Письма в ЭЧАЯ. 2011. Т. 8, № 3(166). С. 506–510

ON THE NATURE OF SUPERSYMMETRIES

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The problem of emergence of supersymmetry is considered. It is argued that putting of a set of harmonic oscillators into a thermal bath gives both quantum mechanics and supersymmetric structures.

PACS: 11.30.Pb; 12.60.Jv

INTRODUCTION

Among the fundamental problems of modern physics the problem of emergence of supersymmetry (SUSY) is one of the most actual. It turns out, however, that SUSY and many other properties of the world follow from simple supposition: Our 3D space is a network made of Bose strings and put into a thermal bath, the state of the network is characterized by the Gibbs distribution. In the large scale limit one receives the Minkowski space. It was shown in [1] that from the Gibbs distribution follow both the classical Hamiltonian mechanics and quantum mechanics, i.e., the probability amplitudes, the Planck constant $\hbar$, the Schroedinger equation, the Fock space, etc. The thermal bath is also responsible for appearance of fermions [1]. Here we discuss in some detail the problem of supersymmetry.

1. THE MINKOWSKI SPACE

It is well known that idealization gives rise to some new properties. For example, planets in the Newton theory are considered as material points and are characterized only by their coordinates and masses. Set of nonrelativistic harmonic oscillators on a line is described by the Lagrangian

$$L = \frac{1}{2}\sum_j [\dot{q}_j^2 - \gamma(q_j - q_{j-1})^2 - m^2 q_j^2], \quad \dot{q} = \frac{dq}{dt}, \quad \gamma > 0. \tag{1}$$

In the continuous limit $a \to 0$, $j \to \infty$, $aj \to x$, $q_j/\sqrt{a} \to \varphi(t,x)$, $\gamma a^2 \to c^2$ (here a is the distance between the oscillators) it becomes

$$L = \frac{1}{2}\int dx(\dot{\varphi}^2 - c^2\varphi'^2 - m^2\varphi^2), \tag{2}$$

where $\dot{\varphi} = \partial\varphi/\partial t$, $\varphi' = \partial\varphi/\partial x$, and c is the velocity of the field φ excitations (if $m = 0$). Thus, in this limit (idealization!) we obtained a relativistic theory of scalar field and can introduce notion of 2D Minkowski space M^2. The corresponding equation of motion ($c = 1$)

$$(\Box - m^2)\varphi = 0, \quad \Box = -\partial_t^2 + \partial_x^2 \tag{3}$$

is Lorentz invariant. It turns out that putting this system into a thermal bath gives rise to quantum mechanics [1] — evolution of nonequilibrium distributions of oscillators with time are described by probability amplitudes. Moreover, besides the bosonic excitations in the case of the Bose strings there appear also the fermionic ones described by spinors. In the next sections we explain why and how there appear supersymmetry.

2. THE GIBBS DISTRIBUTION, THE BOSE STRING, AND FERMIONS

Let

$$G(q,p) = h^{-1}\,\mathrm{e}^{-\beta H(q,p)}, \quad H = \frac{\omega}{2}(p^2 + q^2), \quad \beta = \frac{1}{kT}, \quad h = \frac{2\pi}{\beta\omega} \tag{4}$$

be the Gibbs distribution (k — the Boltzmann constant). Introducing complex canonical variables $z = (q + ip)/\sqrt{2}, \bar{z} = (q - ip)/\sqrt{2}$ (transformation $q, p \to \bar{z}, z$ is not canonical: $\{q, p\} = 1$, $\{\bar{z}, z\} = i$), we come to the following phase space measure:

$$d\mu(q,p) = \frac{dq \wedge dp}{h} \exp\left[\frac{-\beta\omega(p^2 + q^2)}{2}\right] = d\mu(\bar{z}, z) = \frac{d\bar{z} \wedge dz}{ih}\,\mathrm{e}^{-\bar{z}z/\hbar}, \\ H = \omega\bar{z}z, \hbar = \frac{h}{2\pi}. \tag{5}$$

