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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

COMMON STRUCTURES OF QUANTUM FIELD THEORIES AND LATTICE SYSTEMS THROUGH BOUNDARY SYMMETRY

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The sine-Gordon model and affine Toda field theories on the half-line, on the one hand, and the XXZ spin chain with nondiagonal boundary term and interacting many-body lattice systems with a flow, on the other, have a common characteristic. They possess nonlocal conserved boundary charges, generating the Askey–Wilson algebra, a coideal subalgebra of the bulk quantized affine symmetry. We argue that the boundary Askey–Wilson symmetry is the deep algebraic property allowing for integrability of the physical system in consideration.

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A rich family of quantum field theories, known as quantum affine Toda models, possess quantum affine symmetry, solitons, and integrable boundaries. In the presence of general boundaries, the quantum symmetry, and the integrability of the model as well, are broken. However, with suitably chosen boundary conditions, a remnant of the bulk symmetry may survive, and the system possesses hidden boundary symmetries, which determine a K matrix, a solution to the boundary Yang–Baxter equation and allow for the exact solvability. Such nonlocal boundary symmetry charges were originally obtained for the sine-Gordon model [1] and generalized to affine Toda field theories [2], and derived from spin chain point of view as commuting with the transfer matrix for a special choice of the boundary conditions [3] or analogously as the one-boundary Temperley–Lieb algebra centralizer in the «nondiagonal» spin $1/2$ representation [4]. The derivation of the nonlocal charges used the algebraic technique based on the quantum affine symmetry in the bulk and the known boundary reflection K matrices. They were obtained as coideals of the bulk quantum symmetry and interpreted as generating a new symmetry.

In a recent paper [5], we have defined the Askey–Wilson (AW) algebra as a coideal subalgebra of the quantum affine $U_q(\hat{sl}(2))$. We have constructed a K matrix in terms of the Askey–Wilson algebra generators, which satisfies a boundary Yang–Baxter equation (known as a reflection equation). As an example of an Askey–Wilson boundary symmetry, we have considered a model of nonequilibrium physics, the open asymmetric simple exclusion process with most general boundary conditions. This model is exactly solvable in the stationary state within the matrix product ansatz to stochastic dynamics, and it can be shown that the boundary operators generate the Askey–Wilson algebra. The model is known [6] to be equivalent to the integrable spin $1/2$ XXZ chain with general boundary terms (in the proper parameterization), whose bulk Hamiltonian (infinite chain) possesses the quantum affine symmetry $U_q(\hat{su}(2))$. The results of Baseilhac and Koizumi [7] for the XXZ spin chain boundary Askey–Wilson algebraic relations follow from the asymmetric exclusion process boundary algebra for the particular values of the structure constants.

In our opinion, it is quite remarkable that affine Toda field theory (a special case of which is the sine-Gordon model), the XXZ spin chain, and the interacting lattice many-body system with a flow have the common characteristic of possessing a quantum affine symmetry $U_q(\hat{sl}(2))$ (or $U_q(\hat{su}(2))$ in the bulk and boundary nonlocal charges generating an Askey–Wilson algebra, a coideal subalgebra of the quantum group bulk symmetry. The existence of an operator-valued reflection matrix, expressed in terms of the AW algebra generators and satisfying a boundary Yang–Baxter equation, is the deep algebraic property behind these models allowing for integrability. This puts forward a connection to a Bethe Ansatz (BA) solution for the spectrum of the relevant physical quantities.

There is a natural homomorphism to the Askey–Wilson algebra of a tridiagonal algebra (TD), known as deformed Dolan–Grady relations or deformed Onsager algebra. Recently Baseilhac and Koizumi [8] have explicitly constructed the deformed analogue of the Onsager algebra for the XXZ spin chain. Using the representation theory of the q-Onsager algebra they diagonalize the transfer matrix of the spin $1/2$ XXZ chain of L sites, with general integrable boundary conditions, and generic anisotropy parameter q, with $|q| = 1$. They argue to have obtained the complete exact spectrum from the roots of the characteristic polynomial of dimension 2^L.

In this paper, we consider a different spectral problem exact solution for a system with boundary symmetry based on the Askey–Wilson algebra. The importance of the AW algebra related spectral problem is motivated by the identification of one of the generators with the second order difference operator for the AW polynomials in the basic representation. The difference equation for the AW polynomials [9] becomes equivalent to the diagonalization problem for a general quadratic form in the quantum group generators (commonly interpreted as the Hamiltonian of a proper physical system). We present a diagonalization of the ASEP transition rate matrix (Hamiltonian) by Bethe Ansatz procedure for the second order difference operator for the Askey–Wilson polynomials and obtain the complete spectrum for the lattice system with boundary AW symmetry, namely the asymmetric simple exclusion process (the XXZ spin chain). The algebraic scheme can be applied to any system with the boundary Askey–Wilson algebra. In our opinion, however, due to the ultimate relation of the ASEP to the AW polynomials, already manifest in the exact solution at the stationary state, the proposed solution is the most appropriate for this nonequilibrium system.

We would like also to note that the implementation of the second order difference operator for the AW algebra related spectral problem was briefly mentioned in [10]. The careful reader will notice that it was done assuming the constraint $abcd = q$ for the parameters of the AW polynomials. This constraint is the defining condition of a finite-dimensional representation of the Askey–Wilson (tridiagonal) algebra and is *unacceptable* for the ASEP. The nontrivial point in the diagonalization we propose is the construction of a finite-dimensional representation of the tridiagonal algebra without imposing conditions that restrict the physics of the nonequilibrium system. We comment also on the relation to the solution, obtained in [11] for even number of lattice sites.

We will use the definition of the Askey–Wilson algebra as a coideal subalgebra of the quantized affine algebra $U_q(\hat{sl}(N))$, $N \geqslant 3$. (The case $N = 2$ was considered in [5].) Let $u_i, , v_i, , k_i, i = 0, \ldots, N-1$ be some scalars. The AW algebra is defined by the homomorphism to the level zero $U_q(\hat{sl}(N))$ [12] (with Chevalley basis $E_i^{\pm}, H_i$)

$$A_i = u_i E_i^{+} q^{-H_i/2} + v_i E_i^{-} q^{-H_i/2} + k_i q^{-H_i}, \quad i = 0, \ldots, N-1 \tag{1}$$

so that the generators $A_i, i = 0, \ldots, N-1$ satisfy

$$[A_i, A_j] = 0, \quad |i-j| \geqslant 2, \tag{2}$$

$$\begin{aligned} [[A_i, A_j]_q, A_i]_q &= -\rho_i A_j - \omega_j^i A_i - \eta_i^j, \\ [A_j, [A_i, A_j]_q]_q &= -\rho_j A_i - \omega_j^i A_j - \eta_j^i, \end{aligned} \tag{3}$$

where $[A_i, A_j]_q = q^{1/2} A_i A_j - q^{-1/2} A_j A_i$, $0 \leqslant i,\ j \leqslant N$, is the q commutator and the structure constants are representation-dependent

$$-\rho_i = u_i v_i (q - q^{-1})^2, \tag{4}$$

$$\omega_j^i = (q^{1/2} - q^{-1/2})^2 k_i k_j q^{-\mu(i) - \mu(j)}, \tag{5}$$

$$\eta_i^j = (q - q^{-1})^2 u_i v_i k_j q^{-2\mu(i) - \mu(j)}, \quad \eta_j^i = (q - q^{-1})^2 u_j v_j k_i q^{-2\mu(j) - \mu(i)} \tag{6}$$

with either $i = 0, 1, \ldots, N-2$ and $j = i+1$ or $i = 0, j = N-1$. The $U_q(\hat{sl}(N))$ representation module V is of type 1 [13], i.e., $V = \oplus_\mu V_\mu$ with weight space $V_\mu = (\nu \in V | q^{H_i} \nu = q^{\mu(i)} \nu)$, a joint eigenspace of the commuting operators q^{H_i}, $i = 0, \ldots, N-1$. The explicit form of the structure constants for the AW algebra as a coideal of $U_q(\hat{sl}(2))$ is given by the formulae (31)–(34) in [5], where it has been found that the structure constants depend on the quadratic Casimir element [14], l_V^0 (for details, see [5]).

