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INTEGRABLE STRING MODELS WITH CONSTANT SU(3) TORSION

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We used the local invariant chiral currents to obtain new integrable string equations for string WZW model type with $SU(3)$ constant torsion. We solved the Burgers equation of motion for the first invariant current in terms of the Lambert function. We show that string model with $SU(n)$, $n > 3$ constant torsion is not integrable, because procedure of decomposition of nonprimitive invariant chiral currents to primitive currents is the procedure of introduction of infinite-dimensional matrix of the second-kind constraints in the bi-Hamiltonian approch to integrable systems.

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INTRODUCTION

Integrable boson string models in the background fields were considered in the σ-model approach or by the method of effective background fields (Tseytlin, Arutyunov, Frolov [1–3]). In the first approach, the equation of motion for string coordinates is solved for the given background fields. In the second approach, background fields are solutions of the boson part of supergravity equations. In this paper, the third approach based on the method of chiral invariant currents is presented [4–10]. We will consider string model in the curve space of transverse coordinates with constant torsion, defined by metric and additional background constant torsion, defined by topology of string space (e.g., a string model in the background gravity field with constant torsion and the background antisymmetric field). At first, we consider flat string in the conformal and in the light-cone gauges, and after this we consider curve string in the curve space of transverse coordinates. In this case we avoid difficulty connected with additional constraints on the background fields. A string model is described by the Lagrangian

$$L = \frac{1}{2}\int\limits_0^{2\pi}\left[\delta^{\alpha\beta} g_{ab}(X)\frac{\partial X^a}{\partial x^\alpha}\frac{\partial X^b}{\partial x^\beta} + \epsilon^{\alpha\beta} B_{ab}\frac{\partial X^a}{\partial x^\alpha}\frac{\partial X^b}{\partial x^\beta}\right] dx.$$

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The target space local coordinates $X^a(x)$, $a = 1, \dots, 24$ belong to certain given smooth 24-dimensional manifold M^{24} with nondegenerate metric tensor and $x^\alpha = \{t, x\}$

$$g_{ab}(X(x)) = g_{ba}(X(x)), \qquad B_{ab}(X(x)) = -B_{ba}(X(x)).$$

The string equations of motion have the form:

$$g_{ab}(\ddot{X}^b - X''_b) + \Gamma_{abc}(\dot{X}^b\dot{X}^c - X'^b X''^c) + H_{abc}\dot{X}^b X'^c = 0,$$

$$\Gamma_{abc} = \frac{1}{2}\left(\frac{\partial g_{ab}}{\partial X^c} + \frac{\partial g_{ac}}{\partial X^b} - \frac{\partial g_{bc}}{\partial X^a}\right),$$

$$H_{abc} = \frac{\partial B_{ab}}{\partial X^c} + \frac{\partial B_{ca}}{\partial X^b} + \frac{\partial B_{bc}}{\partial X^a},$$

where H_{abc} is the total antisymmetric tensor. Let us introduce the repers e^a_μ such that

$$g_{ab}(X(x)) = \delta_{\mu\nu} e^\mu_a(X(x))) e^\nu_b(X(x)),$$

where $\mu, \nu = 1, \dots, n$ are indexes of tangent space to manifold M^n on some point $X^a(x)$. The repers $e^\mu_a(X)$ and their inverse $e^a_\mu(X)$ satisfy the conditions:

$$e^\mu_a e^b_\mu = \delta^b_a, \qquad e^\mu_a e^{a\nu} = \delta^{\mu\nu}.$$

The Hamiltonian has the form:

$$H = \frac{1}{2}\int_0^{2\pi} [\delta^{\mu\nu} J_{0\mu} J_{0\nu} + \delta_{\mu\nu} J^\mu_1 J^\nu_1]\, dx,$$

