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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

DKP EQUATION UNDER SCALAR AND VECTOR CORNELL INTERACTIONS

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Approximate analytical solutions of Duffin–Kemmer–Petiau (DKP) equation are obtained via the ansatz approach for scalar and vector interactions of Cornell type, and spectrum of the system is numerically reported.

Приближенное решение уравнения Даффина–Кеммера–Петьо (ДКП) и спектр системы получены численно в предположении скалярного и векторного взаимодействий корнеллевского типа.

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INTRODUCTION

The DKP equation (named after Duffin, Kemmer and Petiau) appeared more than seventy years ago and introduced a basis which enabled the theoretical physicists to investigate both spin-zero and spin-one particles on a single equation in the relativistic regime [1–4]. The spin-zero representation of the equation under a vector potential possesses the same mathematical structure of its well-known counterpart, i.e., the Klein–Gordon (KG) equation. As a result, the physical community thought the equations were completely equivalent. Now, however, we know that the equivalence might be generally violated [5–14]. This particularly occurs when symmetry is broken in hadronic processes. In addition, the DKP equation is richer in the study of interactions than the KG equation, and in some cases the results are voting in favor of it [15–20]. To be more precise, the investigations on the equation within the contexts of quantum chromodynamics (QCD) [21] and covariant Hamiltonian [22], Aharonov–Bohm potential [23, 24], five-dimensional Galilean invariance [25], casual approach [26, 27], scattering of some states of kaons [28], Dirac oscillator interaction [29], study of thermodynamics properties [30] and some special shape of interactions raise serious doubts on the equivalence [31]. We can also mention some decay modes of K_{l3} mesons in which the results obtained from KG and DKP equations are different [4–7]. Nevertheless, the question of equivalence is quite open to debate.

As already mentioned, the DKP equation under a vector term [31–35] resembles the KG equation and can be investigated by the well-known techniques of mathematical physics

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such as supersymmetry quantum mechanics (SUSYQM), Nikiforov–Uvarov (NU) technique, asymptotic iteration method (AIM) and series expansion [25–47]. The problem just arises as we intend to explore the equation under a scalar potential. We will later see that this is because we face the Heun equation which is not solvable even for the simplest cases we might think of.

As the DKP equation has been successfully tested in particle physics [48–52], we intend to work on the Cornell potential within the framework of this equation. The latter contains both short and long range terms and is often considered as the quark–antiquark potential [53].

Within the present work, we briefly review the DKP equation under scalar and vector potentials. Next, using successive transformations, we obtain approximate analytical solutions of the equation under the Cornell interaction and report the energy spectrum for some typical values of the potential parameters.

1. DKP EQUATION

The DKP Hamiltonian for scalar and vector interactions is

$$\left(\beta \cdot \mathbf{p}c + mc^2 + U_s + \beta^0 U_v^0\right)\psi(\mathbf{r}) = \beta^0 E\psi(\mathbf{r}), \tag{1}$$

where

$$\psi(\mathbf{r}) = \begin{pmatrix} \psi_{\text{upper}} \\ i\psi_{\text{lower}} \end{pmatrix}, \tag{2}$$

with the upper and lower components respectively being

$$\psi_{\text{upper}} \equiv \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \qquad \psi_{\text{lower}} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{3}$$

β^0 is the usual 5×5 matrix and U_s, U_v^0 respectively represent the scalar and vector interactions. The equation, in $(3+0)$-dimensions, is written as [1–4]

$$\begin{gathered} \left(mc^2 + U_s\right)\phi = \left(E - U_v^0\right)\varphi + \hbar c\boldsymbol{\nabla}\cdot\mathbf{A}, \quad \boldsymbol{\nabla}\phi = \left(mc^2 + U_s\right)\mathbf{A}, \\ \left(mc^2 + U_s\right)\varphi = \left(E - U_v^0\right)\phi, \end{gathered} \tag{4}$$

where $\mathbf{A} = (A_1, A_2, A_3)$. In Eq. (3), ψ is a simultaneous eigenfunction of J^2 and J_3, i.e.,

