

NESTED BETHE ANSATZ FOR $\mathcal{Y}(gl(n))$ OPEN SPIN CHAINS WITH DIAGONAL BOUNDARY CONDITIONS

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In this proceeding we present the nested Bethe ansatz for open spin chains of XXX-type, with arbitrary representations (i.e., «spins») on each site of the chain and diagonal boundary matrices ($K^+(u)$, $K^-(u)$). The nested Bethe ansatz applies for a general $K^-(u)$, but a particular form of the $K^+(u)$ matrix. We give the eigenvalues, Bethe equations and the form of the Bethe vectors for the corresponding models. The Bethe vectors are expressed using a trace formula.

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INTRODUCTION

Recently we proposed a unified formulation for nested Bethe ansatz for closed and open spin chains with «quantum group» [3, 4, 10, 11], or «reflection algebra» [1, 2] related to $gl(n)$ and $gl(n|m)$ Lie algebras [8, 9]. In this proceeding we focus on open anisotropic spin chains, or of XXX-type, related to the Yangian and with the reflection algebra. More precisely, we give Bethe vectors, eigenvalues and Bethe equations for the «universal» transfer matrix, an operator over the tensor product of L highest weight representations of the Yangian. These representations are chosen in the set of irreducible finite dimensional representations. This approach generalizes the fundamental case studies in [6] and needs deeper analysis of the algebraic structure of the reflection algebra to be performed. The main points of this work are the explicit construction of the Bethe vectors as a trace formula, the construction of the Bethe vectors using embedding between different ranks of reflection algebras and the proof of the validity of the Bethe ansatz for arbitrary irreducible finite-dimensional representations (up to some constraint on the boundary). We give here a proof by increasing recursion contrary to the decreasing proof of [9].

The plan of the proceeding is the following. First we recall definitions and property of the Yangian \mathcal{Y}_n and the reflection algebra \mathcal{D}_n . Then we give the finite-dimensional representations of \mathcal{Y}_n and deduce the ones of \mathcal{D}_n . Next we recall the Bethe ansatz for $n = 2$. To perform the nested Bethe ansatz, we present embedding for the reflection algebra of different rank (valid up to some quotient) and the Bethe vectors in two forms, trace formula and recursion formula. Then we give the proof of the nested Bethe ansatz for open spin chains with $K^+(u) = \mathbb{I}$ (the other possibility is briefly discussed). To finish, we give some open problem from this result.

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1. RTT FORMALISM AND YANGIAN

Periodic anisotropic spin chains is closely related to the Yangian \mathcal{Y}_n . Among these realizations [11, 12], the so-called RTT (or FRT) [10] formalism is the more efficient to construct the conserved quantities of the model. These quantities belong to an Abelian subalgebra of the Yangian and are generated from transfer matrix. The explicit construction of local Hamiltonian relevant for physics applications from the transfer matrix is not easy to do in great generality, and we will focus only on its study.

Let us recall the definition of the Yangian in this RTT formalism. \mathcal{Y}_n is a unital associative infinite-dimensional algebra generated by

$$\{t_{ij}^{(p)}; \quad i, j = 1, \dots, n; \quad p \in \mathbb{N}/\{0\}\}. \quad (1)$$

We gather the \mathcal{Y}_n generators for same i, j into a formal series of u^{-1} and then put it in an $n \times n$ matrix acting in an auxiliary space $\mathcal{V} = \mathbb{C}^n$. We obtain the monodromy matrix:

$$T(u) = \sum_{i,j=1}^n E_{ij} \otimes t_{ij}(u) \in \text{End}(\mathcal{V}) \otimes \mathcal{Y}_n, \quad (2)$$

$$t_{ij}(u) = \delta_{ij} + \sum_{n=1}^{\infty} t_{ij}^{(n)} u^{-n}, \quad (3)$$

where E_{ij} are $n \times n$ matrices with 1 at the intersection of line i and column j and 0 otherwise. The commutation between elements of \mathcal{Y}_n are given by the RTT relations:

$$R_{12}(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u-v) \in \text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V}) \otimes \mathcal{Y}_n, \quad (4)$$

where indices 1, 2 label the auxiliary spaces where the operators act nontrivially. The matrix $R \in \text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V})$ is the rational solution of the Yang–Baxter equation:

$$R_{12}(u_1 - u_2) R_{13}(u_1 - u_3) R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3) R_{13}(u_1 - u_3) R_{12}(u_1 - u_2), \quad (5)$$

$$R_{12}(u) = u\mathbb{I} \otimes \mathbb{I} - \hbar P_{12}, \quad P_{12} = \sum_{i,j=1}^n E_{ij} \otimes E_{ji}, \quad (6)$$

written in auxiliary space $\text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V})$. This condition is equivalent to the associativity for the product of monodromy matrices. The R matrix satisfies unitarity relation,

$$R(u)R(-u) = (u - \hbar)(-u - \hbar) \mathbb{I} \otimes \mathbb{I}, \quad (7)$$

crossing unitarity,

$$R^t(u)R^t(-u + n\hbar) = u(-u + n\hbar) \mathbb{I} \otimes \mathbb{I}, \quad (8)$$

and is $GL(n, \mathbb{C})$ group invariant:

$$[R(u), M \otimes M] = 0, \quad M \in GL(n, \mathbb{C}). \quad (9)$$

The transfer matrix is defined as the trace over auxiliary space of the monodromy matrix $t(u) = \text{tr}(T(u))$ and commutes for different values of the formal variable u ,

$$[t(u), t(v)] = 0. \quad (10)$$

This is the main object to study for periodic anisotropic spin chains or, more generally, for two-dimensional quantum integrable models with Yangian symmetry [3–5, 8, 18].

The Yangian have the following automorphism:

— shift of the spectral parameter:

$$\sigma_a : T(u) \rightarrow T(u + a); \quad (11)$$

— product by scalar function:

$$f : T(u) \rightarrow f(u)T(u); \quad (12)$$

antimorphisms:

— matrix inversion:

$$\text{inv} : T(u) \rightarrow T^{-1}(u) = \sum_{i,j=1}^{\mathfrak{n}} E_{ij} \otimes t'_{ij}(u); \quad (13)$$

— spectral parameter inversion:

$$\text{inv} : T(u) \rightarrow T(-u), \quad (14)$$

and a Hopf algebra structure (Δ, S, ϵ) with the coproduct defined as

$$\Delta : \Delta(T(u)) = T(u) \otimes T(u) = \sum_{i,j,k=1}^{\mathfrak{n}} E_{ij} \otimes t_{ik}(u) \otimes t_{kj}(u). \quad (15)$$

More generally, one defines recursively for $L \geq 2$,

$$\Delta^{(L+1)} = (\text{id}^{\otimes(L-1)} \otimes \Delta) \circ \Delta^{(L)} : \mathcal{Y}_{\mathfrak{n}} \rightarrow \mathcal{Y}_{\mathfrak{n}}^{\otimes(L+1)}, \quad (16)$$

with $\Delta^{(2)} = \Delta$ and $\Delta^{(1)} = \text{id}$. The map $\Delta^{(L)}$ is an algebra homomorphism.