where canonical currents have the form

$$J_{0\mu}(X) = e^a_{mu}(X)(p_a - B_{ab}(X)X'^b), J^\mu_1(X) = e^\mu_a X'^a,$$

$$p_a(t,x) = g_{ab}\dot{X}^b + B_{ab}X'^b,$$

here $p_a(x)$ is canonical momentum, and the canonical Poisson bracket (PB) is:

$$\{X^a(x), p_b(y)\} = \delta^a_b \delta(x-y).$$

They satisfy the equation of motion

$$\partial_0 J^\mu_1 - \partial_1 J^\mu_0 = C^\mu_{\nu\lambda} J^\nu_0 J^\lambda_1,$$

$$\partial_0 J^{mu}_0 - \partial_1 J^\mu_1 = -H^{\mu\nu\lambda} J^\nu_0 J^\lambda_1.$$

Last term in the equation of motion describes the anomaly.

The canonical currents satisfy following relations:

$$\{J_{0\mu}(x), J_{0\nu}(y)\} = C^\lambda_{\mu\nu} J_{1\lambda}(x)\delta(x-y) + H_{\mu\nu\lambda} J^\lambda_1 \delta(x-y),$$

$$\{J^\mu_0(x), J^\nu_0(y)\} = C^{\mu\nu}_\lambda J^\lambda_1(x)\delta(x-y) + \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$

$$\{J^\mu_0(x), J^\nu_1(y)\} = C^{\mu\nu}_\lambda(x) J^\lambda_1(x)\delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y).$$

Here $C^{\mu}_{\nu\lambda}$ is the torsion.

$$C^{\mu}_{\nu\lambda} = \frac{\partial e^{\mu}_{a}}{\partial x^{b}}(e^{b}_{\nu}e^{a}_{\lambda} - e^{a}_{\nu}e^{b}_{\lambda}) = \left(\frac{\partial e^{\mu}_{a}}{\partial x^{b}} - \frac{\partial e^{\mu}_{b}}{\partial x^{a}}\right) e^{b}_{\nu}e^{a}_{\lambda}.$$

The chiral currents satisfy the equation of motion:

$$\partial_0 J^{\mu}_{1} - \partial_1 J^{\mu}_{0} = C^{\mu}_{\nu\lambda} J^{\nu}_{0} J^{\lambda}_{1},$$
$$\partial_0 J^{\mu}_{0} - \partial_1 J^{\mu}_{1} = -H^{\mu\nu\lambda} J^{\nu}_{0} J^{\lambda}_{1}.$$

Last term in the equation of motion describes the anomaly. Let us introduce chiral currents:

$$U^{\mu} = \delta^{\mu\nu} J_{0\nu} + J^{\mu}_{1}, \qquad V^{\mu} = \delta^{\mu\nu} J_{0\nu} - J^{\mu}_{1}.$$

Equations of motion in the light-cone coordinates

$$x^{\pm} = \frac{1}{2}(t \pm x), \qquad \frac{\partial}{\partial x^{\pm}} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$$

have the form:

$$\partial_{-} U^{\mu} = -(C^{\mu}_{\nu\lambda}(X) + H^{\mu}_{\nu\lambda}(X)) U^{\nu} V^{\lambda},$$
$$\partial_{-} V^{\mu} = (C^{\mu}_{\nu\lambda}(X) - H^{\mu}_{\nu\lambda}(X)) U^{\nu} V^{\lambda}.$$

The chiral currents satisfy the following relations:

$$\{U^{\mu}(x), U^{\nu}(y)\} = \frac{1}{2}[(3C^{\mu\nu}_{\lambda} + H^{\mu\nu}_{\lambda})U^{\lambda} - (C^{\mu\nu}_{\lambda} + H^{\mu\nu}_{\lambda})V^{\lambda}]\delta(x-y) + \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{V^{\mu}(x), V^{\nu}(y)\} = \frac{1}{2}[(3C^{\mu\nu}_{\lambda} - H^{\mu\nu}_{\lambda})V^{\lambda} - (C^{\mu\nu}_{\lambda} - H^{\mu\nu}_{\lambda})U^{\lambda}]\delta(x-y) - \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{U^{a}(x), V^{b}(y)\} = \frac{1}{2}[(C^{\mu\nu}_{\lambda} + H^{\mu\nu}_{\lambda})U^{\lambda} + (C^{\mu\nu}_{\lambda} - H^{\mu\nu}_{\lambda})V^{\lambda}]\delta(x-y).$$

Here $H_{\mu\nu\lambda}$ is additional external torsion.

