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NUMERICAL AND ANALYTICAL STUDY OF WAVE PROCESSES IN PERIODIC STRATIFIED MEDIA

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We study the behaviour of the solutions of the Cauchy problem with discontinuous initial data for nonstandard linear partial differential equations modeling wave processes in periodic stratified media. Asymptotic formulas at large t are derived. The found asymptotic formulas are in a good agreement with the results of numerical experiments done by using the analytical computation system REDUCE 3.8.

Исследуется поведение решений задачи Коши с разрывными начальными данными для нестандартных линейных дифференциальных уравнений в частных производных, моделирующих волновые процессы в периодических слоистых средах. Получены асимптотические формулы при больших t. Найденные формулы хорошо согласуются с результатами численных экспериментов, проведенных с использованием системы аналитических вычислений REDUCE 3.8.

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INTRODUCTION

This article addresses the study of the following nonstandard linear partial differential equations:

$$U_{tt} = U_{xx} + U_{ttxx}, \tag{1}$$

$$U_{tt} = U_{xx} + ibU_{xxx} + U_{ttxx}, \tag{2}$$

$$U_{tt} = U_{xx} + ibU_{ttx} + U_{ttxx}, \tag{3}$$

$$U_{tt} = U_{xx} - U_{xxxx}, \tag{4}$$

solved for Cauchy problems with discontinuous initial data,

$$u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x \geqslant 0, \end{cases} \quad U_t(x,0) = 0. \tag{5}$$

Such Cauchy problems arise in the study of the wave motion in periodic stratified media [1, 2]. The processes in such media are described by differential equations with rapidly oscillating coefficients [3]. Solutions of these equations could not be obtained by standard numerical methods. After averaging, quite different specific equations of the kind (1)–(4) can arise. The asymptotics developed for these equations is important for fundamental research as well as for numerous applications especially in the mechanics of composite materials.

1. EXISTENCE OF BREATHER-TYPE SOLUTIONS

The existence of a breather-type solution has been proved [4] for the Cauchy problem (1), (5):

$$U(x,t) = \frac{1}{2} + \frac{\operatorname{sgn}(x)}{2}\cos(t) + O(\sqrt{x^2 t}), \quad |x| < \mathrm{ct}^{-1/2}, \quad t \to \infty.$$

The existence of such an exotic solution (for linear equation) has been recovered within numerical experiments. It has been proved in [4] as well that, outside the interval $|x| < t(1+\delta)$ (here and below δ is a small positive constant independent of t), the solution of the Cauchy problem (1), (5) differs from the step (5) only by an exponentially small quantity in t. This enables us to replace the solution of the original problem over an infinite interval by the solution of a boundary value problem on the finite interval: $t \leqslant T, |x| \leqslant le = T + \Delta, \Delta$ is a positive constant.

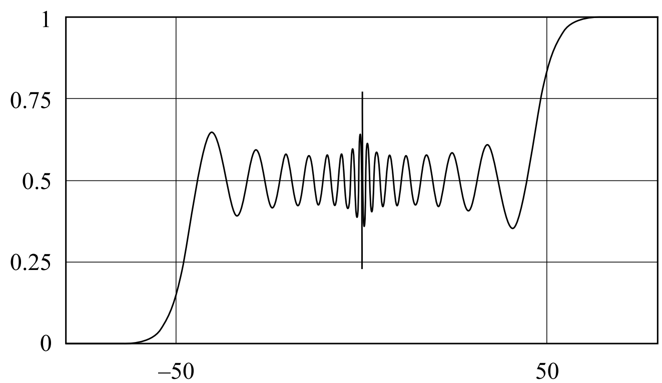

Fig. 1. $b = 0$, Re $U(x, 50)$, $le = 80$

Figure 1 shows the numerical solution to the problem (1), (5) at $t = 50$, derived by the use of a second-order accurate implicit approximation for Eq. (1). At every time step the stepwise pursuit was produced. To implement the matrix stepwise pursuit, the REDUCE computer algebra system [5] was used.

2. PROBLEM CORRECTNESS

The problem (2),(5) is ill posed (in the sense of Hadamard [6]). For all real $b \neq 0$, the function $\exp(at + icx)$ is a solution of Eq. (2), provided the parameters a and c satisfy

$$a^2 = -c^2 + bc^3 - a^2c^2, \quad a = \pm ic\sqrt{\frac{1 - bc}{1 + c^2}} = \pm d.$$

Consider $c_k = bk, d_k = d(c_k)$. The functions

$$U_k = \frac{\exp(d_k t + ibkx) + \exp(-d_k t + ibkx)}{k}$$

are solutions of Eq. (2) for all positive integers k. Remark that the initial data of these functions and their partial derivatives in t tend to zero when $k \to \infty$. At the same time, the asymptotic expansion

$$U_k = \frac{\exp\left(|b|\sqrt{k}(1 + O(k^{-1}))t + ibkx\right)}{k} + O(k^{-1})$$

holds for large k. Hence, U_k does not tend to zero when $k \to \infty$. Thus, the Cauchy problem is ill posed (in the sense of Hadamard) for Eq. (2).