Measure (5) presents the equilibrium distribution for an oscillator. Any other measure

$$d\mu_P = P(x)d\mu, \quad (x_1, x_2) = (q, p), \quad P \geqslant 0, \quad \int d\mu_P = 1 \tag{6}$$

describes a certain nonequilibrium distribution [2, p. 7]. In a general variation of canonical variables $\delta x = \delta x_\perp + \delta x_\parallel$ the first term preserves the Gibbs distribution by definition, i.e., $\delta H = \partial_i H \delta x_\perp^i = 0$, $i = 1, 2$. Solution to this equation is $\delta x_\perp^i = \hat{J}^{ij}\partial_j H \delta t$, where $\hat{J}$ is some antisymmetric matrix, t — some parameter, leads to the standard Hamiltonian equations of motion: $\dot{x} = \{x, H\}$, $\hat{J}$ being the symplectic form. Notice that for dimensionless $\hat{J}$ parameter t has dimension of time. For standard $\hat{J}$ the equations of motion are

$$\dot{z} = -i\omega z, \quad \dot{\bar{z}} = i\omega\bar{z}. \tag{7}$$

The second equation can be obtained from the first one by complex conjugation. Equations (7) are the Hamiltonian equations of motion for classical oscillator. Variations $\delta x_\parallel$ do not preserve the equilibrium distribution. Taking $z \to x + c$ one finds

$$d\mu(\bar{z}, z) \to d\mu_f(\bar{z}, z) = |f_c(z)|^2 d\mu(\bar{z}, z), \quad f_c(z) = \mathrm{e}^{-\bar{c}z/\hbar}\,\mathrm{e}^{-\bar{c}c/2\hbar}. \tag{8}$$

This measure describes a nonequilibrium distribution. Evolution of measure μ_f with time is given by evolution of function f_c in Eq. (8) [1].

$$\dot{f} = \{f, H\} = -i\omega z \frac{df}{dz}. \tag{9}$$

If the relaxation time t_r is large ($t_r \gg \omega^{-1}$), in the time interval $0 < t \ll t_r$ one can use the classical equation of motion (9). Multiplying it by $\hbar$, we come in fact to the Schroedinger equation for harmonic oscillator (as for the «vacuum energy» $\hbar\omega/2$ see [1]). The Bose string is set of harmonic oscillators, so its excitations are described by quantum mechanics.

It is well known that in curved spaces there appears negative quantum potential (in the case of positive curvature). Thus, it is favorable for a string in a thermal bath to curve, and for the Gibbs distribution the proper configurations of the string are those with zero «vacuum energy». The only known quantum oscillator systems with zero lowest energy are the supersymmetric ones.

Evidently, the Hamiltonian of unified SUSY field theory differs from that of the Bose string. To get quantum mechanics one has to put harmonic oscillator into a thermal bath. Are there examples of changing mechanics under influence of temperature? Yes, there are a lot of them.

1. Change of the Hamiltonian — the Higgs phenomenon.
2. Appearance of new dynamical variables. (1) The Goldstone bosons. (2) Superconductivity — pairing of electrons.
3. Other phase transitions in physics. In all these cases the excitations can radically change mechanics (e.g., appearance of transversal excitations in solid states, etc.).

Thus, it does not look absurd to study the possibility that the supersymmetry is also the result of stochasticity of a system. The specific feature of the case is the appearance of new type of dynamical variables (fermions) and anticommuting parameters (the Grassmann algebra).

Compare the square energy operators for strait and curved (circle) strings ($m = 0$ in (2) and (3)). In the case of a circle, we have in fact a 2D problem, and in the quantum theory $\hat{H}^2/c^2 = -\hbar^2\Delta$ (because $E^2 = \mathbf{p}^2c^2$), where Δ is 2D Laplace operator. In the polar coordinates $-\hbar^2\Delta = \hat{P}_r^2 + (\hat{P}_\varphi^2 - \hbar^2/4)/r^2$, where $\hat{P}_r = -i\hbar r^{-1/2}(\partial/\partial r)r^{1/2}$, $\hat{P}_\varphi = -i\hbar\partial/\partial\varphi$ are the radial and angular momenta operators correspondingly. If the radius of the ring is R, then we have constraint $r = R$ and there could be no radial motion, $\hat{P}_r\psi = 0$ [1], so, in the corresponding physical Hilbert subspace

$$\hat{H}^2 = c^2\left(\hat{P}^2 - \frac{\hbar^2}{4R^2}\right) = c\left(\hat{P} + \frac{\hbar}{2R}\right)c\left(\hat{P} - \frac{\hbar}{2R}\right) \equiv \hat{H}_+\hat{H}_-, \quad \hat{P} = R^{-1}\hat{P}_\varphi. \tag{10}$$