The tridiagonal algebra, generated by the elements (1), is obtained by taking the commutator, respectively with A_i and A_j, in the first and second lines of (3) and is defined by the relations

$$\begin{aligned} [A_i, [A_i [A_i, A_j]_q]_{q^{-1}}] &= \rho_i [A_i, A_j], \\ [A_j, [A_j, [A_j, A_i]_q]_{q^{-1}}] &= \rho_j [A_j, A_i], \end{aligned} \tag{7}$$

together with (2) and ρ_i given by (4).

From the explicit realization of the operators A_i, it follows that they generate a linear covariance algebra for the $U_q(\hat{sl}(N))$, which has the property of a coideal subalgebra. From the comultiplication of $U_q(\hat{sl}(N))$ one has

$$\Delta(A_i) = I \otimes A_i + (A_i - k_i I) \otimes q^{-H_i}, \quad i = 0, \ldots, N-1. \tag{8}$$

1. TWO-DIMENSIONAL FIELD THEORY MODELS

The sine-Gordon model is a free bosonic conformal field theory, with the action on the whole line (with $\phi(-\infty, t) = 0$), with the perturbing operator $\Phi^{\rm pert}(x,t) = {\rm e}^{i\hat{\beta}\phi(x,t)} + {\rm e}^{-i\hat{\beta}\phi(x,t)}$, where $\hat{\beta}$ is the Toda coupling constant. The $U_q(\hat{sl}_2)$ symmetry of the sine-Gordon model with deformation parameter $q = \exp\left(\frac{2i\pi(1-\bar{\beta}^2)}{\bar{\beta}^2}\right)$ is generated by the charges $Q_\pm = \frac{1}{4\pi}\int_{-\infty}^{\infty}(J_\pm - H_\pm), \bar{Q}_\pm = \frac{1}{4\pi}\int_{-\infty}^{\infty}(\bar{J}_\pm - \bar{H}_\pm)$ together with the topological charge $T = \frac{\hat{\beta}}{2\pi}\int_{-\infty}^{\infty} dx\, \partial_x\phi$. The explicit expressions of the currents $J_\pm, \bar{J}_\pm, H_\pm \bar{H}_\pm$ are given by formulae (3.5), (3.6), and (3.7) in [2]. The charges are related to the conventional basis in $U_q(\hat{s}l_2)$ $Q_\pm = E^\pm q^H, \bar{Q}_\pm = E^\pm q^{-H}, T = H$. The Neumann boundary conditions $\partial_x\phi = 0$ at $x = 0$ restrict the sine-Gordon model to the half-line $x \geqslant 0$, but the model was found to be classically integrable with rather more general boundary conditions

$$\partial_x\phi = i\beta\lambda_b\left(\epsilon_-\, \mathrm{e}^{i\hat{\beta}\phi(0,t)} - \epsilon_+\, \mathrm{e}^{-i\hat{\beta}\phi(0,t)}\right), \tag{9}$$

which is a perturbation to the Neumann boundary conditions $S_\epsilon = S_{\text{Neumann}} + \frac{\lambda}{2\pi}\times \int dt \Phi^{\text{pert}}_{\text{bound}}(t)$, with the boundary perturbing operator $\Phi^{\text{pert}}_{\text{bound}}(t) = \epsilon_-\, \mathrm{e}^{i\hat{\beta}\phi(0,t)} + \epsilon_+\, \mathrm{e}^{-i\hat{\beta}\phi(0,t)}$. It has been shown in [2] that with these boundary conditions, the nonlocal charges, $\hat{Q}_\mp = Q_\mp + \bar{Q}_\pm + \hat{\epsilon}_\mp q^{\mp T}$, where $\hat{\epsilon}_\pm = \frac{\lambda_b \epsilon_\pm}{2\pi}\frac{(1-\bar{\beta}^2)}{\bar{\beta}^2}$, are conserved and generate a coideal subalgebra of $U_q(\hat{s}l_2)$. It is now straightforward to show that the algebra of the charges $\hat{Q}_\pm$ is the AW algebra with two generators

$$A = Q_+ + \bar{Q}_- + \hat{\epsilon}_+ q^T, \quad A^* = Q_- + \bar{Q}_+ + \hat{\epsilon}_- q^{-T} \tag{10}$$

with structure constants corresponding to the value (of the $U_q(\hat{s}l_2)$) Casimir $l^0_V = q^{1/2} + q^{-1/2}$

$$\rho = \rho^* = -(q - q^{-1})^2, \tag{11}$$

$$\omega = (q^{1/2} - q^{-1/2})^2\left(\hat{\epsilon}_+\hat{\epsilon}_- + (q^{1/2} + q^{-1/2})^2\right), \tag{12}$$

$$\eta = \eta^* = (q - q^{-1})^2(\hat{\epsilon}_+ + \hat{\epsilon}_-). \tag{13}$$

A Hamiltonian, describing a sine-Gordon model on the half-line coupled to a nonlinear oscillator at the boundary (i.e., dynamical boundary), was proposed in [15] and has been shown to be integrable at the classical level. The model was then studied at the quantum level [16], and nonlocal charges $\hat{\mathcal{E}_\pm}$ corresponding to the dynamical case were constructed, which were natural extensions to the known nondynamical ones [2]. These nonlocal charges have been shown to be coideals of $U_q(\hat{s}l_2)$ and to obey the algebraic relations

$$\begin{aligned}(q + q^{-1})\hat{\mathcal{E}}_+\hat{\mathcal{E}}_-\hat{\mathcal{E}}_+ - \hat{\mathcal{E}}_+^2\hat{\mathcal{E}}_- - \hat{\mathcal{E}}_-\hat{\mathcal{E}}_+^2 &= -c^2(q^{1/2} + q^{-1/2})^2\hat{\mathcal{E}}_-,\\ (q + q^{-1})\hat{\mathcal{E}}_-\hat{\mathcal{E}}_+\hat{\mathcal{E}}_- - \hat{\mathcal{E}}_-^2\hat{\mathcal{E}}_+ - \hat{\mathcal{E}}_+\hat{\mathcal{E}}_-^2 &= -c^2(q^{1/2} + q^{-1/2})^2\hat{\mathcal{E}}_+,\end{aligned} \tag{14}$$

where $c^2 = i2\mu(q-1)/\lambda^2$, $\lambda = 2/\hat{\beta}^2 - 1$, μ is the boundary perturbation parameter and $q \equiv \exp(-2\pi i/\hat{\beta}^2)$. These relations define the AW boundary algebra of the dynamical model with structure constants