This relation forms the algebra if both internal and external torsions are constant. Here are two possibilities

$$H^{\mu\nu}_{\lambda} = -C^{\mu\nu}_{\lambda}:$$

$$\{U^{\mu}(x), U^{\nu}(y)\} = C^{\mu\nu}_{\lambda} U^{\lambda}\delta(x-y) + \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{V^{\mu}(x), V^{\nu}(y)\} = C^{\mu\nu}_{\lambda}(2V^{\lambda} - U^{\lambda})\delta(x-y) - \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{U^{\mu}(x), V^{\nu}(y)\} = C^{\mu\nu}_{\lambda} V^{\lambda}\delta(x-y).$$

$$H^{\mu\nu}_{\lambda} = C^{\mu\nu}_{\lambda}:$$

$$\{U^{\mu}(x), U^{\nu}(y)\} = C^{\mu\nu}_{\lambda}(2U^{\lambda} - V^{\lambda})\delta(x-y) + \delta^{ab}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{V^{\mu}(x), V^{\nu}(y)\} = C^{\mu\nu}_{\lambda} V^{\lambda} - \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{U^{\mu}(x), V^{\nu}(y)\} = C^{\mu\nu}_{\lambda} U^{\lambda}\delta(x-y).$$

The chiral currents U^μ in the first case and V^μ in the second case form the Kac–Moody algebras. Equations of motion in light-cone coordinates have the form

$$H^\mu_{\nu\lambda} = -C^\mu_{\nu\lambda}, \qquad \partial_- U^\mu = 0, \qquad \partial_+ V^\mu = 2C^{\mu\nu\lambda} U^\nu V^\lambda,$$
$$H^\mu_{\nu\lambda} = C^\mu_{\nu\lambda}, \qquad \partial_+ V^\mu = 0, \qquad \partial_- U^\mu = -2C^{\mu\nu\lambda} U^\nu V^\lambda.$$

1. INTEGRABLE STRING MODELS WITH CONSTANT TORSION

Let us come back to commutation relations of chiral currents. We use Latin letters in this section instead of Greek for simplicity. Let torsions $C^a_{bc}(X(x)) \neq 0$ and $C_{abc} = f_{abc}$ are structure constants of simple Lie algebra. We will consider string model with constant torsion in light-cone gauge in the target space. This model coincides to principal chiral model on a compact simple Lie group. Evans, Hassan, MacKay, and Mountain [11] constructed local invariant chiral currents as polynomials of initial chiral currents of $SU(n)$, $SO(n)$, $SP(n)$ principal chiral models. Their article was based on the work of de Azcarraga, Macfarlane, MacKay, Perez Bueno [12] about invariant tensors for simple Lie algebras. We have considered total symmetrical invariant chiral currents of $SU(n)$ group. We shown that the infinite set of nonprimitive invariant charges are not commuting and they cannot be considered as Hamiltonians in a bi-Hamiltonian approach to integrable systems. Only consistent system has $SU(3)$ torsion.

