380–396.
4. *Boulton L., García del Moral M. P., Restuccia A.* The Supermembrane with Central Charges: $(2 + 1)$ - D NCSYM, Confinement and Phase Transition // Nucl. Phys. B. 2008. V. 795. P. 27–51.
5. *García del Moral M. P., Pena J. M., Restuccia A.* $N = 1$ 4D Supermembrane from 11D // JHEP. 2008. V. 07. P. 039.
6. *García del Moral M. P., Martin I., Restuccia A.* Nonperturbative $SL(2, Z)(p, q)$ -Strings Manifestly Realized on the Quantum M2. hep-th/0802.0573.
7. *Belhaj A. et al.* The Supermembrane with Central Charges on a G2 Manifold // J. Phys. A. 2009. V. 42. P. 325201.
8. *Bellorin J., Restuccia A.* $D = 11$ Supermembrane Wrapped on Calibrated Submanifolds // Nucl. Phys. B. 2006. V. 737. P. 190–208.
9. *Duff M. J. et al.* Semiclassical Quantization of the Supermembrane // Nucl. Phys. B. 1988. V. 297. P. 515.
10. *Maldacena J. M.* The Large N Limit of Superconformal Field Theories and Supergravity // Adv. Theor. Math. Phys. 1998. V. 2. P. 231–252.
11. *Bagger J., Lambert N.* Modeling Multiple M2's // Phys. Rev. D. 2007. V. 75. P. 045020.
12. *Bagger J., Lambert N.* Gauge Symmetry and Supersymmetry of Multiple M2-Branes // Phys. Rev. D. 2008. V. 77. P. 065008.
13. *Gustavsson A.* Algebraic Structures on Parallel M2-Branes // Nucl. Phys. B. 2009. V. 811. P. 66–76.
14. *Aharony O. et al.* $N = 6$ Superconformal Chern–Simons–Matter Theories, M2-Branes and Their Gravity Duals // JHEP. 2008. V. 10. P. 091.
15. *Bagger J., Lambert N.* Three-Algebras and $N = 6$ Chern–Simons Gauge Theories // Phys. Rev. D. 2009. V. 79. P. 025002.
16. *Bergshoeff E., Sezgin E., Townsend P. K.* Supermembranes and Eleven-Dimensional Supergravity // Phys. Lett. B. 1987. V. 189. P. 75–78.
17. *de Wit B., Hoppe J., Nicolai H.* On the Quantum Mechanics of Supermembranes // Nucl. Phys. B. 1988. V. 305. P. 545.
18. *de Wit B., Marquard U., Nicolai H.* Area Preserving Diffeomorphisms and Supermembrane Lorentz Invariance // Commun. Math. Phys. 1990. V. 128. P. 39.
19. *de Wit B., Peeters K., Plefka J.* Supermembranes with Winding // Phys. Lett. B. 1997. V. 409. P. 117–123.
20. *Martin I., Restuccia A., Torrealba R. S.* On the Stability of Compactified $D = 11$ Supermembranes // Nucl. Phys. B. 1998. V. 521. P. 117–128.
21. *Martin I., Ovalle J., Restuccia A.* Compactified $D = 11$ Supermembranes and Symplectic Non-commutative Gauge Theories // Phys. Rev. D. 2001. V. 64. P. 046001.
22. *Fedosov B.* Deformation Quantization and Index Theory. Berlin: Akademie-Verlag, 1996. 325 p.
23. *Fedosov B.* A Simple Geometrical Construction of Deformation Quantization // J. Diff. Geom. 1994. V. 40. P. 213–238.

24. *Kontsevich M.* Deformation Quantization of Poisson Manifolds, I // *Lett. Math. Phys.* 2003. V. 66. P. 157–216.
25. *Seiberg N., Witten E.* String Theory and Noncommutative Geometry // *JHEP.* 1999. V. 09. P. 032.
26. *Martin I., Restuccia A.* Symplectic Connections, Noncommutative Yang–Mills Theory and Supermembranes // *Nucl. Phys. B.* 2002. V. 622. P. 240–256.
27. *Bergshoeff E., Sezgin E., Townsend P. K.* Properties of the Eleven-Dimensional Supermembrane Theory // *Ann. Phys.* 1988. V. 185. P. 330.
28. *de Wit B., Luscher M., Nicolai H.* The Supermembrane is Unstable // *Nucl. Phys. B.* 1989. V. 320. P. 135.