$$\begin{gathered} J^2 \begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix} = \begin{pmatrix} L^2\psi_{\text{upper}} \\ (L+S)^2\psi_{\text{lower}} \end{pmatrix} = J(J+1)\begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix}, \\ J_3 \begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix} = \begin{pmatrix} L_3\psi_{\text{upper}} \\ (L_3+s_3)\psi_{\text{lower}} \end{pmatrix} = M\begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix}, \end{gathered} \tag{5}$$

and the general solution is considered as

$$\psi_{JM}(r) = \begin{pmatrix} f_{nJ}(r)Y_{JM}(\Omega) \\ g_{nJ}(r)Y_{JM}(\Omega) \\ i\sum\limits_{L} h_{nJL}(r)Y_{JL1}^{M}(\Omega) \end{pmatrix}, \tag{6}$$

where spherical harmonics $Y_{JM}(\Omega)$ are of order J; $Y_{JL1}^{M}(\Omega)$ are the normalized vector spherical harmonics, and f_{nJ}, g_{nJ} and h_{nJL} represent the radial wavefunctions. The above equations yield the coupled differential equations [36–45]

$$\begin{gathered}\left(E_{n,J}-U_v^0\right)F_{n,J}(r)=\left(mc^2+U_s\right)G_{n,J}(r),\\ \left(\frac{dF_{n,J}(r)}{dr}-\frac{J+1}{r}F_{n,J}(r)\right)=-\frac{1}{\alpha_J}\left(mc^2+U_s\right)H_{1,n,J}(r),\\ \left(\frac{dF_{n,J}(r)}{dr}+\frac{J}{r}F_{n,J}(r)\right)=\frac{1}{\zeta_J}\left(mc^2+U_s\right)H_{-1,n,J}(r),\\ -\alpha_J\left(\frac{dH_{1,n,J}(r)}{dr}+\frac{J+1}{r}H_{1,n,J}(r)\right)+\zeta\left(\frac{dH_{-1,n,J}(r)}{dr}-\frac{J}{r}H_{-1,n,J}(r)\right)=\\ =\frac{1}{\hbar c}\left(\left(mc^2+U_s\right)F_{n,J}(r)-\left(E_{n,J}-U_v^0\right)G_{n,J}(r)\right),\end{gathered}\tag{7}$$

which give [47]

$$\begin{aligned}&\frac{d^2F_{n,J}(r)}{dr^2}\left[1+\frac{\zeta_J^2}{\alpha_J^2}\right]-\frac{dF_{n,J}(r)}{dr}\left[\frac{U_s'}{(m+U_s)}\left(1+\frac{\zeta_J^2}{\alpha_J^2}\right)\right]+\\ &\quad+F_{n,J}(r)\Bigg[-\frac{J\left(J+1\right)}{r^2}\left(1+\frac{\zeta_J^2}{\alpha_J^2}\right)+\frac{U_s'}{(m+U_s)}\left(\frac{J+1}{r}-\frac{\zeta_J^2}{\alpha_J^2}\frac{J}{r}\right)-\\ &\qquad-\frac{1}{\alpha_J^2}\left(\left(m+U_s\right)^2-\left(E_{n,J}-U_v^0\right)^2\right)\Bigg]=0,\end{aligned}\tag{8a}$$

where

$$\alpha_J=\sqrt{\frac{J+1}{2J+1}},\quad f_{nJ}(r)=\frac{F(r)}{r},\quad g_{nJ}(r)=\frac{G(r)}{r},\quad h_{nJJ\pm1}=\frac{H_{\pm1}}{r}\quad\text{and}\quad\zeta_J=\sqrt{\frac{J}{2J+1}}.$$

When $U_s=0$, we recover the well-known formula [36–45]

$$\left(\frac{d^2}{dr^2}-\frac{J\left(J+1\right)}{r^2}+\left(E_{n,J}-U_v^0\right)^2-m^2\right)F_{n,J}(r)=0.\tag{8b}$$