Let t_a are $3 \otimes 3$ traceless Hermitian matrix representations of the generators $SU(3)$ Lie algebra:

$$[t_a, t_b] = 2i f_{abc} t_c, \qquad \mathrm{Tr}\,(t_a t_b) = 2\delta_{ab}.$$

Here is additional relation for $su(3)$ algebra:

$$\{t_a, t_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc} t_c,\ \ a = 1, \ldots, 8.$$

Invariant tensors may be constructed as invariant symmetric polynomials on $su(3)$:

$$d_m = d_{(a_1 \ldots a)} = \frac{1}{m!} S\,\mathrm{Tr}\,(t_{a_1} \cdots t_{a_m}),$$

where $S\,\mathrm{Tr}$ means completely symmetry-product of matrices, and $d_{(a_1 \ldots a_m)}$ is totally symmetric tensor and $m = 2, 3, \ldots, \infty$. Another family of invariant symmetric tensors, named «d-family» [14], is based on the product of the symmetric structure constant d_{abc} of $su(3)$ algebra:

$$d_{(a_1 \ldots a_m)} = d^{b_1}_{(a_1 a_2} d^{b_1 b_2}_{a_3} \cdots d^{b_{m-2} b_{m-3}}_{a_{m-2}} d^{b_{m-3}}_{a_{m-1} a_m)},$$

here $m = 4, 5, \ldots, \infty$.

Here are two primitive invariant tensors on $SU(3)$. The invariant tensors for $m \geqslant 3$ are functions of primitive tensors. Evans et al. introduced local chiral currents based on the invariant symmetric polynomials on simple Lie groups:

$$J_m(U) = d_{a_1 \cdots a_m} U^{a_1} \cdots U^{a_m},$$

where $U = t_a U^a$ and $a = 1, \ldots, 8$. It is possible to compose the invariant symmetric polynomials $J_m(U)$ to basis invariant currents $C_m(U)$:

$$C_2(U) = \eta_{ab} U^a U^b, \qquad C_3(U) = d_{abc} U^a U^b U^c,$$
$$C_m(U) = d^{b_1}_{a_1 a_2} d^{b_1 b_2}_{a_3} \cdots d^{b_{m-3}}_{a_{m-1} a_m} U_{a_1} U_{a_2} \cdots U_{a_m},$$

where $m = 4, 5, \ldots, \infty$. The author obtained the following expression for local invariant chiral currents $J_m(U)$:

$$J_2 = 2C_2,\ J_3 = 2C_3, \quad J_4 = 2C_4 + \frac{4}{n} C_2^2, \quad J_5 = 2C_5 + \frac{8}{n} C_2 C_3,$$
$$J_6 = 2C_6 + \frac{4}{n} C_3^2 + \frac{8}{n} C_2 C_4 + \frac{8}{n^2} C_2^3,$$
$$J_7 = 2C_7 + \frac{8}{n} C_3 C_4 + \frac{8}{n} C_2 C_5 + \frac{24}{n^2} C_2^2 C_3,$$
$$J_8 = 2C_8 + \frac{4}{n} C_4^2 + \frac{8}{n} C_3 C_5 + \frac{8}{n} C_2 C_6 + \frac{24}{n^2} C_2 C_3^2 + \frac{24}{n^2} C_2^2 C_4 + \frac{16}{n^3} C_2^4.$$

The commutation relations of invariant chiral currents $J_m(U(x))$ show that these currents are not densities of dynamical Casings operators. We considered the basis family of invariant chiral currents $C_m(U)$, and we proved that invariant chiral currents $C_m(U)$ form closed algebra under canonical PB, and corresponding charges are dynamical Casimir operators. The commutation relations of invariant chiral currents $C_m(U(x))$ and $C_n(U(y))$ for $m, n = 2, 3, 4$ and for $m = 2, n = 2, 3, \ldots, \infty$ are the following:

$$\{C_m(x), C_n(y)\} = -mn C_{m+n-2}(x) \frac{\partial}{\partial x} \delta(x-y) - \frac{mn(n-1)}{m+n-2} \frac{\partial C_{m+n-2}(x)}{\partial x} \delta(x-y).$$