The Yangian has the universal enveloping algebra $\mathcal{U}(gl(\mathfrak{n}))$ as a Hopf subalgebra, the embedding is given by $\mathcal{E}_{ji} \rightarrow t_{ij}^{(1)}$. Where \mathcal{E}_{ij} are the generators of $\mathcal{U}(gl(\mathfrak{n}))$ with commutation relations:

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = \delta_{jk} \mathcal{E}_{ik} - \delta_{il} \mathcal{E}_{kj}. \quad (17)$$

The evaluation homomorphism $\text{ev} : \mathcal{Y}_{\mathfrak{n}} \rightarrow \mathcal{U}(gl(\mathfrak{n}))$ is given by

$$\begin{aligned} \text{ev} : t_{ij}^{(1)} &\rightarrow \mathcal{E}_{ji}, \\ \text{ev} : t_{ij}^{(p)} &\rightarrow 0, \quad p > 1. \end{aligned} \quad (18)$$

This evaluation homomorphism is the key ingredient to construct finite-dimensional representation of $\mathcal{Y}_{\mathfrak{n}}$ [12–14].

Let us introduce some notation used for R matrices in this paper:

— the «normalized» R matrices:

$$\mathbb{R}(u) = \frac{R(u)}{(u - \hbar)} \text{ with } \mathbb{R}(u)\mathbb{R}(-u) = \mathbb{I} \otimes \mathbb{I}, \quad (19)$$

— the «reduced» R matrices $R^{(k,p)}(u)$:

$$R^{(k,p)}(u) = \left(\mathbb{I}^{(k)} \otimes \mathbb{I}^{(p)} \right) R(u) \left(\mathbb{I}^{(k)} \otimes \mathbb{I}^{(p)} \right), \text{ with } \mathbb{I}^{(k)} = \sum_{i=k}^n E_{ii}, \tag{20}$$

$$R^{(k)}(u) = R^{(k,k)}(u).$$

We have $R^{(1)}(u) = R(u)$, and more generally $R^{(k)}(u)$ corresponds to the R matrix of \mathcal{Y}_{n-k} .

2. REFLECTION ALGEBRA AND $K(u)$ MATRICES

The \mathcal{Y}_n algebra is enough to construct a transfer matrix leading to periodic models, but in the context of open spin chains, one needs another algebra, the reflection algebra \mathfrak{D}_n [1], which turns out to be a subalgebra of \mathcal{Y}_n . Indeed, physically, one can interpret the RTT relation as encoding the interaction between the spins of the chain. Hence, it is the only relation needed to describe a periodic chain. On the other hand, in the case of open chain, the interaction with the boundaries has to be taken into account. Following the seminal paper [2], we construct the reflection algebra and the dual reflection equation for the boundary scalar matrices $K^-(u)$ and $K^+(u)$. We first define the matrix $K^-(u)$ to be the solution of the reflection equation in $\text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V})$:

$$R_{12}(u_1 - u_2)K_1^-(u_1)R_{12}(u_1 + u_2)K_2^-(u_2) = K_2^-(u_2)R_{12}(u_1 + u_2)K_1^-(u_1)R_{12}(u_1 - u_2). \tag{21}$$

The scalar solutions to the reflection equation have been classified using the $GL(n, \mathbb{C})$ invariance of R matrix [16]. The diagonal solutions take the form (up to normalization)

$$K^-(u) = \text{diag} \left(\underbrace{u - c_-, \dots, u - c_-}_a, \underbrace{-u - c_-, \dots, -u - c_-}_{n-a} \right) = \sum_{i=1}^n \kappa_i^-(u) E_{ii}, \tag{22}$$

where c_- is a free complex parameter and a is an integer. From this $K^-(u)$ matrix and the monodromy matrix $T(u)$ of \mathcal{Y}_n , we can construct the monodromy matrix of $\mathfrak{D}_n \subset \mathcal{Y}_n$:

$$D(u) = T(u) K^-(u) T^{-1}(-u) = \sum_{i,j=1}^n d_{ij}(u) \otimes E_{ij}, \tag{23}$$

$$d_{ij}(u) = \sum_{a=1}^n \kappa_a^-(u) t_{ia}(u) t'_{aj}(-u). \tag{24}$$

From (4) and (21), we can prove that $D(u)$ also satisfies the reflection equation in $\text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V}) \otimes \mathfrak{D}_n$:

$$R_{12}(u_1 - u_2)D_1(u_1)R_{21}(u_1 - u_2)D_2(u_2) = D_2(u_2)R_{12}(u_1 - u_2)D_1(u_1)R_{21}(u_1 - u_2). \tag{25}$$

The algebra \mathfrak{D}_n is a left coideal [17] of the algebra \mathcal{Y}_n with coproduct action:

$$\Delta(D_{[2]}(u)) = T_{[1]}(u)D_{[2]}(u)T_{[1]}^{-1}(-u) \in \text{End}(\mathcal{V}) \otimes \mathcal{Y}_n \otimes \mathfrak{D}_n, \tag{26}$$

where $[i]$ labels different spaces.

To construct commuting transfer matrices, we introduced a dual equation in $\text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V})$:

$$\begin{aligned} R_{12}(u_2 - u_1)(K_1^+(u_1))^{t_1} R_{12}(-u_1 - u_2 + \mathfrak{n}\hbar)(K_2^+(u_2))^{t_2} = \\ = (K_2^+(u_2))^{t_2} R_{12}(-u_1 - u_2 + \mathfrak{n}\hbar)(K_1^+(u_1))^{t_1} R_{12}(u_2 - u_1). \end{aligned} \quad (27)$$

From isomorphism of the reflection equation and dual reflection equation, we can construct solutions to the dual reflection equation from $K^-(u)$:

$$(K^+(u))^t = K^-\left(-u + \frac{\mathfrak{n}}{2}\hbar\right). \quad (28)$$

With $D(u)$ and $K^+(u)$ we construct the transfer matrix:

$$d(u) = \text{tr} (K^+(u)D(u)). \quad (29)$$

The reflection equation and its dual form ensure the commutation relation:

$$[d(u), d(v)] = 0. \quad (30)$$

Thus, $d(u)$ generates (via an expansion in u^{-1}) a set of commuting conserved quantities and is related to boundaries anisotropy spin chains models and, more generally, to boundaries quantum integrable models related to \mathfrak{D}_n .