Before proceeding further, it should be noted DKP equation in some cases appears just as different types of Heun equation. For example, for vanishing scalar term and a Kratzer vector interaction, the equation resembles the Double Confluent Heun (DCH) equation, i.e. [53]

$$y''+\left(A+\frac{B}{x}+\frac{C}{x^2}+\frac{D}{x^3}+\frac{A}{x^4}\right)y=0.\tag{9a}$$

In the presence of the scalar term, we are actually dealing with a rather general form of the Heun equation

$$y''+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\varepsilon}{x-a}\right)y'+\frac{\alpha\beta x-q}{x\left(x-1\right)\left(x-a\right)}y=0,\tag{9b}$$

where $\gamma+\delta+\varepsilon=\alpha+\beta+1$, $a\neq0$ and $a\neq1$ [53]. The Heun differential equation is the more general form of hypergeometric equations. Therefore, it goes without saying that the difficulty doubles as the Heun equation has not been yet solved even for some cases we frequently face.

We consider the following Cornell vector and scalar potentials:

$$U_s = a_s r + \frac{b_s}{r}, \tag{10a}$$

$$U_v^0 = a_v r + \frac{b_v}{r}, \tag{10b}$$

which has been successfully tested in particle and nuclear physics [52]. The inverse term arises from the one-gluon exchange between the quark and its antiquark, and the linear term is included to take into account confining phenomena. In its spin-averaged form, the Cornell potential is represented as $-(4/3)(\alpha_s/r) + kr$, where r, k and α_s respectively represent the interquark distance, confinement constant and the quark–gluon (or strong) coupling [52, 54].

Substitution of the potentials in Eq. (8) gives

$$\begin{aligned}\frac{d^2F_{n,J}(r)}{dr^2} - \frac{dF_{n,J}(r)}{dr}\left[\frac{a_s - b_s/r^2}{m + a_s r + b_s/r}\right] + \\ + F_{n,J}(r)\Bigg[- \frac{J(J+1)}{r^2} + \frac{C}{A}\left(\frac{a_s - b_s/r^2}{r[m + a_s r + b_s/r]}\right) - \\ - \frac{1}{\alpha_J^2 A}\left\{\left(m + a_s r + \frac{b_s}{r}\right)^2 - \left(E_{n,J} - a_v r - \frac{b_v}{r}\right)^2\right\}\Bigg] = 0,\end{aligned} \tag{11}$$

where

$$A = 1 + \frac{\zeta_J^2}{\alpha_J^2}, \tag{12a}$$

$$C = J + 1 - \frac{\zeta_J^2}{\alpha_J^2} J. \tag{12b}$$

2. SOLUTION OF THE PROBLEM VIA THE ANSATZ APPROACH

The change of variable

$$F_{n,J}(r) = \sqrt{m + a_s r + \frac{b_s}{r}}\ \phi_{n,J}(r) \tag{13}$$

brings Eq. (11) into the form

$$\begin{aligned}\frac{d^2\phi_{n,J}(r)}{dr^2} + \Bigg[\frac{b_s}{a_s}\frac{1}{r^2\left(r^2 + \frac{m}{a_s}r + \frac{b_s}{a_s}\right)} - \frac{3r^2}{4\left(r^2 + \frac{m}{a_s}r + \frac{b_s}{a_s}\right)^2} + \\ + \frac{3b_s}{2a_s}\frac{1}{\left(r^2 + \frac{m}{a_s}r + \frac{b_s}{a_s}\right)^2} - \frac{Cb_s}{Aa_s}\frac{1}{r^2\left(r^2 + \frac{m}{a_s}r + \frac{b_s}{a_s}\right)} - \frac{3b_s^2}{4a_s^2}\frac{1}{r^2\left(r^2 + \frac{m}{a_s}r + \frac{b_s}{a_s}\right)^2} -\end{aligned}$$

$$-\left(J\left(J+1\right)+\frac{(b_s^2-b_v^2)}{A\alpha_J^2}\right)\frac{1}{r^2}+\frac{C}{A}\frac{1}{\left(r^2+\frac{m}{a_s}r+\frac{b_s}{a_s}\right)}-$$
$$-\frac{1}{A\alpha_J^2}(m^2+2a_sb_s-E_{n,J}^2-2a_vb_v)-\frac{(a_s^2-a_v^2)}{A\alpha_J^2}r^2-$$
$$\left.-\frac{(2ma_s+2E_{n,J}a_v)}{\alpha_J^2A}r-\frac{(2mb_s+2E_{n,J}b_v)}{\alpha_J^2A}\frac{1}{r}\right]\phi_{n,J}(r)=0. \quad (14)$$