The commutation relations for $m \geqslant 5$, $n \geqslant 3$ are the following (we show one formula for C_8 on the right side as an example only):

$$\{C_5(x), C_3(y)\} = -[12C_6(x) + 3C_{6,1}(x)] \frac{\partial}{\partial x} \delta(x-y) - \\ - \frac{1}{3} \frac{\partial}{\partial x} [12C_6(x) + 3C_{6,1}(x)] \delta(x-y),$$

$$\{C_5(x), C_4(y)\} = -[16C_7(x) + 4C_{7,1}(x)] \frac{\partial}{\partial x} \delta(x-y) - \\ - \frac{3}{7} \frac{\partial}{\partial x} [16C_7(x) + 4C_{7,1}(x)] \delta(x-y),$$

$$\{C_6(x), C_3(y)\} = -[12C_7(x) + 6C_{7,1}(x)] \frac{\partial}{\partial x} \delta(x-y) - \\ - \frac{2}{7} \frac{\partial}{\partial x} [12C_7(x) + 6C_{7,1}(x)] \delta(x-y),$$

$$\{C_5(x), C_5(y)\} = -[16C_8(x) + 8C_{8,1}(x) + C_{8,2}(x)] \frac{\partial}{\partial x} \delta(x-y) - \\ - \frac{1}{2} \frac{\partial}{\partial x} [16C_8(x) + 8C_{8,1}(x) + C_{8,2}(x)] \delta(x-y),$$

$$\{C_6(x), C_4(y)\} = -[16C_8(x) + 8C_{8,3}(x)]\frac{\partial}{\partial x}\delta(x-y) - \\ -\frac{3}{8}\frac{\partial}{\partial x}[16C_8(x) + 8C_{8,3}(x)]\delta(x-y),$$

$$\{C_7(x), C_3(y)\} = -[12C_8(x) + 6C_{8,1}(x) + 3C_{8,3}]\frac{\partial}{\partial x}\delta(x-y) - \\ -\frac{1}{4}\frac{\partial}{\partial x}[12C_8(x) + 6C_{8,1}(x) + 3C_{8,3}(x)]\delta(x-y).$$

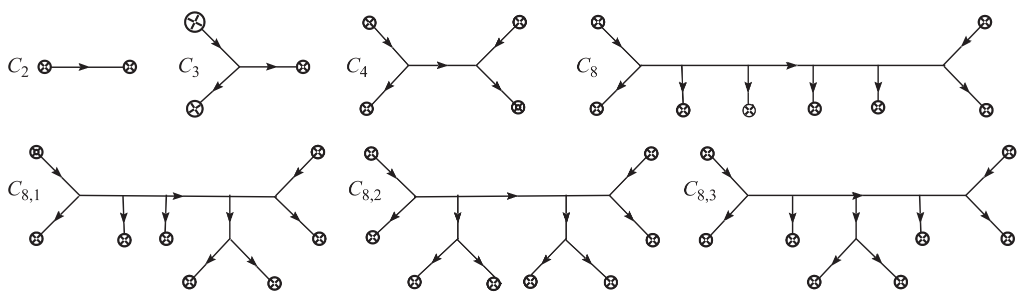
The new dependent invariant chiral currents $C_{6,1}$, $C_{7,1}$, $C_{8,1} - C_{8,3}$, $C_{9,1} - C_{9,4}$, $C_{10,1} - C_{10,7}$ have the form (we show formulas for C_8 on the right side as an example only):

$$C_{8,1} = [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho}][d^m_{\sigma\varphi}][d^p_{\tau\theta}] d^{nmp}(U^8)\mu\nu\lambda\rho\sigma\varphi\tau\theta,$$

$$C_{8,2} = [d^k_{\mu\nu}][d^l_{\lambda\rho}][d^n_{\sigma\varphi}][d^m_{\tau\theta}] d^{klp} d^{nmp}(U^8)\mu\nu\lambda\rho\sigma\varphi\tau\theta,$$

$$C_{8,3} = [d^k_{\mu\nu} d^{kl}_{\lambda}][d^n_{\rho\sigma} d^{nm}_{\varphi}][d^p_{\tau\theta}] d^{lmp}(U^8)\mu\nu\lambda\rho\sigma\varphi\tau\theta.$$

In the Figure we show the graphic image of invariant chiral currents $C_m(U)$ and the difference between C_8 and $C_{81} - C_{83}$.