3. HIGHEST WEIGHT REPRESENTATIONS

The fundamental point in using the ABA is to know a pseudo-vacuum for the model. In the mathematical framework it is equivalent to know a highest weight representation for the algebra which underlies the model. Since the generators of the algebra \mathfrak{D}_n can be constructed from the \mathcal{Y}_n ones, see Eq. (23), we first describe how to construct highest representations for the infinite-dimensional algebras \mathcal{Y}_n from highest weight representations of the finite-dimensional Lie algebras $gl(\mathfrak{n})$. Next, we show how these representations induce (for diagonal $K^-(u)$ matrix) a representation for \mathfrak{D}_n with same highest weight vector.

Definition 3.1. *A representation of \mathcal{Y}_n is called highest weight if there exists a nonzero vector Ω such that*

$$t_{ii}(u)\Omega = \lambda_i(u)\Omega \quad \text{and} \quad t_{ij}(u)\Omega = 0 \quad \text{for} \quad i > j, \quad (31)$$

for some scalars $\lambda_i(u) \in \mathbb{C}[[u^{-1}]]$. $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$ is called the highest weight and Ω the highest weight vector.

It is known that any finite-dimensional irreducible representation of \mathcal{Y}_n is highest weight and that it contains a unique (up to scalar multiples) highest weight vector [13, 14].

To construct such representations, one uses the evaluation morphism, which relates the infinite-dimensional algebra \mathcal{Y}_n to its finite-dimensional subalgebra $\mathcal{U}(gl(\mathfrak{n}))$ and a finite-dimensional irreducible highest weight representation $\pi_\mu : \mathcal{U}(gl(\mathfrak{n})) \rightarrow \text{End}(\mathcal{V}_\mu)$ with highest weight $\Omega \in \mathcal{V}_\mu$:

$$\pi_\mu(\mathcal{E}_{ij})\Omega = 0, \quad 1 \leq i < j \leq n, \quad \pi_\mu(\mathcal{E}_{ii})\Omega = \mu_i\Omega, \quad 1 \leq i \leq n, \quad \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+. \quad (32)$$

The evaluation representations of \mathcal{Y}_n are constructed by the following composition of maps:

$$\rho_a^\mu = \pi_\mu \circ \text{ev} \circ \sigma_a : \mathcal{Y}_n \xrightarrow{\sigma_a} \mathcal{Y}_n \xrightarrow{\text{ev}} \mathcal{U}(\mathfrak{gl}(n)) \xrightarrow{\pi_\mu} \text{End}(\mathcal{V}_\lambda). \tag{33}$$

The weight of this evaluation representation is given by $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$, with

$$\lambda_j(u) = u - a - \hbar \mu_j, \quad j = 1, \dots, n, \tag{34}$$

More generally, one constructs tensor products of evaluation representations using the coproduct of \mathcal{Y}_n ,

$$\left(\otimes_{i=1}^L \rho_{a_i}^{\mu^{(i)}} \right) \circ \Delta^{(L)}(T(u)) = \rho_{a_1}^{\mu^{(1)}}(T(u)) \dot{\otimes} \rho_{a_2}^{\mu^{(2)}}(T(u)) \dot{\otimes} \dots \dot{\otimes} \rho_{a_L}^{\mu^{(L)}}(T(u)), \tag{35}$$

where $\mu^{(i)} = (\mu_1^{(i)}, \dots, \mu_n^{(i)})$, $i = 1, \dots, L$, are the weights of the $\mathcal{U}(\mathfrak{gl}(n))$ representations. This provides an \mathcal{Y}_n representation with weight,

$$\lambda_j(u) = \prod_{i=1}^L \lambda_j^{(i)}(u), \quad j = 1, \dots, n, \tag{36}$$

where $\lambda_j^{(i)}(u)$ have the form (34). Evaluation representations are central in the study of representations because all finite-dimensional irreducible representations of \mathcal{Y}_n can be constructed from tensor products of evaluation representations (see [8] for references).

To obtain representation of $D(u)$, we also need to give $T^{-1}(u)$ in terms of the $T(u)$ elements. It could be done using the quantum determinant $q \det(T(u))$ and the comatrix $\widehat{T}(u)$, see [15].

The quantum determinant $q \det(T(u))$ which generates the center of \mathcal{Y}_n is defined as

$$q \det(T(u)) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n t_{i\sigma(i)}(u + (i - n)\hbar), \tag{37}$$

where S_n is the permutation group of n elements and σ a permutation with signature $\text{sign}(\sigma)$.

The quantum comatrix $\widehat{T}(u)$ satisfies

$$\widehat{T}(u) T(u - (n - 1)\hbar) = q \det(T(u)), \tag{38}$$

this equation allows one to relate $T^{-1}(u)$ to $\widehat{T}(u)$:

$$T^{-1}(u) = \frac{\widehat{T}(u + (n - 1)\hbar)}{q \det(T(u + (n - 1)\hbar))}. \tag{39}$$

From the exact form of $\widehat{T}(u)$ in terms of $t_{ij}(u)$ we can find that Ω is also a highest weight vector for $T^{-1}(u)$ with weights:

$$t'_{ii}(u)\Omega = \lambda'_i(u)\Omega, \quad \lambda'_i(u) = \left(\prod_{k=1}^{i-1} \frac{\lambda_k(u + k\hbar)}{\lambda_k(u + (k - 1)\hbar)} \right) \frac{1}{\lambda_i(u + (i - 1)\hbar)}. \tag{40}$$

The representations of the reflection algebra \mathfrak{D}_n could be studied from previous results [17]. For $K^-(u)$ diagonal, the finite-dimensional irreducible highest weight representations follow from the ones of \mathcal{Y}_n and lead to the following theorem:

Theorem 3.1. *If Ω is a highest weight vector of $\mathcal{Y}_{\mathfrak{n}}$, with eigenvalue $(\lambda_1(u), \dots, \lambda_{\mathfrak{n}}(u))$, then, when $K(z)$ is a diagonal matrix with $\kappa_i(u)$ diagonal elements, Ω is also a highest weight vector for $\mathfrak{D}_{\mathfrak{n}}$,*

$$d_{ij}(u)\Omega = 0 \quad \text{for } i > j, \text{ and } d_{ii}(u)\Omega = \Lambda_i(u, \{\kappa^-(u), \lambda(u), \lambda'(-u)\})\Omega, \quad (41)$$

with eigenvalues:

$$\Lambda_i(u) = \mathcal{K}_i(u) \lambda_i(u) \lambda'_i(-u) - \sum_{k=1}^{i-1} \frac{\hbar}{2u - \frac{(k-1)\hbar}{2}} \mathcal{K}_k(u) \lambda_k(u) \lambda'_k(-u), \quad (42)$$

$$\mathcal{K}_i(u) = \kappa_i(u) + \sum_{k=1}^{i-1} \kappa_k(u) \frac{\hbar}{2u - \frac{(i-2)\hbar}{2}}. \quad (43)$$