$$\rho = \rho^* = -c^2(q^{1/2} + q^{-1/2})^2, \tag{15}$$

$$\omega = 0, \qquad \eta = \eta^* = 0. \tag{16}$$

The more general class of models is the affine Toda field theory associated to every affine Lie algebra of rank N and defined by the Euclidean action for an N-component boson field in two dimensions

$$S = \frac{1}{4\pi}\int d^2z\,\partial\phi\,\bar{\partial}\phi + \frac{\lambda}{2\pi}\int d^2z \sum_{j=0}^{N-1} \exp\left(-i\hat{\beta}\frac{1}{|\alpha_j^2|}\alpha_j\cdot\phi\right), \tag{17}$$

where the exponential interaction potential is expressed by the simple roots $\alpha_j, j = 0, \ldots,$ $N-1$, λ is the mass parameter, and $\hat{\beta}$ is the coupling constant. The quantum symmetry $U_q(\hat{sl}(N))$ is generated by the topological charges $T_j = \frac{\hat{\beta}}{2\pi}\int\limits_{-\infty}^{\infty} dx\,\alpha_j\partial_x\phi$ and the nonlocal conserved charges $Q_j = \frac{1}{4\pi}\int\limits_{-\infty}^{\infty}(J_j - H_j)$, $\bar{Q}_j = \frac{1}{4\pi}\int\limits_{-\infty}^{\infty}(\bar{J}_j - \bar{H}_j)$, where $j = 0, \ldots, N-1$, $Q_i \equiv E_i^+ q^{H_i/2}$, $\bar{Q}_i \equiv E_i^- q^{H_i/2}$, $T_i \equiv H_i$. The explicit expressions for $J_j, \bar{J}_j, H_j, \bar{H}_j$ are given by the formulae (4.4), (4.5), and (4.6) in [2]. The linear combinations $Q_j + \bar{Q}_j$ are parity invariant and conserved on the half-line with Neumann boundary conditions. Adding to the action a boundary perturbation $S_\epsilon = S_{\text{Neumann}} + \frac{\lambda}{2\pi}\int dt\,\Phi_{\text{bound}}^{\text{pert}}(t)$, where $\Phi_{\text{bound}}^{\text{pert}}(t) = \sum\limits_{j=0}^{N-1}\epsilon_j \exp\left(-\frac{i\hat{\beta}}{2}\alpha_j\cdot\phi(0,t)\right)$, we obtain a more general boundary condition

$$\partial_x\phi = -i\hat{\beta}\lambda_b\sum_{j=0}^{N-1}\epsilon_j\alpha_j\exp\left(-\frac{i\hat{\beta}}{2}\alpha_j\cdot\phi(0,t)\right), \quad x = 0. \tag{18}$$

The new conserved charges are

$$\hat{Q}_i = Q_i + \bar{Q}_i + \hat{\epsilon}_i q^{T_i}, \quad \hat{\epsilon}_i = \frac{\lambda_b\epsilon_i}{2\pi c}\frac{(1-\hat{\beta}^2)}{\hat{\beta}^2}, \quad i = 0, \ldots, N-1. \tag{19}$$

The nonlocal charges of the affine Toda field theory generate the AW algebra with N generators and with structure constants given by (no summation over repeated indices)

$$\rho_i = \rho_{i+1} = -(q - q^{-1})^2, \tag{20}$$

$$\omega_{i+1} = (q^{1/2} - q^{-1/2})^2\hat{\epsilon}_i\hat{\epsilon}_{i+1}q^{-\mu(i)-\mu(i+1)}, \tag{21}$$

$$\eta_i^{i+1} = (q - q^{-1})^2\hat{\epsilon}_{i+1}q^{-2\mu(i)-\mu(i+1)}, \quad \eta_{i+1}^{i} = (q - q^{-1})^2\hat{\epsilon}_i q^{-2\mu(i+1)-\mu(i)}. \tag{22}$$

The nonlocal charges of the affine Toda field theory generate the tridiagonal algebra with structure constants ρ_i given by Eq. (20).

2. THE XXZ SPIN CHAIN AND THE OPEN ASYMMETRIC SIMPLE EXCLUSION PROCESS (ASEP)

The ASEP is an interacting many-body system with wide range of applications [17, 18]. It is described in terms of a probability distribution $P(s_i, t)$ of a stochastic variable s_i at a site $i = 1, 2, \ldots, L$ of a linear chain. On successive sites, particles hop with probability

$g_{01}dt$ to the left, and $g_{10}dt$ to the right. The event of hopping occurs if out of two adjacent sites one is a vacancy, $s_i = 0$, and the other, $s'_i = 1$, is occupied by a particle. The symmetric process is the lattice gas model of particles hopping between the nearest-neighbour sites with a constant rate g. The asymmetric simple exclusion process with hopping in a preferred direction is the driven diffusive lattice gas of particles moving under the action of an external field. The process is partially asymmetric if there is a different nonzero probability of both left and right hopping, $g_{01} = q$, $g_{10} = 1$, and totally asymmetric if all jumps occur in one direction only, $q = 0$. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density, and additional processes can take place at the boundaries. Namely, at the left boundary $i = 1$ a particle can be added with probability $\alpha\, dt$ and removed with probability $\gamma\, dt$, and at the right boundary $i = L$ it can be removed with probability $\beta\, dt$ and added with probability $\delta\, dt$. The time evolution of the system is governed by the master equation $dP(s,t)/dt = \sum_{s'} \Gamma(s,s')P(s',t)$ which is mapped to a Schrödinger equation in imaginary time for a quantum Hamiltonian with the nearest-neighbour interaction in the bulk and single-site boundary terms. A relation to the integrable spin $1/2$ XXZ quantum spin chain is obtained through the similarity transformation for the transition rate matrix $\Gamma = -qU_\mu^{-1}H_{XXZ}U_\mu$ (for details, see [6]); H_{XXZ} is the Hamiltonian of the $U_q(su(2))$ invariant quantum spin chain with anisotropy $\Delta = -1/2(q+q^{-1})$ and with the added nondiagonal boundary terms B_1 and B_L: $H_{XXZ} = -1/2\sum_{i=1}^{L-1}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \Delta\sigma_i^z\sigma_{i+1}^z + h(\sigma_{i+1}^z - \sigma_i^z) + \Delta) + B_1 + B_L$, where $h = 1/2(q-q^{-1})$. The explicit relation of the boundary terms to the transition rates of the ASEP is given in [11].