The problem (3), (5) is posed correctly for $|b| < 2$ (in the sense of Petrovskii [7]). A system $\partial\bar{U}/\partial t = P(i\partial/\partial x)\bar{U}$ with constant coefficients is well posed (in the sense of Petrovskii) if $\operatorname{Re}\lambda_j(\sigma) < c$, where $s = \sigma + i\tau$, $\lambda_j(\sigma)$ are roots of the equation $\det||P(s) - \lambda E|| = 0$. I. G. Petrovskii has proved [7] that the Cauchy problem for the considered system is well posed in a class of restricted initial data. In the case of Eq. (3), $\lambda_{1,2} = \pm is/\sqrt{(1 - bs + s^2)}$. For $b > 2$ and $\sigma < \sigma_0 = b/2 + \sqrt{b^2/4 - 1}$ the characteristic polynomial has two real roots

$$\pm\frac{\sigma}{\sqrt{b^2/4 - 1 - (\sigma - b/2)^2}},$$

which are growing infinitely when $\sigma \to \sigma_0$. If $b < 2$, the eigenvalues $\lambda_j(\sigma)$ are imaginary for all real σ, and this guarantees the well-posedness of the problem (3), (5).

3. DEFORMATION OF THE BREATHER-TYPE SOLUTION

Numerical experiments at $b \neq 0$ pointed to the appearance of the breather-type solutions again. Figures 2 and 3 show the real part of the solution to the problem (3), (5) at $t = 50$ for $b = 1$ and $b = 1.5$, respectively. Numerically instead of the original Cauchy problem we solve an initial-boundary value problem on the interval $|x| \leqslant le$. At sufficiently large t, there are obvious differences between the solution in the case $b = 0$ and the real part of the solution in the case $b \neq 0$. In the case $b = 0$ the breather stands out against oscillations of a smaller amplitude (Fig. 1). These oscillations are located between the characteristics $x \pm t = 0$. Outside the region between the characteristics, the solution tends exponentially rapidly to the

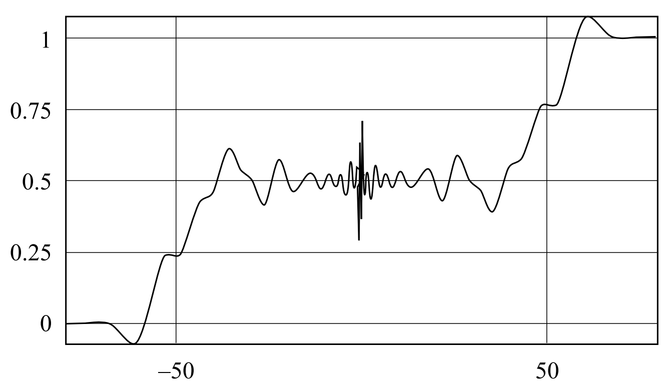

Fig. 2. $b = 1$, Re $U(x, 50)$, $le = 80$

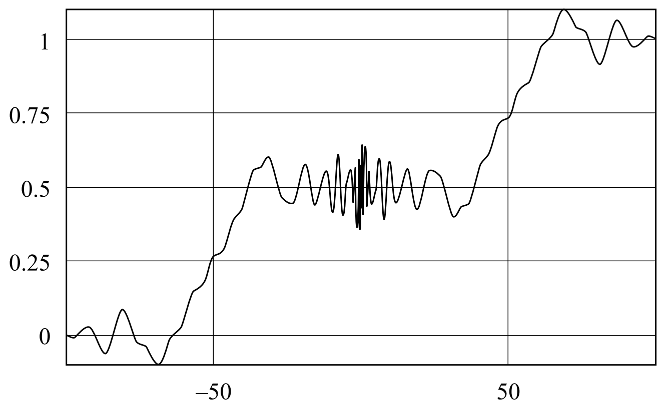

Fig. 3. $b = 1.5$, Re $U(x, 50)$, $le = 100$

limit values 0 when $x \to -\infty$ and 1 when $x \to +\infty$. Even at $t = 9$ (in the case $b = 1$), steps near characteristics are present. When $b \neq 0$ the amplitude oscillations decrease, while the region of oscillations extends. No distinct exponential decay to limiting values is observed any more at $|x| > t$.