Now we can introduce what we call «general transfer matrix» $\mathfrak{d}(u; L, \{a\}, \{\mu\})$:

$$\mathfrak{d}(u; L, \{a\}, \{\mu\}) = \left(\otimes_{i=1}^L \rho_{a_i}^{\mu^{(i)}} \right) \circ \Delta^{(L)}(d(u)). \quad (44)$$

In the next section we will give the proof of the nested Bethe ansatz for this «general transfer matrix». To simplify notation, we will use $d(u)$ for $\mathfrak{d}(u; L, \{a\}, \{\mu\})$ in the next sections.

4. ALGEBRAIC BETHE ANSATZ FOR $\mathfrak{D}_{\mathfrak{n}}$ WITH $\mathfrak{n} = 2$

In this section, we remind the framework of the Algebraic Bethe Ansatz (ABA) [3] introduced in order to compute transfer matrix eigenvalues and eigenvectors. The method follows the same steps as in the closed chain case, up to a preliminary step. We will only consider the case $K^+(u) = \mathbb{I}$ which is relevant for the nested Bethe ansatz. In the open case the transfer matrix has the form

$$d(u) = \text{tr} (D_a(u)) = d_{11}(u) + d_{22}(u). \quad (45)$$

We perform a change of basis and a shift,

$$d_{11} \left(u + \frac{\hbar}{2} \right) = \widehat{d}_{11}(u), \quad d_{12} \left(u + \frac{\hbar}{2} \right) = \widehat{d}_{12}(u), \quad d_{21} \left(u + \frac{\hbar}{2} \right) = \widehat{d}_{21}(u), \quad (46)$$

$$d_{22} \left(u + \frac{\hbar}{2} \right) = \widehat{d}_{22}(u) - \frac{\hbar}{2u} \widehat{d}_{11}(u). \quad (47)$$

This change of basis leads to symmetric exchange relations:

$$[\widehat{d}_{12}(u) \widehat{d}_{12}(v)] = 0, \quad (48)$$

$$\begin{aligned} \widehat{d}_{11}(u) \widehat{d}_{12}(v) &= \frac{(u-v+\hbar)(u+v+\hbar)}{(u-v)(u+v)} \widehat{d}_{12}(v) \widehat{d}_{11}(u) - \\ &\quad - \frac{\hbar(2v+\hbar)}{2v(u-v)} \widehat{d}_{12}(u) \widehat{d}_{11}(v) + \frac{\hbar}{u+v} \widehat{d}_{12}(u) \widehat{d}_{22}(v), \end{aligned} \quad (49)$$

$$\begin{aligned} \widehat{d}_{22}(u)\widehat{d}_{12}(v) &= \frac{(u-v-\hbar)(u+v-\hbar)}{(u-v)(u+v)}\widehat{d}_{12}(v)\widehat{d}_{22}(u)+ \\ &+ \frac{\hbar(2u-\hbar)}{2u(u-v)}\widehat{d}_{12}(u)\widehat{d}_{22}(v) - \frac{\hbar(2u-\hbar)(2v+\hbar)}{4uv(u+v)}\widehat{d}_{12}(u)\widehat{d}_{11}(v). \end{aligned} \quad (50)$$

In the new basis, Ω is still a pseudo-vacuum:

$$\widehat{d}_{11}(u)\Omega = \widehat{\Lambda}_1(u)\Omega = \mathcal{K}_1\left(u + \frac{\hbar}{2}\right)\lambda_1\left(u + \frac{\hbar}{2}\right)\lambda'_1\left(-u - \frac{\hbar}{2}\right)\Omega, \quad \widehat{d}_{21}(u)\Omega = 0, \quad (51)$$

$$\begin{aligned} \widehat{d}_{22}(u)\Omega = \widehat{\Lambda}_2(u)\Omega &= \left(\Lambda_2\left(u + \frac{\hbar}{2}\right) + \frac{\hbar}{2u}\Lambda_1\left(u + \frac{\hbar}{2}\right)\right)\Omega = \\ &= \mathcal{K}_2\left(u + \frac{\hbar}{2}\right)\lambda_2\left(u + \frac{\hbar}{2}\right)\lambda'_2\left(-u - \frac{\hbar}{2}\right)\Omega, \end{aligned} \quad (52)$$

and we can use the algebraic Bethe ansatz as in the closed chain case. The transfer matrix rewrites:

$$d\left(u + \frac{\hbar}{2}\right) = \frac{2u-\hbar}{2u}\widehat{d}_{11}(u) + \widehat{d}_{22}(u) \equiv \widehat{d}(u). \quad (53)$$

Applying M creation operators $\widehat{d}_{12}(u_j)$ on the pseudo-vacuum, we generate a Bethe vector:

$$\Phi(\{u\}) = \widehat{d}_{12}(u_1)\cdots\widehat{d}_{12}(u_M), \quad \Omega. \quad (54)$$

where $\{u\} = \{u_1, \dots, u_M\}$. Demanding $\Phi(\{u\})$ to be an eigenvector of $\widehat{d}(u)$ leads to a set of algebraic relations on the parameters $\{u\}$, the so-called Bethe equations:

$$\begin{aligned} \frac{\mathcal{K}_1(u_k + \hbar/2)\lambda_1(u_k + \hbar/2)\lambda'_1(-u_k - \hbar/2)}{\mathcal{K}_2(u_k + \hbar/2)\lambda_2(u_k + \hbar/2)\lambda'_2(-u_k - \hbar/2)} &= \\ &= \frac{2u_k}{2u_k + \hbar} \prod_{i \neq k}^l \frac{(u_k - u_i - \hbar)(u_k + u_i - \hbar)}{(u_k - u_i + \hbar)(u_k + u_i + \hbar)}. \end{aligned} \quad (55)$$

Then, the eigenvalues of the transfer matrix read

$$d(u)\Phi(\{u\}) = \Lambda(u)\Phi(\{u\}), \quad (56)$$

$$\begin{aligned} \Lambda(u) &= \frac{2u-2\hbar}{2u-\hbar}\mathcal{K}_1(u)\lambda_1(u)\lambda'_1(-u) \prod_{k=1}^M \frac{(u-u_k+\hbar/2)(u+u_k+\hbar/2)}{(u-u_k-\hbar/2)(u+u_k-\hbar/2)} + \\ &+ \mathcal{K}_2(u)\lambda_2(u)\lambda'_2(-u) \prod_{k=1}^M \frac{(u-u_k-3\hbar/2)(u+u_k-3\hbar/2)}{(u-u_k-\hbar/2)(u+u_k-\hbar/2)}. \end{aligned} \quad (57)$$

Note that Bethe equations correspond to the vanishing of the residue of $\widehat{\Lambda}(u; \{u\})$ at $u = u_j$.