The steady state properties of the ASEP are studied within the Matrix Product State Ansatz (MPA). The idea is [19] that the stationary probability of a given configuration $(s_1, s_2, \ldots, s_L)$ can be exactly obtained as the expectation value $P(s) = \dfrac{\langle w|D_{s_1}D_{s_2}\ldots D_{s_L}|v\rangle}{Z_L}$, in terms of matrices $D_{s_i} = D_1$ if $s_i = 1$, and $D_{s_i} = D_0$ if $s_i = 0$, satisfying the quadratic (bulk) algebra $D_1D_0 - qD_0D_1 = x_1D_0 - D_1x_0$, with boundary conditions of the form and $x_0 + x_1 = 0$

$$(\beta D_1 - \delta D_0)|v\rangle = x_0|v\rangle, \quad \langle w|(\alpha D_0 - \gamma D_1) = \langle w|x_0. \tag{23}$$

The exact solution in the stationary state was related to Askey–Wilson polynomials [20]. Emphasizing the equivalence of the open ASEP to the $U_q(\hat{s}u(2))$ XXZ invariant quantum spin chain with added general boundary terms, we have shown [21] that the boundary operators generate the AW algebra with the structure constants

$$\rho = x_0^2\beta\delta q^{-1}(q^{1/2}+q^{-1/2})^2, \quad \rho^* = x_0^2\alpha\gamma q^{-1}(q^{1/2}+q^{-1/2})^2, \tag{24}$$

$$-\omega = x_0^2(\beta-\delta)(\gamma-\alpha) - x_0^2(\beta\gamma+\alpha\delta)(q^{1/2}-q^{-1/2})Q, \tag{25}$$

$$\begin{aligned} \eta &= q^{1/2}(q^{1/2}+q^{-1/2})x_0^3\left(\beta\delta(\gamma-\alpha)Q + \frac{(\beta-\delta)(\beta\gamma+\alpha\delta)}{q^{1/2}-q^{-1/2}}\right), \\ \eta^* &= q^{1/2}(q^{1/2}+q^{-1/2})x_0^3\left(\alpha\gamma(\beta-\delta)Q + \frac{(\alpha-\gamma)(\alpha\delta+\beta\gamma)}{q^{1/2}-q^{-1/2}}\right), \end{aligned} \tag{26}$$

where Q is the central element of the finite-dimensional $U_q(su(2))$ representation. The left boundary operator and right boundary operator, being shifted AW algebra generators

$D^L = A^* + \alpha - \gamma$ and $D^R = A + \beta - \delta$, have a diagonal and a tridiagonal (infinite-dimensional) representations, respectively, with basis the AW polynomials p_n, $n = 0, 1, \ldots$ The parameters a, b, c, d of the AW polynomials are uniquely related to the four boundary rates, namely $a = k_+(\alpha, \gamma), b = k_+(\beta, \delta), c = k_-(\alpha, \gamma), d = k_-(\beta, \delta)$, where

$$k_\pm(u, v) = \frac{-u + v + (1-q) \pm \sqrt{(u - v - (1-q))^2 + 4uv}}{2u}. \tag{27}$$

The corresponding AW algebra for the XXZ spin chain, proposed and studied in [10], is a particular case of the ASEP boundary AW algebra. It is generated by the operators

$$A = \frac{1}{c_0} Q_+ + \bar{Q}_-, \quad A^* = Q_- + \frac{1}{c_0} \bar{Q}_+, \tag{28}$$

where c_0 is an arbitrary parameter and structure constants are given by

$$\rho = \rho^* = \frac{(q^{1/2} + q^{-1/2})^2}{c_0}, \tag{29}$$

$$\omega = -\frac{\omega^{(j)}}{c_0(q - q^{-1})}, \qquad \eta = \eta^* = 0, \tag{30}$$

where $\omega^{(j)} = (q^{j+1/2} + q^{-j-1/2})$ is the value of the $U_q(sl_2)$ Casimir in the spin j representation.

The tridiagonal algebra as a coideal subalgebra of the $U_q(\hat{s}l(2))$ for the XXZ chain with general boundary terms was considered in [22] with $\rho = \rho^* = k_+ k_-$, where k_+, k_- belong to the boundary parameters at the left end of the chain.

We point out the different form of the structure constants in the boundary AW algebras for the ASEP and XXZ spin chain. Despite the equivalence of the ASEP to the XXZ spin chain, through a similarity transformation, they describe different physics. A relation among the structure constants of the type $\rho = \rho^*$ is unacceptable for a model of nonequilibrium physics, as the ASEP is, because it will restrict the physics of the system. On the other hand, we have proved in [5] that the solution of the AW operator-valued K matrix to the boundary Yang–Baxter equation requires $\rho \equiv \rho^*$. The AW operator-valued K matrix is essential for the transfer matrix formalism. Within the framework of the latter, the exact spectrum of the XXZ spin chain was derived in [8] in the parameterization of six (two real and two complex) boundary parameters and q — a phase. It may seem likely that using the property of the ASEP AW (TD) algebras with $\rho \neq \rho^*$ one can rescale the generators to obtain equal structure constants and then try to implement the construction of [8], taking care of the proper ASEP parameterization. The AW algebra is very important for the exact solution of the ASEP in the stationary state, where the form of the functions of the model parameters defining the phase diagram were determined for the first time from the boundary symmetry representation properties. In view of this, we develop an exact spectral problem solution based on the boundary algebra and using the zeros of the AW polynomials to provide an independent treatment of the stochastic dynamics.

The tridiagonal algebra approach provides a unifying scheme for the exact description of the various versions of the asymmetric simple exclusion process. The AW algebra of the open ASEP with incoming and outgoing particles at both boundaries, whose structure constants are given by (25)–(27), is the general algebra from which the boundary algebras of the particular processes follow as limit cases. This is due to its irreducible modules, namely the Askey–Wilson polynomials, that are the most extensive generalizations known of orthogonal polynomials. Hereafter, we denote the boundary algebras of the ASEP i) with general boundary conditions — $AW(\alpha, \gamma, \beta, \delta; q)$; ii) with only incoming particles at site 1 and only outgoing particles at site L — $AW(\alpha, \beta, q)$; iii) $AW(PBC; q)$ — with periodic boundary conditions. The boundary algebra $AW(\alpha, \beta, q)$ with generators A, A^* is represented in the space with basis the Al-Salam–Chihara polynomials depending on two parameters a, b. The structure constants of the algebra have the form

$$\rho = 0, \qquad \rho^* = abq^{-1}(q - q^{-1})^2, \qquad \omega = \eta = \eta^* = 0 \tag{31}$$

with

$$a = \kappa_+(\alpha, 0), \qquad b = \kappa_+(\beta, 0). \tag{32}$$

The unique relation to the model parameters follows as a special case of (27) for $\gamma = \delta = 0$. The exact stationary solution of the process was obtained in [23] without any relevance to the boundary algebra. From the point of view of the present algebraic treatment, the Jacobi matrix $C = D_0 + D_1$ solving the eigenvalue equation for the Al-Salam–Chihara polynomials in [23] coincides upto a shift with the operator A in the tridiagonal representation. The other particular case is the ASEP on a chain of L sites with periodic boundary conditions. This is given by the algebra $AW(\bar{\rho}, q;)$ in a space with basis the big q-Hermite polynomials for the representation of the generators A, A^*. In the periodic case, the number of particles N is conserved and the density parameter in the bulk is $\bar{\rho} = \langle N \rangle / L$. The relation of the parameter a of the big q-Hermite polynomials $H_n(x; a)$ to the density parameter $a = \dfrac{-\bar{\rho} - (1-q)}{\bar{\rho}}$ is the limit case of the corresponding relation for the AW polynomials with $b = c = d = 0$. The structure constants of the boundary algebra $AW(PBC; q)$ have the form

$$\rho = 0, \qquad \rho^* = aq^{-1}(q - q^{-1})^2, \qquad \omega = \eta = \eta^* = 0. \tag{33}$$