In [9] we derived asymptotics (as $t \to \infty$) for the problem (3), (5) in the case $b = 1$. They confirm the validity of the breather deformation processes detected in the numerical simulation. Specifically, it was found that the support of the breathers is reduced to $|x| < c/t$ (against $|x| < c/\sqrt{t}$ in the case $b = 0$) and the exponential decay to the limiting values occurs at $|x| = 1.215\ldots t$ (against $|x| = t$ at $b = 0$). As b increases, the oscillation zone expands and in the limit case $|b| = 2$ occupies the whole real line. The boundaries of the oscillation zone are easily calculated via the multiple critical points of the function $s/\sqrt{1 - bs + s^2}$. The asymptotics are derived using complex variable methods — in particular the stationary phase method and the saddle point method [10]. To derive the asymptotic expansions [10], following [11], the solution to the problem (3), (5) is represented in the form of the contour integral

$$u = -\frac{1}{4\pi i} \int\limits_{\Gamma} \left(\exp\left(-\frac{ist}{\sqrt{1 - bs + s^2}} - ixs \right) + \exp\left(\frac{ist}{\sqrt{1 - bs + s^2}} - ixs \right) \right) \frac{ds}{s}.$$

The contour Γ goes along the real line except a neighborhood of zero: the pole is rounded in the upper half plane over a semicircle of small radius.

4. EXCITATION OF AN EXTENDED ZONE OF SLOWLY DAMPED OSCILLATIONS

The solution of the Cauchy problem (4), (5) shows an extended zone of slowly damped oscillations for large t [8]:

$$U(x,t) = \frac{1 + \operatorname{sgn}(x)}{2} - \frac{\operatorname{sgn}(x)}{\sqrt{\pi x^2/t}} \sin\left\{ \frac{x^2}{4t} - \frac{\pi}{4} \right\} (1 + O(t^{-1})), \quad t \to \infty, \quad |x| > t^{1+\delta}.$$

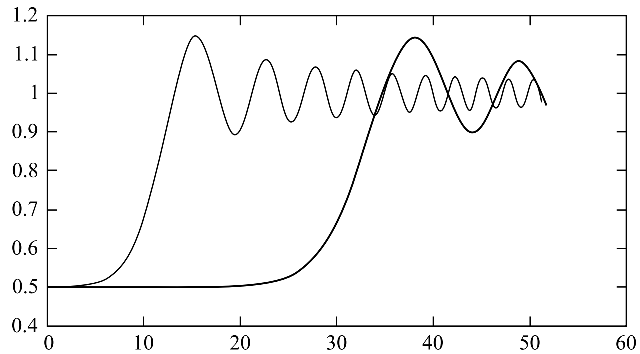

Fig. 4

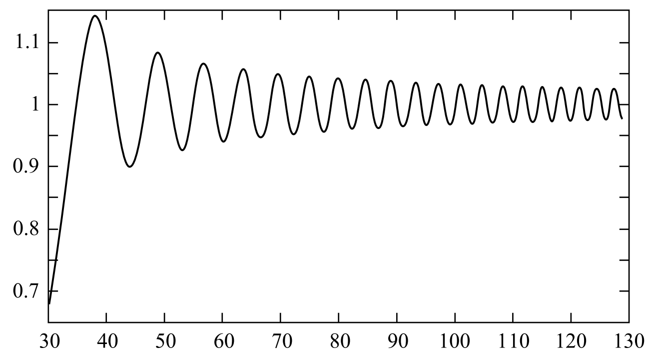

Fig. 5

These oscillations are excited for $|x| > t$. When $|x| < (1-\delta)t$ the solution differs from 1/2 by an exponentially small value in t. The oscillations happen near the limit values 0 for $x < -t$ and 1 for $x > t$.

The existence of a wide zone of slowly damping oscillations for this Cauchy problem was recovered within the outputs of the numerical experiments. In the numerical simulation Eq. (4) was approximated by the second-order accuracy explicit difference scheme

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} - \frac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-1}^n}{h^4},$$

which is stable in L_2 for $\tau < h^2/\sqrt{4+h^2}$. In the computation we used $h = 0.1$ and $\tau = h^2/\sqrt{2(4+2h^2)}$.

Figure 4 shows the numerical solutions over $0 < x < 55$ at $t = 10$ and $t = 30$, respectively. The graph corresponding to $t = 10$ increases faster in the neighborhood of $x = 0$. In this graph, one can clearly see slowly damping oscillations to the right of the characteristic $x - t = 0$. In the graph corresponding to $t = 30$ there is a distinct horizontal segment to the left of the characteristic. In Fig. 5, slowly damping oscillations of $u(x, 30)$ to the right of the characteristic are clearly seen, for $x > 30$.

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