After decomposition of fractions, we arrive at

$$\frac{d^2\phi_{n,J}}{dr^2}+\left[\left(-\frac{3b_2}{4}+\frac{3b_sb_3}{2a_s}-\frac{3b_s^2h}{4a_s^2}\right)\frac{1}{(r+r_1)^2}+\left(-\frac{3d_2}{4}+\frac{3b_sd_3}{2a_s}-\frac{3b_s^2N}{4a_s^2}\right)\frac{1}{(r+r_2)^2}+\right.$$
$$+\left(\frac{b_sd_1}{a_s}-\frac{3c_2}{4}+\frac{3b_sc_3}{2a_s}-\frac{3b_s^2M}{4a_s^2}+\frac{cb_5}{A}-\frac{cb_sd_1}{a_sA}\right)\frac{1}{(r+r_2)}+$$
$$+\left(\frac{b_sc_1}{a_s}-\frac{3a_2}{4}+\frac{3b_sa_3}{2a_s}-\frac{3b_s^2g}{4a_s^2}+\frac{ca_5}{A}-\frac{cb_sc_1}{Aa_s}\right)\frac{1}{(r+r_1)}+$$
$$+\left(\frac{b_sa_1}{a_s}-\frac{3b_s^2s}{4a_s^2}-\frac{(2mb_s+2E_{n,J}b_v)}{A\alpha_J^2}-\frac{cb_sa_1}{a_sA}\right)\frac{1}{r}+$$
$$+\left(\frac{b_sb_1}{a_s}-\frac{3b_s^2f}{4a_s^2}-J(J+1)+\frac{(-b_s^2+b_v^2)}{A\alpha_J^2}-\frac{cb_sb_1}{Aa_s}\right)\frac{1}{r^2}-$$
$$-\frac{(2ma_s+2E_{n,J}a_v)}{A\alpha_J^2}r-\frac{(a_s^2-a_v^2)}{A\alpha_J^2}r^2-$$
$$\left.-\frac{1}{A\alpha_J^2}(m^2+2a_sb_s-E_{n,J}^2-2a_vb_v)\right]\phi_{n,J}(r)=0, \quad (15)$$

where

$$r_1=\frac{\frac{m}{a_s}+\sqrt{\frac{m^2}{a_s^2}-4\frac{b_s}{a_s}}}{2}, \quad r_2=\frac{\frac{m}{a_s}-\sqrt{\frac{m^2}{a_s^2}-4\frac{b_s}{a_s}}}{2},$$
$$a_1=-\frac{r_1+r_2}{(r_1r_2)^2}, \quad d_1=\frac{1}{(r_1-r_2)r_2^2}, \quad b_1=\frac{1}{r_1r_2}, \quad c_1=-a_1-d_1,$$
$$a_2=\frac{2r_2r_1}{(r_1-r_2)^3}, \quad c_2=-\frac{2r_2r_1}{(r_1-r_2)^3}, \quad b_2=\frac{r_1^2}{(r_1-r_2)^2}, \quad d_2=\frac{r_2^2}{(r_1-r_2)^2},$$
$$a_3=\frac{2}{(r_1-r_2)^3}, \quad c_3=-\frac{2}{(r_1-r_2)^3}, \quad b_3=\frac{1}{(r_2-r_1)^2}, \quad d_3=\frac{1}{(r_2-r_1)^2}, \quad (16)$$

$$s=-\frac{2(r_2+r_1)}{r_1^3r_2^3},\quad f=\frac{1}{(r_1r_2)^2},\quad h=\frac{1}{r_1^2(r_1-r_2)^2},$$
$$g=\frac{2(2r_1-r_2)}{r_1^3(r_1-r_2)^3},\quad M=\frac{2(-2r_2+r_1)}{r_2^3(r_1-r_2)^3},\quad N=\frac{1}{r_2^2(r_1-r_2)^2},$$
$$a_5=\frac{1}{r_2-r_1},\quad b_5=\frac{1}{r_1-r_2}.$$