5. NESTED BETHE ANSATZ

In this section we will give the step for a direct recursion for the Bethe equations and eigenvalues of a «general open spin chain» of rank $n + 1$. This proof uses the knowledge of the recursion formula for the Bethe vectors and the embedding $\mathfrak{D}_n \rightarrow \mathfrak{D}_{n+1}$. More precisely, one has to consider the quotient of \mathfrak{D}_{n+1} by \mathcal{I} , the left ideal generated by $\{d_{ij}(u), i > j\}$. First we give the theorem about this embedding and next we prove the nested Bethe ansatz for \mathfrak{D}_{n+1} from the \mathfrak{D}_n one. This formulation gives an alternative proof of the one given in [9] for a more general case.

5.1. Embeddings of \mathfrak{D}_n Algebras. The algebraic cornerstone for the nested Bethe ansatz is a recursion relation on the \mathfrak{D}_n algebraic structure:

$$\mathfrak{D}_n \rightarrow \mathfrak{D}_{n-1} \rightarrow \cdots \rightarrow \mathfrak{D}_3 \rightarrow \mathfrak{D}_2. \quad (58)$$

In this section we give two important properties of the algebra \mathfrak{D}_n , described in the following theorem:

Theorem 5.1. *For $k = 1, 2, \dots, n-1$, let $F^{(k)}$ be a linear combination of $d_{i_1 j_1}(u_1) \cdots d_{i_l j_l}(u_l)$ with all indices $k-1 < i_p \leq j_p$, and let \mathcal{I} be the left ideal generated by $d_{ij}(u)$ for $i > j$. Then, we have the following properties:*

$$d_{ij}(u) F^{(k)} \equiv 0 \pmod{\mathcal{I}} \text{ for } i > j \text{ and } j < k, \quad (59)$$

$$[d_{ii}(u), F^{(k)}] \equiv 0 \pmod{\mathcal{I}} \text{ for } i < k. \quad (60)$$

We introduce the generators:

$$\widehat{D}^{(k)}(u) = \sum_{i,j=k}^n E_{ij} \otimes d_{ij}^{(k)}(u), \quad (61)$$

$$d_{ij}^{(k)}(u) = d_{ij} \left(u + \frac{(k-1)\hbar}{2} \right) + \delta_{ij} \sum_{a=1}^{k-1} \frac{\hbar}{2u} d_{aa} \left(u + \frac{(k-1)\hbar}{2} \right). \quad (62)$$

They satisfy in $\mathfrak{D}_n/\mathcal{I}$ the reflection equation for $\mathfrak{D}_{m-k+1|n}$:

$$\begin{aligned} R_{12}^{(k)}(u_1 - u_2) \widehat{D}_1^{(k)}(u_1) R_{12}^{(k)}(u_1 + u_2) \widehat{D}_2^{(k)}(u_2) &\equiv \\ &\equiv \widehat{D}_2^{(k)}(u_2) R_{12}^{(k)}(u_1 + u_2) \widehat{D}_1^{(k)}(u_1) R_{12}^{(k)}(u_1 - u_2). \end{aligned}$$

Let us give two useful relations from this theorem for the nested Bethe ansatz:

— the action of $d_{kk}^{(k)}(u)$ on Ω :

$$d_{kk}^{(k)}(u)\Omega = \mathcal{K}_i \left(u + \frac{(k-1)\hbar}{2} \right) \lambda_k \left(u + \frac{(k-1)\hbar}{2} \right) \lambda'_k \left(-u - \frac{(k-1)\hbar}{2} \right); \quad (63)$$

— the embedding $\tau : \mathfrak{D}_n/\mathcal{I}_n \rightarrow \mathfrak{D}_{n+1}/\mathcal{I}_{n+1}$ given by

$$\tau(d_{ij}(u)) = d_{i+1 j+1}^{(2)}(u) = d_{i+1 j+1} \left(u + \frac{\hbar}{2} \right) + \delta_{i,j} \frac{\hbar}{2u} d_{11} \left(u + \frac{\hbar}{2} \right); \quad (64)$$

it follows

$$\tau(d_{ij}^{(k)}(u)) = d_{i+1,j+1}^{(k+1)}(u). \tag{65}$$

This morphism will be crucial for the computation of the nested Bethe ansatz. We will use it in the form

$$\tau(D(u)) = \widehat{D}^{(2)}(u). \tag{66}$$

Choosing the form (23) for the operator D , we can compute the action of the coproduct of \mathcal{Y}_n on $\widehat{D}^{(k)}(u)$.

Theorem 5.2. *In the coset $\mathcal{Y}_n/\mathcal{J} \otimes \mathfrak{D}_n/\mathcal{I}$, where \mathcal{J} is the left ideal generated by*

$$\{t_{ij}(u), t'_{ij}(-u), i > j\},$$

the coproduct takes the form

$$\Delta(\widehat{D}_{[2]}^{(k)}(u)) \equiv T_{[1]}^{(k)}(u)\widehat{D}_{[2]}^{(k)}(u)(T_{[1]}^{-1})^{(k)}(-u) \text{ mod } \mathcal{J}, \tag{67}$$

$$T^{(k)}(u) = \sum_{i,j=k}^n E_{ij} \otimes t_{ij}^{(k)}(u) \text{ and } (T^{-1})^{(k)}(-u) = \sum_{i,j=k}^n E_{ij} \otimes t'_{ij}{}^{(k)}(-u), \tag{68}$$

$$t_{ij}^{(k)}(u) = t_{ij} \left(u + \frac{(k-1)\hbar}{2} \right) \text{ and } t'_{ij}{}^{(k)}(-u) = t'_{ij} \left(-u - \frac{(k-1)\hbar}{2} \right), \tag{69}$$

where [1] labels the space $\mathfrak{D}_n/\mathcal{I}$, [2] labels the space $\mathcal{Y}_n/\mathcal{J}$, and Δ is the coproduct of \mathcal{Y}_n .