In all three cases the exact solution in the stationary case is due to the calculation of the relevant physical quantities exploiting the orthogonality relation of the corresponding polynomials with respect to a positive measure. These are polynomials orthogonal (on the unit circle) and with a resolution of unity with respect to the corresponding positive measure (see [24] for details). The calculation of the relevant quantities has similar structure. To find, e.g., the normalization factor to the stationary probability distribution, one considers the orthogonality condition as the contour integral and performs the asymptotic analysis. In this way, the expressions for the normalization factor Z_L were obtained in [20] for the general case of ASEP boundary conditions and in [23] for only incoming particles at the left boundary and only outgoing at the right one. We present here the exact stationary solution of the asymmetric process on a chain of L sites with periodic boundary conditions $i = L + i$. In this case, the normalization factor to the stationary probability distribution is given in the form $\mathrm{Tr}\,(D_0 + D_1) \equiv \mathrm{Tr}\,A$, where the trace has to be taken in the auxiliary Hilbert space with respect to the vector $|1\rangle_0$, i.e., $\mathrm{Tr}\,A = \langle 1|A|1\rangle$. Using the resolution of unity in the space with the big q-Hermite polynomials as the basis we have

$$\mathrm{Tr}\,(D_0+D_1)^L = \\ = \oint\limits_C \frac{dz}{4\pi i z} w\left(\frac{z+z^{-1}}{2}\right)\langle 1|(H\left(\frac{z+z^{-1}}{2}\right);a|q0\rangle\langle H\left(\frac{z+z^{-1}}{2}\right);a|q)(D_0+D_1)^L|1\rangle, \quad (34)$$

where the measure w is according to Eq. (3.18.2), Sec. 3.18 in [24]. There is no essential difference in the technique already applied in [20, 23], and the result for the case of periodic boundary conditions is

$$Z_L \simeq (1/a^2;q)_\infty \frac{(a+a^{-1}+2)^L}{(1-q)^L}. \quad (35)$$

For the current $J \simeq (1-q)\dfrac{a}{(1+a)^2}$ one has

$$J = (1-q)\bar{\rho}(\bar{\rho}-1) \quad (36)$$

for $0 < \bar{\rho} < 1/2$ and $1 > \bar{\rho} > 1/2$ with $J_{\max} = (1-q)/4$ at $\bar{\rho} = 1/2$. The uniform distribution in the case of a partial ASEP on a ring is known. Apart from the existence of a current in the steady state there are no correlations and no phase transitions. However the physics becomes less trivial for the time-dependent process. We will show that within the tridiagonal approach one obtains a solution to the ASEP to describe the dynamics of the afore-mentioned processes.

The dynamics of the ASEP is governed by the master equation with the transition rate matrix Γ whose non-negative off-diagonal elements are the probability rates. As was pointed out in [18], if one can diagonalize the matrix Γ, all probabilities at all times can be found due to the formal solution of the master equation $|P(s,t)\rangle = \exp\,(t\Gamma)|P(s,t=0)\rangle$. Γ is a stochastic intensity matrix with the property that the columns (rows) sum up to zero. Hence it can be related to a positive Markov matrix $1+\Gamma$ with the largest eigenvalue $\lambda_{\max} = 1$ due to probability conservation and all other eigenvalues (or their real parts) $|\lambda| \leqslant 1$ according to Perron–Frobenius theorem.

The importance of the AW algebra related spectral problem is motivated by the representation of the generator A^* as the second order difference operator for the AW polynomials, commonly interpreted as the Hamiltonian (H) of a proper physical system [9]. For the transition matrix of the ASEP we have $\Gamma = -H$.

We first summarize the most important formulae and notations about the representations of the AW algebra with two generators A, A^* which we will need in the following. Let $p_n = p_n(x;a,b,c,d)$ denote the nth Askey–Wilson polynomial [25] depending on four parameters a, b, c, d

$$p_n = {}_4\Phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ay, ay^{-1} \\ ab, ac, ad \end{matrix}\ |q;q\right) \quad (37)$$

with $p_0 = 1$, $x = y + y^{-1}$ and $0 < q < 1$. There is a basic representation of the AW algebra [26] in the space of symmetric Laurent polynomials $f[y] = f[y^{-1}]$ with a basis $(p_0, p_1, \ldots)$ as follows:

$$Af[y] = (y+y^{-1})f[y], \quad A^*f[y] = \mathcal{D}f[y], \quad (38)$$

where $\mathcal{D}$ is the second order q-difference operator [25] having the Askey–Wilson polynomials p_n as eigenfunctions. It is a linear transformation given by $\mathcal{D}f[y] = (1 + abcdq^{-1})f[y] + \varphi(y)(f[qy] - f[y]) + \varphi(y^{-1})(f[q^{-1}y] - f[y])$ with

$$\varphi(y) = \frac{(1-ay)(1-by)(1-cy)(1-dy)}{(1-y^2)(1-qy^2)} \tag{39}$$

and $\mathcal{D}(1) = 1 + abcdq^{-1}$. The eigenvalue equation for the joint eigenfunctions p_n reads

$$\mathcal{D}p_n = \lambda_n^* p_n, \quad \lambda_n^* = q^{-n} + abcdq^{n-1}, \tag{40}$$

and the operator A^* is represented by an infinite-dimensional matrix $\mathrm{diag}\,(\lambda_0^*, \lambda_1^*, \lambda_2^*, \ldots)$. The operator $Ap_n = xp_n$ is represented by a tridiagonal matrix, denoted $\mathcal{A}$, whose matrix elements enter the three-term recurrence relation for the Askey–Wilson polynomials

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \quad p_{-1} = 0. \tag{41}$$

We will only need the explicit form of the matrix elements b_n of A

$$b_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}. \tag{42}$$

In diagonalizing the transition matrix we have to keep in mind that Γ is not Hermitian and has different left and right eigenvectors. In [21], we proved that the quantum affine $U_q(\hat{su(2)})$ is the hidden symmetry of the ASEP in the bulk. Hence the boundary conditions define the boundary operators D^L and D^R as shifted coideal elements of the bulk quantum affine $U_q(\hat{su(2)})$. Accordingly, the left boundary vector is the left eigenvector of the shifted generator A^* of the AW algebra in the diagonal representation, and the right boundary vector is the right eigenvector of the shifted generator A of the AW algebra in the diagonal representation. The importance of the AW algebra for the steady state exact solution is the identification of the bulk matrix $D_0 + D_1$ with the generator in the tridiagonal representation.

Proposition I. The exact solution of the ASEP is achieved in the auxiliary space of symmetric Laurent polynomials, where the operator

$$D^L + D^R = (A^* + \alpha - \gamma) + (A + \beta - \delta) \tag{43}$$

is interpreted as the transition rate matrix of the process. It is assumed that A is in the tridiagonal representation of the AW algebra while A^* is in the diagonal representation (or equivalently A is diagonal and A^* is tridiagonal in the dual representation of the AW algebra).