Let us now consider an ansatz of the form

$$\phi_{n,J}(r)=f_n(r)\exp\left(g_J(r)\right), \tag{17}$$

with

$$f_n(r)=\begin{cases}1, & n=0\\ \prod_{i=1}^{n}(r-\alpha_i^n), & n>0,\end{cases} \tag{18a}$$

$$g_J(r)=\beta\ln(r+r_1)+\gamma\ln(r+r_2)+\xi\ln(r)+\delta r+\eta r^2. \tag{18b}$$

Substitution of the above ansatz in Eq. (15) yields

$$\begin{aligned}\phi''_{0,J}(r)=\Bigg[&(\beta^2-\beta)\frac{1}{(r+r_1)^2}+(\gamma^2-\gamma)\frac{1}{(r+r_2)^2}+\left(2\delta\gamma-4\gamma\eta r_2+2\beta\gamma b_5-\frac{2\xi\gamma}{r_2}\right)\frac{1}{(r+r_2)}+\\&+\left(2\beta\delta-4\eta\beta r_1-\frac{2\xi\beta}{r_1}+2\beta\gamma a_5\right)\frac{1}{(r+r_1)}+\left(2\xi\delta+\frac{2\beta\xi}{r_1}+\frac{2\gamma\xi}{r_2}\right)\frac{1}{r}+(\xi^2-\xi)\frac{1}{r^2}+\\&+4\eta^2r^2+(4\delta\eta)r+(\delta^2+2\eta+4\eta\beta+4\eta\gamma+4\eta\xi)\Bigg]\phi_{0,J}(r).\end{aligned} \tag{19}$$

Equating the coefficients on both sides, we find

$$\beta^2-\beta=-\left(-\frac{3b_2}{4}+\frac{3b_sb_3}{2a_s}-\frac{3b_s^2h}{4a_s^2}\right), \tag{20a}$$

$$\gamma^2-\gamma=-\left(-\frac{3d_2}{4}+\frac{3b_sd_3}{2a_s}-\frac{3b_s^2N}{4a_s^2}\right), \tag{20b}$$

$$2\delta\gamma-4\gamma\eta r_2+2\beta\gamma b_5-\frac{2\xi\gamma}{r_2}=-\left(\frac{b_sd_1}{a_s}-\frac{3c_2}{4}+\frac{3b_sc_3}{2a_s}-\frac{3b_s^2M}{4a_s^2}+\frac{cb_5}{A}-\frac{cb_sd_1}{a_sA}\right), \tag{20c}$$

$$2\beta\delta-4\eta\beta r_1-\frac{2\xi\beta}{r_1}+2\beta\gamma a_5=-\left(\frac{b_sc_1}{a_s}-\frac{3a_2}{4}+\frac{3b_sa_3}{2a_s}-\frac{3b_s^2g}{4a_s^2}+\frac{ca_5}{A}-\frac{cb_sc_1}{Aa_s}\right), \tag{20d}$$

$$2\xi\delta+\frac{2\beta\xi}{r_1}+\frac{2\gamma\xi}{r_2}=-\left(\frac{b_sa_1}{a_s}-\frac{3b_s^2s}{4a_s^2}-\frac{(2mb_s+2E_{n,J}b_v)}{A\alpha_J^2}-\frac{cb_sa_1}{a_sA}\right), \tag{20e}$$

$$\xi^2-\xi=-\left(\frac{b_sb_1}{a_s}-\frac{3b_s^2f}{4a_s^2}-J(J+1)+\frac{(-b_s^2+b_v^2)}{A\alpha_J^2}-\frac{cb_sb_1}{Aa_s}\right), \tag{20f}$$