Only motion with $P^2 \geqslant \hbar^2/4R^2$ is allowed. For a ring the state with zero energy is impossible — the spectrum of $\hat{P}$ is $p_n = n\hbar/R$, $n = 0, 1, 2, \ldots$ because of periodicity condition $\psi(\varphi + 2\pi) = \psi(\varphi)$. To get the momentum $p = \hbar/2R$ one should take a «double ring» ($\psi(\varphi + 4\pi) = \psi(\varphi)$) with spectrum $\tilde{p}_n = n\hbar/2R$. We observe that angular momentum for motion with $\tilde{p}_1 = \hbar/2R$ is $\hbar/2$ ($\mathbf{M} = \mathbf{r} \times \tilde{\mathbf{p}}_1$, $|\mathbf{r} \times \tilde{\mathbf{p}}_1| = R\hbar/2R = \hbar/2$), and the corresponding state vector $\psi_\pm(\varphi) = \mathrm{e}^{\pm i\varphi/2}$ transforms under rotations like a spinor: $\psi_\pm(\varphi + \alpha) = \mathrm{e}^{\pm i\alpha/2}\psi_\pm(\varphi)$. Notice that the spinor excitations in case of M^2 (no angular motion) are given by $u_\pm = (t \pm z)^{1/2}$ [1].

If we take R small enough and assume that only the first two lowest energy levels are essential for the model, then there appear fermions. Indeed, spectrum of operator $\hat{P}$ coincides with that of harmonic oscillator, i.e., $\hat{P} = \hat{a}^+\hat{a}/R$, $[\hat{a}, \hat{a}^+] = \hbar$. In the corresponding matrix representation for operators $\hat{a}$, $\hat{a}^+$ in the case of two lowest levels we should take only the 2×2 matrices in the upper left corners of matrices $\hat{a}$, $\hat{a}^+$. They are proportional to

$$\hat{f} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{f}^+ \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and $\hat{f}\hat{f}^+ + \hat{f}^+\hat{f} = \hbar$. Thus, we have 1) spinors ($\psi_\pm(\varphi + \alpha) = \mathrm{e}^{\pm i\alpha/2}\psi_\pm(\varphi)$), 2) half-integer angular momenta ($|\mathbf{r} \times \tilde{\mathbf{p}}_1| = \hbar/2$), and 3) the anticommuting operators $\hat{f}, \hat{f}^+$. The latter in the classical limit $\hbar \to 0$ become the Grassmann variables. We conclude that helical excitations with periodicity condition $\psi(\varphi + 4\pi) = \psi(\varphi)$ model fermions.

3. THE RAMON–NEVEU–SCHWARZ MODEL AND SUPERSYMMETRY

A helix has different kinds of excitations (see, e.g., [3, 4]). If we take into consideration only oscillations of centers of rings (bosonic degrees of freedom) and motion of excitations over rings with angular momenta $\hbar/2$, then we get in fact the Ramon–Neveu–Schwarz (RNS) model. Indeed, if the Nambu–Goto–Polyakov string is formulated in the D26 space-time, then instead of operators $\hat{a}, \hat{a}^+$ one should take $\hat{a}^\mu, \hat{a}^{+\mu}$, $\mu = 0, 1, \ldots, 25$, and, evidently, instead of operators $\hat{f}, \hat{f}^+$ one should take $\hat{f}^\mu, \hat{f}^{+\mu}$. Introducing corresponding fields $X^\mu(\tau, \sigma)$, $S^\mu_\alpha(\tau, \sigma)$, $\alpha = 1, 2$, we obtain the RNS Lagrangian [1]

$$\mathcal{L} = -\frac{\partial X^\mu}{\partial \sigma^+}\frac{\partial X^\mu}{\partial \sigma^-} + \bar{S}^\mu i\gamma_a \partial_a S^\mu, \quad \sigma_\pm = \tau \pm \sigma, \quad a = 0, 1, \quad \gamma_0 = \sigma_3, \quad \gamma_1 = -i\sigma_2. \tag{11}$$

Here σ_i are the Pauli matrices, and $\bar{S} = S^*\gamma_0$. Fields X, S correspond to operators $\hat{a}, \hat{a}^+$ and $\hat{f}, \hat{f}^+$, respectively.