The diagonal generator of the AW algebra is responsible for the eigenvalue zero in the stationary state, while the tridiagonal generator is relevant for the time-dependent description of the process. This is justified by the observation that in the basic representation the tridiagonal Jacobi matrix when properly rescaled by $\sqrt{-\gamma/\alpha}$ and shifted by $(\alpha-\gamma-(1-q))/\alpha$ has the property of an intensity stochastic matrix. We have

$$a_n + b_n + c_n = 0 \tag{44}$$

due to

$$a + a^{-1} + (\alpha - \gamma - (1-q)) = 0 \tag{45}$$

with solution

$$a_{\pm} = \frac{-(\alpha - \gamma - (1-q)) \pm \sqrt{(\alpha - \gamma - (1-q))^2 + 4\alpha\gamma}}{2\alpha}. \tag{46}$$

This is consistent with the unique relation of the four parameters of the AW polynomials to the ASEP model parameters obtained in the previous work as a consequence of the boundary vector definition in the stationary state, namely $a_+ = k_+(\alpha, \gamma)$. Thus the Askey–Wilson polynomials belong to the class of random walk polynomials [27] providing a basis for the stochastic matrix $1 + \Gamma$ with the property $a_n + b_n + c_n = 1$.

We consider now the eigenvalue equation (42) for the operator $\mathcal{D}$ for a polynomial of a given finite degree n (with $\varphi(y)$ from (41))

$$\varphi(y)(p_n(qy) - p_n(y)) + \varphi(y^{-1})(p_n(q^{-1}y) - p_n(y)) = (q^{-n} - 1)(1 - q^{n-1}abcd)p_n(y) \tag{47}$$

and use the procedure of algebraic Bethe Ansatz [9]. Expanding the function p_n as a product of its zeros

$$p_n(y) = \prod_{m=1}^{n} (y - y_m)(y - y_m^{-1}) \tag{48}$$

gives the Bethe-Ansatz equation for the zeros of the Askey–Wilson polynomials

$$\frac{(y_k - a)(y_k - b)(y_k - c)(y_k - d)}{(ay_k - 1)(by_k - 1)(cy_k - 1)(dy_k - 1)} = \prod_{l=1, l \neq k}^{L} \frac{(qy_k - y_l)(qy_k y_l - 1)}{(y_k - qy_l)(y_k y_l - q)}. \tag{49}$$

The second order AW difference operator is exactly solvable and these equations are valid for any $L < 2j + 1$, so that for any L there is exactly one polynomial (48). This means that for each $L(< 2j + 1)$ the Bethe equations have exactly one solution for the set $y_k, k = 1, \ldots, L$.

The diagonalization procedure can be summarized in the following steps:

1. We use the unique solution for the AW zeros Bethe-Ansatz equation to obtain a discrete set of AW polynomials in the space of Laurent polynomials of a given degree.

2. We impose a condition for the finite-dimensional representations of the TD algebra in the space with discrete set of AW polynomials as the basis.

3. We relate the finite-dimensional representation of the TD algebra with the irreducible 2^L-dimensional representation of the quantum affine symmetry in the bulk.

To terminate the three-term recurrence relation at any finite $(n+1) \equiv L$ for a discrete set of AW polynomials ($p[y] = p[y^{-1}]$), due to $p_L[y] = 0$, we have to set $b_n = 0$ in the matrix representing the operator A, without imposing restrictive conditions on the model parameters. Note that for the XXZ spin chain this can be done directly by the vanishing of any of the factors in the numerator of b_n, e.g., $1 - abcdq^{n-1} = 0$ (with the proper identification of the AW parameters a, b, c, d with the parameters of the XXZ boundary terms, this is the second factor of the XXZ BA condition [11]). For the ASEP one can use the parameter $x_0 \equiv \zeta$ in the stationary state to rescale $a \to \zeta a$, $b \to \zeta b$, $c \to \zeta c$, $d \to \zeta d$. With $|\zeta| \leqslant 1$, this has no effect on the Bethe equations and does not change the identification of the parameters a, b, c, d with the boundary probability rates. In the factor $(1 - abq^n)$ of the matrix element b_n we can redefine $\zeta^2 ab$ as a new parameter t treating it independent of a, b, and q, so that the factor $(1-\zeta^2 abq^n)$ in the numerator of b_n becomes $(1-tq^n)$. The condition to terminate the AW algebra ladder representation due to $b_n = 0$ becomes

$$tq^n = 1. \tag{50}$$

We thus obtain a discrete set of AW polynomials $p_n(x_k, a, b, c, d|t, q); n = 0, 1, \ldots, L-1$, such that $\sum_{k=0}^{L-1} w_k p_n(x_k) p_m(x_k) = 0$ for distinct n, m. Then we will have, if $x = x_k$,

$$xp_{L-1}(x) = a_{L-1}p_{L-1}(x) + c_{L-1}p_{L-2}(x), \tag{51}$$

which for general x will define a polynomial

$$p_L(x) = \mathrm{const} \prod_{k=0}^{L-1} (x - x_k). \tag{52}$$

For each x_k, the condition (50) with $n = L-1$ determines a finite-dimensional representation (of dimension L) of the AW (and the TD) algebra. The representing matrices for A, A^* in the tridiagonal, diagonal representation are finite $L \times L$ square matrices. The eigenvalues of the diagonal matrix have eigenspaces of dimension 1. We want to relate this representation to the highest weight irreducible representation of the $U_q(\hat{s}u(2))$ with deformation parameter q. (Note the change of the deformation parameter from $q^{1/2}$ to q.)

With each zero y_i from the unique solution to the Bethe equations we associate a two-dimensional irreducible representation $V_1(x_i; p_0, p_1)$ of $U_q(\hat{s}u(2))$. According to the theorem of Chari and Pressley [28], the tensor product representation of dimension 2^L $V_1(x_1) \otimes V_1(x_2) \otimes \ldots \otimes V_1(x_L)$ is irreducible and possesses the highest weight vector Ω generating an $L+1$ dimensional subrepresentation whose associated unique polynomial is such, that with $x_k \to x_k^{-1}$ it coincides with (42) for the choice $\mathrm{const} = (-1)^L \prod_{k=0}^{L-1} x_k^{-1}$. (To simplify notations in what follows we keep x_k to denote $x_k = (y_k + y_k^{-1})^{-1}$.)

From the action of A, $Ap_n(x_k) = x_k p_n(x_k), n = 0, 1$, it follows that the module V_1 is an eigenvector of A (but not of A^*)

$$AV_1(x_k) = x_k V_1(x_k). \tag{53}$$

On the tensor product of two irreducible modules $V_1(x_i) \otimes V_1(x_k)$ the operator A will act by means of the coproduct

$$\Delta(A) = A_{i_1} \otimes I + I \otimes A_{k_2} + A_{i_1} \otimes A_{k_2}. \tag{54}$$

Iterating the coproduct we obtain the action of the operator A on the tensor product. (We denote the n-fold iteration by $\Delta^{(1)} = \Delta, \Delta^{(n)} = (\Delta \otimes I^{(n-2)})\Delta^{(n-1)}$ with $I^{(n-2)} = I \otimes \ldots \otimes I(n-2$ times).) To make the formulae more transparent, we denote the first two terms in (54) by $\Delta_P(A)$. We have

$$\Delta_P^{(n)}(A) = \sum_{k=1}^{n} I^{(k-1)} \otimes A_{i_k} \otimes I^{(n-k)}. \tag{55}$$

The complete set of eigenvalues of A on the tensor product of n representations V_1 will be given by subsequent action of the operators

$$\Delta_P^{(n)}A, A\otimes\Delta_P^{(n-1)}A+\Delta_P^{(n-1)}A\otimes A, \dots, A^{(k)}\otimes\Delta_P^{(n-k)}A^{(k)}+\Delta_P^{(n-k)}A^{(k)}\otimes A^{(k)}, A^{(n)}, \quad (56)$$

where $k = 1, \dots, n-1$ and $A^{(l)} \equiv A \otimes A \otimes \dots \otimes A$ (l times), for $l = k$ or $l = n$.