Invariant chiral currents

Let us note that these PBs are PBs of hydrodynamic type. The ultralocal term with antisymmetric structure constant f_{abc} in commutation relation of chiral currents U^a does not contribute to commutation relations of invariant chiral currents because of totally symmetric invariant tensors $d_{(a_1 \ldots a_m)}$. Therefore chiral currents $C_m(U(x))$ form closed algebra under canonical PB.

The new dependent invariant chiral currents and the new dependent totally symmetric invariant tensors for $SU(3)$ group can be obtained under different order of calculation of trace of the product of the generators of $su(n)$ algebra. Let us mark the matrix product of two generators t_a, t_b in round brackets:

$$(t_a t_b) = \frac{2}{3}\delta_{ab} + (d^c_{ab} + i f^c_{ab}) t_c.$$

The expression of invariant chiral currents $J_m(U)$ depends on the order of the matrix product of two generators in general list of generators. For example:

$$J_8 = \mathrm{Tr}\,[t(\underline{tt})tt(\underline{tt})t] = 2C_8 + \frac{4}{n}C_4^2 + \frac{8}{n}C_3C_5 + \\ + \frac{8}{n}C_2C_6 + \frac{24}{n^2}C_2C_3^2 + \frac{24}{n^2}C_2^2C_4 + \frac{16}{n^3}C_2^4,$$

$$J_8 = \mathrm{Tr}\,[(\underline{tt})(\underline{tt})t(\underline{tt})t] = 2C_{8,1} + \frac{4}{n}C_4^2 + \frac{4}{n}C_3C_5 + \\ + \frac{24}{n^2}C_2C_3^2 + \frac{12}{n}C_2C_6 + \frac{24}{n^2}C_2^2C_4 + \frac{16}{n^3}C_2^4,$$

$$J_8 = \mathrm{Tr}\,[(\underline{tt})(\underline{tt})(\underline{tt})(\underline{tt})] = 2C_{8,2} + \frac{4}{n}C_4^2 + \frac{16}{n}C_2C_{6,1} + \frac{32}{n^2}C_2^2C_4 + \frac{16}{n^3}C_2^4,$$

$$J_8 = \mathrm{Tr}\,[t(\underline{tt})(\underline{tt})(\underline{tt})t] = 2C_{8,3} + \frac{12}{n}C_2C_6 + \frac{8}{n}C_3C_5 + \frac{24}{n^2}C_2^2C_4 + \frac{24}{n^2}C_2C_3^2 + \frac{16}{n}C_2^4,$$

where $t = t_aU^a$. Because the result of calculation does not depend on the order of calculation, we can obtain relations between new invariant chiral currents and basis invariant currents $C_m(U)$:

$$C_{6,1} = C_6 + \frac{2}{n}C_3^2 - \frac{2}{n}C_2C_4,$$
$$C_{8,1} = C_8 + \frac{2}{n}C_3C_5 - \frac{2}{n}C_2C_6,$$
$$C_{8,2} = C_8 + \frac{4}{n}C_3C_5 - \frac{4}{n}C_2C_6 - \frac{4}{n^2}C_2C_3^2 + \frac{4}{n^2}C_2^2C_4,$$
$$C_{8,3} = C_8 + \frac{2}{n}C_4^2 - \frac{2}{n}C_2C_6.$$

The family of invariant chiral currents $C_m(U(x))$ satisfies conservation equations

$$\partial_- C_m(U(x)) = 0.$$

2. NEW INTEGRABLE STRING EQUATIONS

Let us apply hydrodynamic approach to integrable string models with constant torsion. In this case we must consider the conserved primitive chiral currents $C_m(U(x))$, $m = 2, 3$ as local fields of the Riemann manifold. The nonprimitive local charges of invariant chiral currents with $m \geqslant 3$ form the hierarchy of new Hamiltonians in bi-Hamiltonian approach to integrable systems. The commutation relations of invariant chiral currents are local PBs of hydrodynamic type.