From this result and using the fundamental representation $\bar{\pi}_a$ of \mathcal{Y}_i , we can obtain the convenient relation for $i < k$ (see [9]):

$$\begin{aligned} \bar{\pi}_a^{(i)}(T^{(k)}(u)) &= \mathbb{R}^{(i,k)} \left(u + a + \frac{(k-i)\hbar}{2} \right), \\ \bar{\pi}_a^{(i)}((T^{-1})^{(k)}(-u)) &= \mathbb{R}^{(i,k)} \left(u - a + \frac{(k-i)\hbar}{2} \right). \end{aligned} \tag{70}$$

These formulas will be used to prove the recursion of the nested Bethe ansatz.

5.2. Bethe Vectors. We present here a generalization to open spin chains of the recursion and trace formulas for Bethe vectors, obtained in [18, 19] for closed spin chains.

Let us introduce the following trace formula for the Bethe vectors (or weight function) of \mathfrak{D}_n universal «diagonal» open spin chains. We introduce a family of Bethe parameters u_{kj} , $j = 1, \dots, M_k$, the number M_k of these parameters being a free integer. The partial unions of these families will be noted as

$$\{u_\ell\} = \bigcup_{i=1}^{\ell} \{u_{ij}, j = 1, \dots, M_i\}, \tag{71}$$

so that the whole family of Bethe parameters is $\{u\} = \{u_{n-1}\}$ with cardinal $M = \sum_{k=1}^{n-1} M_k$.

Theorem 5.3. We denote by $A_1; \dots; A_{n-1}$ the ordered sequence of auxiliary spaces $a_1^1, \dots, a_{M_1}^1; a_1^2, \dots, a_{M_2}^2; \dots; a_1^{n-1}, \dots, a_{M_{n-1}}^{n-1}$. Then

$$\Phi_M^n(\{u\})\Omega = \text{tr}_{A_1 \dots A_{n-1}} \left(\prod_{i=1}^{n-1} \widehat{\mathbb{D}}_{A_i}^{(i)}(\{u_i\}) E_{n,n-1}^{\otimes M_{n-1}} \otimes \dots \otimes E_{21}^{\otimes M_1} \right) \Omega, \quad (72)$$

where

$$\widehat{\mathbb{D}}_{A_i}^{(i)}(\{u_i\}) = \prod_{j=1}^{M_i} \overleftarrow{\mathcal{R}}_{A_{<i}, a_j^i}(\{u_{i-1}\}, u_{ij}) \widehat{D}_{a_j^i}^{(i)} \left(u_{ij} + \frac{\hbar}{2} \right) \mathcal{R}_{a_j^i, A_{<i}}^{(i)}(\{u_{i-1}\}, u_{ij}), \quad (73)$$

$$\overleftarrow{\mathcal{R}}_{A_{<i}, a_j^i}(\{u_{i-1}\}, u_{ij}) = \overleftarrow{\prod}_{b<i} \overleftarrow{\prod}_{c=1 \dots M_b} \mathbb{R}_{a_j^i a_c^b}^{(i,b+1)} \left(u_{ij} + u_{bc} + \frac{(i-b+1)\hbar}{2} \right), \quad (74)$$

$$\mathcal{R}_{a_j^i, A_{<i}}^{(i)}(\{u_{i-1}\}, u_{ij}) = \overleftarrow{\prod}_{b<i} \overleftarrow{\prod}_{c=1 \dots M_b} \mathbb{R}_{a_j^i a_c^b}^{(i,b+1)} \left(u_{ij} - u_{bc} + \frac{(i-b+1)\hbar}{2} \right), \quad (75)$$

$$\overleftarrow{\prod}_{i=1, \dots, n} X_i = X_1 \dots X_n, \quad \overleftarrow{\prod}_{i=1, \dots, n} X_i = X_n \dots X_1. \quad (76)$$

This formula is invariant under the same permutation of elements of A_i and $\{u_{i1}, \dots, u_{iM_i}\}$.

The proof of the last assertion does not clearly appear in [9] and will be published elsewhere.

From the trace formula, we can extract a recurrent form for the Bethe vectors,

$$\Phi_M^n(\{u\})\Omega = \widehat{B}_{a_1^1}^{(1)}(u_{11}) \dots \widehat{B}_{a_{M_1}^1}^{(1)}(u_{1M_1}) \widehat{\Psi}_{\{u_1\}}^{(1)}(\Phi_{M-M_1}^{n-1}(\{u\}/\{u_1\}))\Omega, \quad (77)$$

$$\widehat{\Psi}_{\{u_1\}}^{(1)} = v^{(2)} \circ (\overleftarrow{\pi}_{u_{11}}^{(2)} \otimes \dots \otimes \overleftarrow{\pi}_{u_{1M_1}}^{(2)} \otimes \mathbb{I}) \circ \Delta^{(M_1)} \circ \tau, \quad (78)$$

$$\widehat{B}^{(1)}(u) = \sum_{j=1}^n e_j^t \otimes d_{1j}^{(1)}(u), \quad (79)$$

where $\overleftarrow{\pi}_a^{(2)}$ is the fundamental representation evaluation homomorphism normalized as in (70), $v^{(2)}$ is the application of the highest weight vector e_2 for the space A_1 :

$$v^{(2)}(X) = X(e_2)^{\otimes M_{k-1}}. \quad (80)$$

The proof is given in [9].

5.3. Eigenvalues and Bethe Equations. We state the following commutation relation between $d(u)$ and $\Phi_M^n(\{u\})$ for $K^+(u) = \mathbb{I}$:

$$d(u)\Phi_M^n(\{u\}) = \text{UWT} + \Phi_M^n(\{u\}) \left(\sum_{k=1}^n \frac{2u - n\hbar}{2u - k\hbar} \prod_{i=1}^{M_k} f \left(u - \frac{k\hbar}{2}, u_{kj} \right) \times \right. \\ \left. \times \prod_{i=1}^{M_{k-1}} \tilde{f} \left(u - \frac{(k-1)\hbar}{2}, u_{k-1j} \right) d_{kk}^{(k)} \left(u - \frac{(k-1)\hbar}{2} \right) \right),$$

$$f(u, v) = \frac{(u - v + \hbar)(u + v + \hbar)}{(u - v)(u + v)}, \quad \tilde{f}(u, v) = \frac{(u - v - \hbar)(u + v - \hbar)}{(u - v)(u + v)} \quad (81)$$

with the convention $M_0 = M_n = 0$. The UWT contains terms with u in the vector. We will prove the following theorem:

Theorem 5.4. For $K^+(u) = \mathbb{I}$ we have

$$d(u)\Phi_M^n(\{u\})\Omega = \Lambda(u)\Phi_M^n(\{u\})\Omega. \quad (82)$$

If the following set of Bethe equations is satisfied:

$$\begin{aligned} \frac{\mathcal{K}_k(u_{kj} + k\hbar/2)\lambda_k(u_{kj} + k\hbar/2)\lambda'_k(-u_{kj} - k\hbar/2)}{\mathcal{K}_{k+1}(u_{kj} + k\hbar/2)\lambda_{k+1}(u_{kj} + k\hbar/2)\lambda'_{k+1}(-u_{kj} - k\hbar/2)} &= \frac{2u_{kj}}{2u_{kj} + \hbar} \times \\ \times \prod_{i=1}^{M_{k-1}} \frac{(u_{kj} - u_{k-1,i} - \hbar/2)(u_{kj} + u_{k-1,i} - \hbar/2)}{(u_{kj} - u_{k-1,i} + \hbar/2)(u_{kj} + u_{k-1,i} + \hbar/2)} &\prod_{i \neq j}^{M_k} \frac{(u_{kj} - u_{ki} - \hbar)(u_{kj} + u_{ki} - \hbar)}{(u_{kj} - u_{ki} + \hbar)(u_{kj} + u_{ki} + \hbar)} \times \\ \times \prod_{i=1}^{M_{k+1}} \frac{(u_{kj} - u_{k+1,i} - \hbar/2)(u_{kj} + u_{k+1,i} - \hbar/2)}{(u_{kj} - u_{k+1,i} + \hbar/2)(u_{kj} + u_{k+1,i} + \hbar/2)}, & \quad (83) \end{aligned}$$

$$j = 1, \dots, M_k, \quad k = 1, \dots, n-1,$$

then the eigenvalues of the transfer matrix have the form

$$\begin{aligned} \Lambda(u) &= \sum_{k=1}^n \frac{2u - n\hbar}{2u - k\hbar} \mathcal{K}_k(u)\lambda_k(u)\lambda'_k(-u) \prod_{j=1}^{M_k} \frac{(u - u_{kj} - k\hbar/2 + \hbar)(u + u_{kj} - k\hbar/2 + \hbar)}{(u - u_{kj} - k\hbar/2)(u + u_{kj} - k\hbar/2)} \times \\ &\times \prod_{j=1}^{M_{k-1}} \frac{\left(u - u_{k-1j} - \frac{(k-1)\hbar}{2} - \hbar\right) \left(u + u_{k-1j} - \frac{(k-1)\hbar}{2} - \hbar\right)}{\left(u - u_{k-1j} - \frac{(k-1)\hbar}{2}\right) \left(u + u_{k-1j} - \frac{(k-1)\hbar}{2}\right)}. \quad (84) \end{aligned}$$

Proof: For $n = 2$ we find the result of section 4. We will prove the case $n + 1$, assuming the case n is true. We decompose the transfer matrix:

$$d(u) = d_{11}(u) + d^{(2)}(u), \quad d^{(2)}(u) = \text{tr} (D^{(2)}(u)). \quad (85)$$

We make a transformation of the operator and a shift of the spectral parameter to have symmetric commutation relations:

$$\begin{aligned} d_{11} \left(u + \frac{\hbar}{2}\right) &= \widehat{d}_{11}(u), \quad B_a^{(1)} \left(u + \frac{\hbar}{2}\right) = \widehat{B}_a^{(1)}(u), \\ D_a^{(2)} \left(u + \frac{\hbar}{2}\right) &= \widehat{D}_a^{(2)}(u) - \frac{\hbar}{2u} \mathbb{I}_a^{(2)} \otimes \widehat{d}_{11}(u). \end{aligned} \quad (86)$$

From this transformation we get a new form for the transfer matrix:

$$d \left(u + \frac{\hbar}{2}\right) = \frac{2u - n\hbar}{2u} \widehat{d}_{11}(u) + \text{tr}_a \left(\widehat{D}_a^{(2)}(u)\right). \quad (87)$$

The commutation relations between $\widehat{d}_{11}(u)$, $\widehat{D}^{(2)}(u)$ and $\widehat{B}^{(1)}(u)$ are obtained from the reflection equation (25):

$$\widehat{B}_a^{(1)}(u)\widehat{B}_b^{(1)}(v) = \widehat{B}_b^{(1)}(v)\widehat{B}_a^{(1)}(u)\mathbb{R}_{ab}^{(2)}(u-v), \quad (88)$$

$$\begin{aligned} \widehat{d}_{11}(u)\widehat{B}_b^{(1)}(v) &= \frac{(u-v+\hbar)(u+v+\hbar)}{(u-v)(u+v)}\widehat{B}_b(v)\widehat{d}_{11}(u) - \frac{\hbar(2v+\hbar)}{(u-v)2v}\widehat{B}_b^{(1)}(u)\widehat{d}_{11}(v) + \\ &\quad + \frac{\hbar}{u+v}\widehat{B}_b^{(1)}(v)\widehat{D}_b^{(2)}(v), \end{aligned} \quad (89)$$

$$\begin{aligned} \widehat{D}_a^{(2)}(u)\widehat{B}_b^{(1)}(v) &= \frac{(u-v-\hbar)(u+v-\hbar)}{(u-v)(u+v)}\widehat{B}_b^{(1)}(v)\mathbb{R}_{ab}^{(k+1)}(u+v)\widehat{D}_a^{(2)}(u)\mathbb{R}_{ab}^{(2)}(u-v) - \\ &\quad - \frac{\hbar(2v+\hbar)}{4uv(u+v)}\widehat{B}_b^{(1)}(u)R_{ab}^{(2)}(2u)P_{ab}^{(2)}\widehat{d}_{kk}(v) + \\ &\quad + \frac{\hbar}{(u-v)2u}\widehat{B}_b^{(k)}(u)R_{ab}^{(2)}(2u)\widehat{D}_a^{(2)}(v)P_{ba}^{(2)}. \end{aligned} \quad (90)$$