To obtain a compete set of 2^L eigenvectors with 2^L eigenvalues for any finite n, $0 \leqslant n \leqslant L$, we associate with each lattice site i a basis vector $p_0(x_k)$ if a site is empty (occupation number $s_i = 0$) or $p_1(x_k)$ if there is a particle on the site (occupation number $s_i = 1$). A state $\psi(x_1, x_2, \dots, x_L)$ of the ASEP on the lattice of L sites corresponding to any configuration given by the set $s_{i_1}, s_{i_2}, \dots, s_{i_L}$ is identified with the $U_q(\hat{s}u(2))$ irreducible tensor product representation

$$\psi(x_1, x_2, \dots, x_L) = V_1(x_1) \otimes \dots \otimes V_1(x_L) \quad (57)$$

with the highest weight vector generating the $2j = L$ subrepresentation.

The highest weight vector of the considered $U_q(\hat{s}u(2))$ tensor product evaluation representation has the form $\Omega = p_0(x_1)p_0(x_2)\cdots p_0(x_L)$. The discrete set of AW polynomials satisfy the three-term recurrence relation (41) with $p_0(x) = 1$ for $x = x_k$. Hence Ω is a constant vector and is an eigenvector of the operator A^* with the eigenvalue determined by the condition $\mathcal{D}(1) = 1 + abcdq^{-1}$

$$A^*\Omega = (1 + abcdq^{-1})\Omega. \quad (58)$$

This property is related to the ground state of the system. Namely, a proper shift of A^* will produce a unique state with eigenvalue zero (the steady state of the ASEP).

By construction the state $\psi(x_i)$ becomes an eigenvector of the operator A to be interpreted as the Hamiltonian in the auxiliary space of the physical system. It acts on it by means of the coproduct. Namely, the action of the iterated coproduct according to (56) gives the eigenvalues $\sum_{k=1}^{L} x_k$, in the one occupation number *zero* $s_i = 0$ (one spin down) sector, the $A \otimes A$-type operator terms in (56) give the values $\sum_{i<j} x_i x_j$ in the two occupation numbers *zero* (two-spin down) sector and so on. The action of A according to (56) yields all the eigenvalues whose number is

$$\sum_{n=1}^{L} \frac{L!}{n!(L-n)!} = 2^L - 1 \quad (59)$$

from which the eigenvalue equation with the corresponding distinct eigenvalues for the state $\psi(x_1, x_2, \dots, x_L)$ follows:

$$A\psi(x_1, x_2, \dots, x_L) = \left(\sum_{i=1}^{L} x_i + \sum_{i<j} x_i x_j + \dots + x_1 x_2 \cdots x_L \right) \psi(x_1, x_2, \dots, x_L). \quad (60)$$

With the interpretation of A as the Hamiltonian, Eq. (60) yields the energy eigenvalues.

The considered algebraic BA, based on the unique solution of the Bethe equations (49), yields for any $n+1=L$ an exactly solvable two-boundary value spectral problem with the identification of L with the spin value $2j$ of the finite-dimensional highest weight evaluation representation of $U_q(\hat{s}u(2))$. There are two limit cases. The first one is $n\to\infty$, L fixed. This limit is obtained by treating ab dependent on a,b, $1-abq^n\neq 1$ so that $b_n\neq 0$ when the infinite-dimensional representation of the AW algebra is restored corresponding to a finite-dimensional representation of $U_{q^{1/2}}(\hat{s}u(2))$. The thermodynamic limit for finite lattice systems with added boundary terms is conventionally obtained by letting $L\to\infty$. In our scheme we start from the very beginning with models in the infinite volume/infinite chain with quantum affine $U_{q^{1/2}}(\hat{s}u(2))$ symmetry which is manifest. Boundary conditions break the infinite volume symmetry. However, with suitably imposed boundary conditions, a remnant of this symmetry survives and is encoded in the nonlocal conserved charges, elements of the coideal AW subalgebra of $U_{q^{1/2}}(\hat{s}u(2))$, defined through the homomorphism to the quantized affine $U_{q^{1/2}}(\hat{s}u(2))$.

We can now use this prescription to obtain the complete set of eigenvectors with distinct eigenvalues for the transition matrix of the open ASEP. For a finite-dimensional representation of the AW (TD) algebra from the infinite-dimensional one, we use the set of zeros, denoted $\hat{x}_i, i=1,\ldots,L$, of an AW polynomial of degree L for the particular choice of parameters in terms of the boundary probability rates, $a=k_+(\alpha,\gamma)$, $b=k_+(\beta,\delta)$, $c=k_-(\alpha,\gamma)$, $d=k_-(\beta,\delta)$. Result: In the auxiliary space of symmetric Laurent polynomials p_n of order n, $0\leqslant n\leqslant L-1$, the ASEP transition matrix is identified with the representation of the right boundary operator $A+\beta-\delta$ and the left boundary operator $A^*+\alpha-\gamma$ in the dual representation. There is a representation of dimension 2^L for any finite L, where the transition rate matrix Γ (the Hamiltonian H, respectively) has a unique ground state $(\Omega,0,0,\ldots,0)$ of eigenvalue zero which is the eigenstate of the right boundary operator, to be identified with the ASEP stationary state and 2^L-1 eigenstates of the right boundary operator with real distinct eigenvalues given by

$$E=\alpha-\gamma-(1-q)+ \\ +\left((1-q)\sum_{i=1}^{L}\hat{x}_i+(1-q)^2\sum_{i<j}\hat{x}_i\hat{x}_j+\ldots+(1-q)^L\hat{x}_1\hat{x}_2\cdots\hat{x}_L\right), \quad (61)$$

where $\hat{x}_i^{-1}=\hat{y}_i+\hat{y}_i^{-1}$ and $\hat{y}_i$ satisfy the Bethe-Ansatz equation

$$\frac{(\hat{y}_i-\zeta a)(\hat{y}_i-\zeta b)(\hat{y}_i-\zeta c)(\hat{y}_i-\zeta d)}{(\zeta a\hat{y}_i-1)(\zeta b\hat{y}_i-1)(\zeta c\hat{y}_i-1)(k_-(\zeta d\hat{y}_i-1)}=\prod_{l=1,l\neq i}^{L}\frac{(qy_i-y_l)(qy_iy_l-1)}{(y_i-qy_l)(y_iy_l-q)} \quad (62)$$

with a suitable choice for $\zeta=\dfrac{q^{-1/2}}{\sqrt{k_+(a,c)k_+(b,d)}}$ and $k_\pm(u,v)$ given by (27). In this representation the time-dependent ASEP transition matrix is a left stochastic Markov matrix $\Gamma_M=1+t\Gamma$ for the (one step) infinitesimal time dt transition.

There is a dual representation in the auxiliary space of symmetric Laurent polynomials p_n, $0\leqslant n\leqslant L-1$, of dimension 2^L, where the transfer matrix Γ_M (the Hamiltonian H, respectively) has a unique eigenstate $(\Omega,0,0,\ldots,0)^t$ of eigenvalue zero which is the eigenstate of the left boundary operator and $2^L - 1$ nonzero eigenvalues. The complete set of eigenvalues are the same as (61) up to the shift term where $\alpha - \gamma \to \beta - \delta$. In the dual representation the ASEP transition matrix is a right stochastic Markov matrix $\Gamma_M = 1 + t\Gamma$.