The invariant chiral currents C_m with $m \geqslant 3$ for the $SU(3)$ group can be obtained from the following relation:

$$d_{kln}d_{kmp} + d_{klm}d_{knp} + d_{klp}d_{knm} = \frac{1}{3}(\delta_{ln}\delta_{mp} + \delta_{lm}\delta_{np} + \delta_{lp}\delta_{nm}).$$

The corresponding invariant chiral currents for $SU(3)$ group have the form:

$$C_{2n} = \frac{1}{3^{n-1}}(\eta_{\mu\nu}U^\mu U^\nu)^n = \frac{1}{3^{n-1}}(C_2)^n,$$
$$C_{2n+1} = \frac{1}{3^{n-1}}(\eta_{\mu\nu}U^\mu U^\nu)^{n-1}d_{kln}U^k U^l U^n = \frac{1}{3^{N-1}}(C_2)^{N-1}C_3.$$

The invariant chiral currents C_2, C_3 are local coordinates of the Riemann manifold M^2. The local charges C_{2n}, $n \geqslant 2$ are the invariant subalgebra of algebra charges C_{2n},C_{2n+1} and they form hierarchy of Hamiltonians. Let us introduce new notation

$$f(x) = C_2(U(x)),\ g(x) = C_3(U(x)),$$

$$H_n(t_n) = \frac{1}{2n(2n-1)}\int_0^{2\pi} f^n(x,t_n)dx,\ n = 2,\ldots,\infty.$$

The new nonlinear equations of motion for chiral currents are the following:

$$\frac{\partial f}{\partial t_n} + f^{n-1}\frac{\partial f}{\partial x} = 0,$$
$$\frac{\partial g}{\partial t_n} + \frac{3}{2n-1}gf^{n-2}\frac{\partial f}{\partial x} + \frac{1}{2N-1}f^{n-1}\frac{\partial g}{\partial x} = 0.$$

The first equation for function $f(x,t_n)$ is generalized unviscid Burgers' equations [15]; in component

$$\frac{\partial f}{\partial t_2} + f\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial t_3} + f^2\frac{\partial f}{\partial x} = 0.$$

They have the following solutions:

$$f(x,t_n) = h_n[x - t_n f^{n-1}(x,t_n)]$$

and $h(x)$ is the periodical arbitrary function of x.

For simplicity we introduce new variables

$$f^n = y_n, \qquad n = 2,\ldots,\infty.$$

New equations of motion for functions $y(x)_n$ coincide, and they do not depend on n:

$$\frac{\partial y_n}{\partial t_n} + y_n\frac{\partial y_n}{\partial x} = 0$$

(no sum); in component

$$\frac{\partial y_2}{\partial t_2} + y_2\frac{\partial y_2}{\partial x} = 0, \qquad \frac{\partial y_3}{\partial t} + y_3\frac{\partial y_3}{\partial x} = 0,\ldots$$

The first equations are n Burgers equations. General solutions are:

$$y_n = h_n[x - t_n y_n(x,t_n)].$$

Here there are a lot of solutions in terms of rational functions for h_n which led to algebraic equations for functions y_n. But these solutions do not satisfy periodical bound conditions. In general case, these equations are solved by numerical methods. However, here are weak solutions for small-times intervals. Let us introduce periodical initial conditions

$$y_n(x,0) = h_0(x,0),\ h_0(x) = h_0(x+2\pi),$$
$$y_n(x,t) = h_0[x - t_n h_0(x)] + O(t_n).$$

For example, in the first approximation on t we obtained the following solutions:

$$\underline{h_0(x) = e^{ix}}, \qquad y_n = e^{ix} - t_n e^{2ix}, \qquad \underline{h_0(x) = \sin x}, \qquad y_n = \sin x - \frac{1}{2} t_n \sin 2x.$$