From these commutation relations and using the fact that the Bethe vector is globally invariant, if we permute \widehat{B} we obtain two types of terms: the wanted and unwanted. Let us consider first the wanted terms. For $\widehat{d}_{11}(u)$ we have

$$\prod_{i=1}^{M_1} f(u, u_{1i})\Phi_M^{n+1}(\{u\})\widehat{d}_{11}(u)\Omega, \quad (91)$$

where we have used theorem 5.1 to put $\widehat{d}_{11}(u)$ in the right. For $\widehat{D}^{(2)}(u)$ we have

$$\prod_{i=1}^{M_1} \widetilde{f}(u, u_{1i})\widehat{B}_{a_1^1}^{(1)}(u_{11})\cdots\widehat{B}_{a_{M_1}^1}^{(1)}(u_{1M_1})\widehat{\Psi}_{\{u_1\}}^{(1)}(d(u)\Phi_{M-M_1}^n(\{u\}/\{u_1\})), \quad (92)$$

where $d(u)$ is the transfer matrix for \mathfrak{D}_n . We have used the definition of $\widehat{\Psi}_{\{u_1\}}^{(1)}$ and the relations (66), (70) to find

$$\prod_{i=1\dots M_1}^{\rightarrow} \mathbb{R}_{aa_i^1}^{(2)}(u+u_{1i})\widehat{D}_a^{(2)}(u) \prod_{i=1\dots M_1}^{\leftarrow} \mathbb{R}_{aa_i^1}^{(2)}(u-u_{1i})\widehat{\Psi}_{\{u_1\}}^{(1)}(X) = \widehat{\Psi}_{\{u_1\}}^{(1)}(D_a(u)X). \quad (93)$$

Using (81), we can commute $d^{(2)}(u)$. It remains to compute the action of $\widehat{\Psi}_{\{u_1\}}^{(1)}$. The formulas (65), (70) and the fact that $\widehat{\Psi}_{\{u_1\}}^{(1)}$ is a morphism (up to $\nu^{(2)}$) allow one to find ($1 < k \leq n$)

$$\begin{aligned} \widehat{\Psi}_{\{u_1\}}^{(1)}(d_{11}^{(1)}(u)) &= d_{22}^{(2)}(u), \\ \widehat{\Psi}_{\{u_1\}}^{(1)}\left(d_{kk}^{(k)}\left(u - \frac{(k-1)\hbar}{2}\right)\right) &= d_{k+1k+1}^{(k+1)}\left(u - \frac{(k-1)\hbar}{2}\right)\prod_{i=1}^{M_1} \widetilde{f}^{-1}(u, u_{1i}). \end{aligned} \quad (94)$$

Using these formulas and making a reverse shift, the theorem is proved for the wanted term.

Let us now consider the unwanted terms. First we give the reason why it is not possible to deal with a general diagonal $K^+(u)$ matrix. To factorize the unwanted term for $\widehat{d}_{11}(u)$, we must have $\text{tr}((K_a^+)^{(2)}(u)R_{ab}^{(2)}(2u)P_{ab}^{(2)}) \propto \mathbb{I}^{(2)}$. The only possibility is $(K_a^+)^{(2)}(u) \propto \mathbb{I}^{(2)}$. Here we will just prove the case $K^+(u) = \mathbb{I}$ and let the reader consult [9] for the other case: $K^+(u) = k(u)E_{11} + \mathbb{I}^{(2)}$.

Using the commutation relations and looking for the term with $\widehat{d}_{11}(u_{11})$ — the other terms are similar using the invariance by permutation of the Bethe vector, see theorem 5.3 — we find

$$-\frac{(2u - n\hbar)(2u_{11} + \hbar)}{2u_{11}(u^2 - u_{11}^2)} \prod_{i=2}^{M_1} f(u_{11}, u_{1i}) \Phi_M^{n+1}(\{u\}, u_{11} \rightarrow u) \widehat{d}_{11}(u_{11}).$$

Looking now for the term $\widehat{D}^{(2)}(u_{11})$, after using the trick $\text{tr}(R_{ab}^{(2)}(2u)P_{ab}^{(2)}) = (2u - n\hbar)\mathbb{I}^{(2)}$ to obtain a good form for commuting with $\widehat{\Psi}_{\{u_1\}}^{(1)}$, we find

$$\frac{(2u - n\hbar)(2u_{11} - \hbar)\hbar}{(2u_{11} - n\hbar)(u^2 - u_{11}^2)} \prod_{i=2}^{M_1} \widetilde{f}(u_{11}, u_{1i}) \widehat{B}_{a_1^1}^{(1)}(u) \cdots \widehat{B}_{a_{M_1}^1}^{(1)}(u_{1M_1}) \widehat{\Psi}_{\{u_1\}}^{(1)}(d(u_{11})\Phi_{M-M_1}^n(\{u\}/\{u_1\})).$$

From (94) we see that only the first term of the eigenvalue is nonzero. We can obtain the Bethe equation for u_{11} :

$$\begin{aligned} \frac{\mathcal{K}_1(u_{11} + \hbar/2)\lambda_1(u_{11} + \hbar/2)\lambda_1'(u_{11} + \hbar/2)}{\mathcal{K}_2(u_{11} + \hbar/2)\lambda_2(u_{11} + \hbar/2)\lambda_2'(u_{11} + \hbar/2)} &= \\ &= \frac{2u_{11}}{(2u_{11} + \hbar)} \prod_{i=2}^{M_1} \frac{\widetilde{f}(u_{11}, u_{1i})}{f(u_{11}, u_{1i})} \prod_{i=1}^{M_2} f^{-1}(u_{11} - \hbar/2, u_{2i}). \end{aligned}$$

Using the invariance by permutation, the Bethe equations for the other u_{ij} follow. We must also modify the other Bethe equations. The only change comes from the relations (94) which change the eigenvalues of the $d_{kk}^{(k)}\left(u - \frac{(k-1)\hbar}{2}\right)$. This modification only affects the first family of Bethe equations, adding a term to the right product. This ends the recursion and proves the theorem. \blacksquare

CONCLUSION

In this proceeding we give the nested Bethe ansatz for open spin chains of XXX-type with diagonal boundary conditions. This result could be extended to the case of nondiagonal boundary conditions but with some constraints between $K^+(u)$ and $K^-(u)$. To do this, we use the $GL(n, \mathbb{C})$ invariance of the Yangian [6, 16] and take for an arbitrary invertible M :

$$\widetilde{K}^+(u) = MK^+(u)M^{-1}, \quad \widetilde{K}^-(u) = MK^-(u)M^{-1}. \quad (95)$$

It is equivalent to the assertion that $K^+(u)$ and $K^-(u)$ are diagonalizable in the same basis, otherwise the nested Bethe ansatz does not work and the diagonalization of the transfer matrix remains an open problem.

We also give a *trace formula* for the Bethe vector of the open chain. This formulation could be a starting point for the investigation of the quantized Knizhnik–Zamolodchikov equation following the work [19]. For such a purpose, the coproduct properties of Bethe vectors for open spin chains remain to be studied. Defining a scalar product and computing the norm of these Bethe vectors is also a point of fundamental interest.

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