We note that we can shift by $2\gamma + (1 - q) + 2\delta$ the boundary operators to produce the terms $\alpha + \beta + \gamma + \delta$ in (61) which will correspond to the solution in [11], in a different basis, as given for the energy eigenvalues in the one spin sector. Apart from the shift, the latter is inferred from the XXZ BA solution for even L and needs, in addition, a wavenumber counting function. Exploring the boundary symmetry of the lattice system we have obtained the complete set of eigenvalues of the transition matrix (the Hamiltonian) for any finite L.

The special cases are immediately obtained from (61). The transition matrix of the process with particles only incoming at left and only outgoing at right is the limit case $\gamma = \delta = 0$ in (62) with $\hat{x_i}$ being the simple zeros of the Al-Salam–Chihara polynomials. There is a left and right stochastic Markov matrix, whose eigenvalues are related by a shift and should not be considered as different ones. In the case of the ASEP on a ring the tansition matrix is a double stochastic Markov matrix, where the eigenvalues of Γ are obtained from (61) with $\alpha = \bar{\rho}$, $\beta = \gamma = \delta = 0$, and $\hat{x_i}$ being the simple zeros of the big q-Hermite polynomials.

It is well known that the transition matrix of the ASEP being a positive stochastic matrix has real eigenvalues or, if complex, they appear in conjugate pairs. In our scheme the ASEP transition matrix is related by a (real) shift to the operator $A + A^*$ in the space of Laurent polynomials which is self-dual with respect to the AW duality. In view of this property, the zeros of the AW polynomials and their limit cases provide the most adequate fitting for modelling the stochastic dynamics. Our study strengthens the conviction [20] about the intimate relationship of the ASEP with the AW polynomials.

We stress once again the difference in the way the finite-dimensional representation of the AW (TD) algebra needed for the BA is obtained. We have used the general scheme where no relation among the model parameters appears so that we can apply it to models of nonequilibrium physics. For the XXZ chain the condition for the finite-dimensional representation of the AW algebra follows directly from the three-term recurrence relation and coincides with the previously found BA condition [11]. The considered prescription for the diagonalization of the transition matrix in this paper provides an opportunity for independent treatment of stochastic dynamics which will be very useful for description of the many-species systems.

To summarize, we have developed an algebraic Bethe Ansatz based on Bethe equations for the Askey–Wilson polynomials with a unique solution for any $n + 1 = L$ (including $L = 0$, any finite $L = 2j$ and $L = \infty$), which yields a complete set of 2^L eigenvectors with distinct eigenvalues and a unique ground state of transition matrix (equivalently Hamiltonian) operator. We have illustrated the algebraization of the difference eigenvalue equation for the AW polynomials on lattice systems, but the procedure is rather general and should work for any system with boundary AW symmetry. The developed BA scheme will produce a diagonalization of the proper Hamiltonian in the auxiliary space for the sine-Gordon model and quantum affine Toda field theory as well.

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APPENDIX

We show the validity of the derived formula for the ASEP for the cases $L = 2, \gamma = \delta = 0$, $0 < q < 1$ and $q = 0$. In the configuration space the transition rate matrix [11] has the form

$$\Gamma = \begin{pmatrix} -\alpha & \beta & 0 & 0 \\ 0 & -\alpha - \beta - q & 1 & 0 \\ \alpha & q & -1 & \beta \\ 0 & \alpha & 0 & -\beta \end{pmatrix} \tag{63}$$

and is diagonalized with eigenvalues $\lambda_0 = 0$, a pair of complex conjugate roots λ_1, λ_2 and a real λ_3. The nonzero eigenvalues are the distinct roots of the cubic characteristic polynomial equation which is solved by using the Cardano formulae after the substitution $\lambda_i = \tilde{\lambda_i} - 1/3(2\alpha + 2\beta + q + 1)$, with $\tilde{\lambda_3} = -\tilde{\lambda_1} - \tilde{\lambda_2}$. We spare the details and omit the rather long explicit expressions for the roots $\tilde{\lambda_i}$. In the (auxiliary) representation space of the boundary deformed Dolan–Grady algebra, the transition rate matrix Γ_{AW} is diagonalized with the discrete set of AW polynomials $p_0(x), p_1(x)$ as the basis. It has the form diag $(x_0, x_1, x_2, x_1 x_2)$, where $x_0 = 0$ and x_1, x_2 are the simple roots of the (second order) AW polynomial $p_2(x, a, b, |q)$ [25] in the interval $(0, \pi/2)$. The transition rate matrix Γ and the diagonalized matix Γ_{AW} have determinants equal to zero. The one-to-one correspondence requires that both matrices have equal traces

$$\text{Tr}\,(\Gamma) = -2\alpha - 2\beta - q - 1. \tag{64}$$

Equality of the traces is achieved due to the rescaling and shifting properties of the boundary operators in the representation space. By separating the constant $C_0 = -\alpha - \beta$, corresponding to a shift of the eigenvalue zero, the equality of the traces requires the relation

$$-(1 + q)(a + b) - q(a + b)^2 - (1 - q)^2 = -\alpha - \beta - q - 1 \tag{65}$$

(where a, b are the rescaled parameters by $(qab)^{-1/2}$); on the r.h.s. of (65) $-\alpha - \beta - q - 1 \equiv \text{Tr}\,(\Gamma) - C_0$. From the stationary state we have $a = (1 - q)\alpha^{-1} - 1, b = (1 - q)\beta^{-1} - 1$. These relations follow from the representation of the boundary algebra where the stationary state corresponds to the eigenvalue zero of the transition matrix. The shift of the eigenvalue zero amounts to a change of the steady state parameterization. We recall that the dominant contributions in the physical quantities of the exact stationary solution [20] in terms of the AW polynomials are expressed in terms of the parameter a only, if $a > b$ (or in terms of b only if $b > a$). For the time-dependant process, as a consequence of (65), the parameters a, b obey

$$\frac{q^{1/2}(a + b)}{\sqrt{ab}} = -(1 + q) \pm 2\sqrt{3 + \alpha + \beta - 2(1 - q)} \tag{66}$$

and one has either $a = (1 - q)\alpha^{-1} - 1$, or $b = (1 - q)\beta^{-1} - 1$. The relation (66) is equivalently written as

$$\frac{q^{1/2}(a + b)}{\sqrt{ab}} = -(1 + q) \pm 2\sqrt{1 - (\text{Tr}\,(\Gamma) - C_0)}. \tag{67}$$

The correspondence for the totally asymmetric process follows straightforward as the $q = 0$ limit of the above formulae. Besides, it can be verified independently with the help of the parameterization $a/b = (1/3)\ \exp\left(\sqrt{2\alpha + 2\beta}\right)$ (up to $O(\alpha^2, \beta^2)$). After the proper rescaling by $1+q$ this parameterization works also in the $q \neq 0$ case with $q = (1/3)\ \exp\left(-\sqrt{h}\right)$, which amounts to adding $-2h$ under the square root for a/b. This proves that the $L = 2$ transition rate matrices in the configuration and the auxiliary space describe the same process.

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