$$4\eta^2=\frac{(a_s^2-a_v^2)}{A\alpha_J^2}, \tag{20g}$$

$$4\delta\eta = \frac{(2ma_s + 2E_{n,J}a_v)}{A\alpha_J^2}, \tag{20h}$$

$$\delta^2 + 2\eta + 4\eta\beta + 4\eta\gamma + 4\eta\xi = \frac{1}{A\alpha_J^2}(m^2 + 2a_sb_s - E_{n,J}^2 - 2a_vb_v), \tag{20i}$$

where

$$\beta = \frac{1 \pm \sqrt{1 - 4\left(-\frac{3b_2}{4} + \frac{3b_sb_3}{2a_s} - \frac{3b_s^2h}{4a_s^2}\right)}}{2}, \tag{21a}$$

$$\gamma = \frac{1 \pm \sqrt{1 - 4\left(-\frac{3d_2}{4} + \frac{3b_sd_3}{2a_s} - \frac{3b_s^2N}{4a_s^2}\right)}}{2}, \tag{21b}$$

$$\xi = \frac{1 \pm \sqrt{1 - 4\left(\frac{b_sb_1}{a_s} - \frac{3b_s^2f}{4a_s^2} - J(J+1) + \frac{(-b_s^2 + b_v^2)}{A\alpha_J^2} - \frac{cb_sb_1}{Aa_s}\right)}}{2}, \tag{21c}$$

$$\eta = \pm\sqrt{\frac{(a_s^2 - a_v^2)}{4A\alpha_J^2}}, \tag{21d}$$

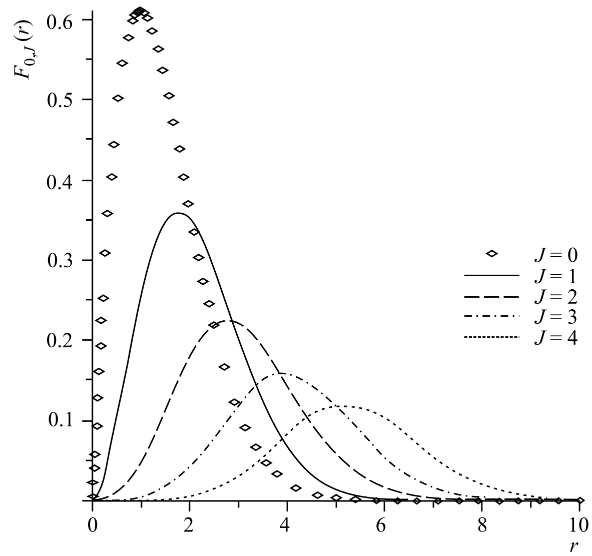
$$\delta = -\frac{1}{2\beta}\left(\frac{b_sc_1}{a_s} - \frac{3a_2}{4} + \frac{3b_sa_3}{2a_s} - \frac{3b_s^2g}{4a_s^2} + \frac{ca_5}{A} - \frac{cb_sc_1}{Aa_s} - 4\eta\beta r_1 - \frac{2\xi\beta}{r_1} + 2\beta\gamma a_5\right). \tag{21e}$$

By combining Eqs. (20c), (20h), (20e) and (20i), one can find

$$\begin{aligned}2\xi\delta + \frac{2\beta\xi}{r_1} + \frac{4\gamma\xi}{r_2} - 2\delta\gamma + 4\gamma\eta r_2 - 2\beta\gamma b_5 + 4\delta\eta + \delta^2 + 2\eta + 4\eta\beta + 4\eta\gamma + 4\eta\xi = \\ = -\left(\frac{b_sa_1}{a_s} - \frac{3b_s^2s}{4a_s^2} - \frac{(2mb_s + 2E_{n,J}b_v)}{A\alpha_J^2} - \frac{cb_sa_1}{a_sA}\right) + \\ + \left(\frac{b_sd_1}{a_s} - \frac{3c_2}{4} + \frac{3b_sc_3}{2a_s} - \frac{3b_s^2M}{4a_s^2} + \frac{cb_5}{A} - \frac{cb_sd_1}{a_sA}\right) + \\ + \frac{(2ma_s + 2E_{n,J}a_v)}{A\alpha_J^2} + \frac{1}{A\alpha_J^2}(m^2 + 2a_sb_s - E_{n,J}^2 - 2a_vb_v).\end{aligned} \tag{22}$$