Here is one exact solution of the Burgers equation with periodical bound conditions in terms of the Lambert function [16]. Let

$$y_n(t,x) = h_n[x - t_n y_n(x,t_n)],$$

where a is arbitrary parameter. The Burgers equation can be rewritten in the following form:

$$Y_n = Z_n\, \mathrm{e}^{Z_n}, \qquad Y_n = it_n\, \mathrm{e}^{a+ix}, \qquad Z_n = it_n y_n.$$

The inverse transformation $Z_n = Z_n(Y_n)$ is defined by the W Lambert function $Z_n = W(Y_n)$:

$$y_n(x,t) = \frac{-i}{t_n} W(it_n\, \mathrm{e}^{a+ix}).$$

Let us remember that $f^n(x,t_n) = y_n,\ f(x) = U^a U^a$, and $U^a,\ a = 1,\ldots,8$ are initial chiral currents. The equation of motion for the second invariant currents $g(x)$ has the form

$$\frac{\partial g}{\partial t_n} + \frac{3g}{2n-1}\frac{\partial y_{n-1}}{\partial x} + \frac{y_{n-1}}{2n-1}\frac{\partial g}{\partial x} = 0,$$

$$\frac{\partial g}{\partial t_2} + g\frac{\partial y_1}{\partial x} + \frac{y_1}{3}\frac{\partial g}{\partial x} = 0, \qquad \frac{\partial g}{\partial t_3} + \frac{3g}{5}\frac{\partial y_2}{\partial x} + \frac{y_2}{5}\frac{\partial g}{\partial x} = 0,$$

The construction of integrable equations with $SU(n)$ symmetries for $n \geqslant 4$ has difficulties of reduction of nonprimitive invariant currents to primitive currents. Let us consider the generating function

$$A(x,\lambda) = \det(1 - \lambda U(x)) = \exp \mathrm{Tr}\, \ln(1 - \lambda U^a(x) t_a).$$

We obtained the following expressions for nonprimitive chiral currents C_n:

$$SU(4): C_5 \to \frac{2}{3} C_2 C_3,\ C_6 \to \frac{1}{6} C_3^2 + \frac{1}{2} C_2 C_4, \quad C_7 \to \frac{1}{3} C_2^2 C_3 + \frac{1}{6} C_3 C_4,$$

$$C_8 \to \frac{7}{36} C_2 C_3^2 + \frac{1}{4} C_2^2 C_4, \quad C_9 \to \frac{1}{6} C_2^3 C_3 + \frac{1}{36} C_3^3 + \frac{1}{6} C_2 C_3 C_4.$$

$$SU(5): C_6 \to -\frac{3}{50} C_2^3 + \frac{4}{15} C_3^2 + \frac{7}{10} C_2 C_4, \quad C_7 \to -\frac{3}{50} C_2^2 C_3 + \frac{11}{30} C_3 C_4 + \frac{3}{5} C_2 C_5,$$

$$C_8 \to -\frac{9}{250}C_2^4\frac{4}{25}C_2C_3^2 + \frac{9}{25}C_2^2C_4 + \frac{1}{10}C_4^2 + \frac{4}{15}C_3C_5,$$
$$C_9 \to -\frac{13}{250}C_2^3C_3 + \frac{16}{225}C_3^3 + \frac{61}{150}C_2C_3C_4 + \frac{3}{10}C_2^2C_5 + \frac{1}{10}C_4C_5.$$

However, the nonprimitive charges are not commuting. They are not Casimirs and we cannot consider them as Hamiltonians. Consequently, only string model with constant $SU(3)$ torsions is integrable one. The reason of noncommutativity of nonprimitive currents is explained by procedure of decomposition of nonprimitive currents to functions of primitive currents. This procedure in the formalism of Poisson brackets is the procedure of introduction of infinite number of the second-kind constraints. It is very hard or it is impossible to obtain the infinite-dimension matrix of the second-kind constraints and its inverse to construct Dirac brackets, in terms of which nonprimitive charge must commute.

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