As is well known, a theory is supersymmetric if corresponding Hamiltonian is invariant under substitutions $\hat{a} \rightleftharpoons \hat{f}$, $\hat{a}^+ \rightleftharpoons \hat{f}^+$, i.e., if, for example,

$$\hat{H} = \omega(\hat{a}^+\hat{a} + \hat{f}^+\hat{f}). \tag{12}$$

There are two generators of supersymmetric transformations: $\hat{G}_1 = i(\hat{a}^+\hat{f} - \hat{a}\hat{f}^+)$ and $\hat{G}_2 = (\hat{a}^+\hat{f} + \hat{a}\hat{f}^+)$. The corresponding operators are

$$\hat{U} = \mathrm{e}^{\epsilon_1\hat{G}_1 + \epsilon_2\hat{G}_2}, \quad \hat{U}^+ = \mathrm{e}^{-(\epsilon_1\hat{G}_1 + \epsilon_2\hat{G}_2)}, \tag{13}$$

where $\hat{\epsilon}_{1,2}$ are the Grassmann parameters: $\hat{\epsilon}_i^2 = 0$, $\hat{\epsilon}_i\hat{f} = -\hat{f}\hat{\epsilon}_i$, $\hat{\epsilon}_i^+ = \hat{\epsilon}_i$. Further, $\hat{G}_i^2 = \hat{H}/\omega$, and the transformation rule for some operator $\hat{A}$ is

$$\hat{A}' = \hat{U}\hat{A}\hat{U}^+ = \hat{A} + [\epsilon_i\hat{G}_i, \hat{A}]_-. \tag{14}$$

As is easily seen, $\hat{H}' = \hat{H}$, and this is the principal property of supersymmetric theory. It is to stress that from the Lagrangian (11) follows the supersymmetric Hamiltonian (of type (12)). It imposes certain conditions on fields X and S velocities: the corresponding oscillator frequencies should be identical.

CONCLUSION

The main idea of the report is: It is a thermal bath which is responsible for supersymmetry. The Gibbs distribution in fact introduces the notion of Hamiltonian. In a thermal bath only the most probable configurations of a system survive. In Sec. 2 we have given examples of changing mechanics — there are preferable Hamiltonians. This phenomenon is known as «phase transition» in solid state. In the field theory there is another terminology, e.g., the Higgs phenomenon. One may argue that there is no thermal bath in quantum field theory. It was shown in [1] that probability amplitudes describe evolution of small deviations from equilibrium distribution for harmonic oscillator in a thermal bath (of course, we speak on physics at the Planck distances). All fields and strings are sets of oscillators, so this rule is universal.

The spinors appear as follows. In (1+1) space-time there is no angular momenta, the quantities $u_\pm = (t \pm z)^{1/2}$ transform under rotations like spinors: $u'_\mu = \mathrm{e}^{\pm\vartheta/2} u_\pm$, where ϑ is the $SO(1,1)$ group parameter. In the 2D space (a plane) there are rotations generated by angular momentum, so $\zeta_\pm = (x \pm iy)^{1/2}$ transform like spinors ($\zeta'_\pm = \mathrm{e}^{\pm i\varphi/2}\zeta_\pm$). The direct product of $u_\pm$ and $\zeta_\pm$ composes a spinor in M^4 Minkowski space.

Helical excitations are described by both coordinates x^μ and fermionic variables $\hat{f}, \hat{f}^+$. But in the classical limit $\hat{f}, \hat{f}^+$ become θ, θ^+ — the Grassmann parameters. Motion over ring may be characterized by infinitesimal displacement $\theta \to \theta + \epsilon$, the latter is connected with displacement in z direction, so the rule $x^\mu \to x^\mu - i\bar{\epsilon}\gamma^\mu\theta$ looks natural. This is the basic rule for supersymmetric generalization of the Poincare algebra [5–8].

We conclude: The Gibbs distribution with supersymmetric Hamiltonian is an optimal distribution for the Bose strings (thermal bath!). Probability amplitudes and supersymmetry have one «father» — a thermostat. As for the nature of this thermal bath — it is a separate problem. This hypothesis gives too many important consequences, among others — the cosmological constant, the dark matter, etc. For example, the 10D superstring appears only in the limit $R \to 0$. Thus, the helix is an object in $(10+2)$D space-time. In case of the Bose string there are two unphysical degrees of freedom, and $26 - 2 = 24 = 5^2 - 1$ is dimension of $SU(5)$ group. Superstrings also have two unphysical degrees of freedom, and $10 - 2 = 8 = 3^2 - 1$ gives the dimension of $SU(3)$ group. Of course, it cannot be just a coincidence.

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