Energy and normalization constant for
$a_s = 0.4$**,** $b_s = 0.3$**,** $a_v = 0.2$**,** $b_v = 0.1$**,** $m = 1$
$\eta = -0.1732050808$**,** $\gamma = -0.5000000005$**,** $\beta = -0.500000000$

$\|n, J\rangle$	$E_{n,J}$	ξ	δ	$N_{n,J}$
$\|0,0\rangle$	2.637824618	1.539230485	–0.7270276355	2.372602902
$\|0,1\rangle$	3.495550051	2.254992878	–0.3943296503	0.7201493353
$\|0,2\rangle$	4.112492347	3.160826939	0.0267167337	0.0829224123
$\|0,3\rangle$	4.430201988	4.116628264	0.4709887127	0.004017137757
$\|0,4\rangle$	4.480450270	5.091296113	0.9240301600	0.000085978771



Wave function vs. r

Therefore, the spectrum and the eigenfunctions of the system are obtained. We have reported the energy of the system and the normalization constant in the table. The wavefunction is depicted in the figure. We wish to mention one important point before giving our concluding remarks. We think working on the Heun equation stemming from the scalar interaction is an interesting basis to investigate the equivalence or nonequivalence of KG and DKP equations. This is, however, too deep to be the purpose of a single manuscript as the general solution of the Heun equation is an unsolved problem within the annals of differential equations.

CONCLUSION

After applying some appropriate transformations and introducing an elegant ansatz, we have obtained approximate analytical solutions of DKP equation under the Cornell potential. Apart from the application of Cornell potential in particle physics, the solution is mathematically interesting as we have in fact obtained the solutions of the corresponding Heun equation which is, if not the most difficult, among the very difficult differential equations of mathematical physics which is yet quite open to debate. We hope our work motivates further studies on the DKP equation under scalar interaction and some oriented studies on the corresponding Heun equation.

APPENDIX

The nonrelativistic limit of Eq. (8b) can be easily checked by the way we do for the KG equation. For the sake of simplicity, let us consider the case of vanishing scalar and vector interactions, i.e.,

$$\left(\nabla^2 + E_{n,J}^2 - m^2\right) \psi_{n,J}(\mathbf{r}, t) = 0, \tag{A.1}$$

or equivalently

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2\right)\psi_{n,J}(\mathbf{r},t) = 0. \tag{A.2}$$

Let us define the relations

$$\psi_{n,J}(\mathbf{r},t) = \varphi_{n,J}(\mathbf{r},t)\exp\left(-imt\right), \tag{A.3}$$

$$E_{n,J} = E'_{n,J} - mc^2, \tag{A.4}$$

or

$$\left|i\hbar\frac{\partial\varphi_{n,J}(\mathbf{r},t)}{\partial t}\right| \approx E_{n,J}\varphi_{n,J}(\mathbf{r},t) - mc^2\varphi_{n,J}(\mathbf{r},t). \tag{A.5}$$

Using the above equations, the derivates can be written as

$$\frac{\partial\psi}{\partial t} = \left(\frac{\partial\varphi}{\partial t} - im\varphi\right)\exp(-imt) \approx -im\varphi\exp\left(-imt\right), \tag{A.6a}$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\left(\frac{\partial\varphi}{\partial t} - im\varphi\right)\exp\left(-imt\right)\right) \approx -\left[2im\frac{\partial\varphi}{\partial t} + m^2\varphi\right]\exp\left(-imt\right). \tag{A.6b}$$

Substitution of Eq. (A.6b) in Eq. (A.3) gives

$$\left(\nabla^2 + \left[2im\frac{\partial}{\partial t} + m^2\right] - m^2\right)\varphi(\mathbf{r},t)\exp\left(-imt\right) = 0, \tag{A.7}$$

or

$$-\frac{1}{2m}\nabla^2\varphi(\mathbf{r},t) = i\frac{\partial}{\partial t}\varphi(\mathbf{r},t), \tag{A.8}$$

which is, as we expect, the nonrelativistic Schrödinger equation.

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