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SOLUTIONS TO THE DIRAC EQUATION FOR SYMMETRIC AND ASYMMETRIC TRIGONOMETRIC ROSEN–MORSE POTENTIAL USING SUSYQM

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In this paper, we calculate the energy spectra and the corresponding wave function for the symmetric and asymmetric trigonometric Rosen–Morse potential of the Dirac equation within the framework of spin and pseudospin symmetry limits including the tensor interaction using the supersymmetric quantum mechanics (SUSYQM) formalism. We have also reported some numerical results and figures to show the effect of tensor interaction.

В статье выполнен расчет спектра энергий и соответствующей ему волновой функции для симметричного и асимметричного тригонометрического потенциала Розена–Морзе в уравнении Дирака в предположении симметрии по спину и псевдоспину. В расчете учитывалось тензорное взаимодействие с помощью формализма суперсимметричной квантовой механики (SUSYQM). Также приводятся некоторые численные результаты и рисунки, иллюстрирующие эффект тензорного взаимодействия.

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INTRODUCTION

Dirac equation is the well-known equation that describes spin-half particles in relativistic quantum mechanics. Also, it is well known that the study of exactly solvable problems in quantum mechanics has attracted great attention in recent times [1]. In order to investigate nuclear shell model, the study of spin and pseudospin symmetries of the Dirac equation has been an important area of research in nuclear physics [2]. Within the framework of the Dirac theory, the spin symmetry occurs when the difference of the potential between the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant; that is, $\Delta(r) = V(r) - S(r) = \text{const}$. On the other hand, the pseudopsin symmetry arises when the sum of the potential of the repulsive Lorentz vector potential

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$V(r)$and attractive Lorentz scalar potential $S(r)$ is a constant; that is, $\Sigma(r) = V(r) + S(r) = \text{const}$. By considering relativistic theories with scalar and vector potentials, Ginocchio showed that these symmetries could explain degeneracies in meson spectra or in single-particle energy levels [3–5]. The spin symmetry in nuclear theory is usually referred to as a quasi-degeneracy of the single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n, l, j = l - 1/2)$, where n, l and j are the single-nucleon radial, orbital and total angular momentum quantum numbers for a single particle, respectively [6]. Also, the pseudospin symmetry implies that $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$ states are degenerate [7]. Pseudospin and spin symmetries concept on a number of potentials have been investigated by many researchers in the field. These potentials include Woods–Saxon [8], Manning–Rosen [9], Eckart potential [10], harmonic oscillator [11], and Coulomb and Hartmann potentials [12] among others [13]. These concepts, under various phenomenological potentials, have been investigated by many authors, using various methods such as asymptotic iteration (AIM) [14], Nikiforov–Uvarov (NU) [15], supersymmetric quantum mechanics (SUSYQM) [16–18] and shape invariance (SI) [19], and exact quantization rule [20]. Similarly, the tensor interaction term was introduced into the Dirac equation with the replacement $\mathbf{p} \to \mathbf{p} - iM\omega\beta \cdot \hat{r}U(r)$ and a spin-orbit coupling is added to the Dirac Hamiltonian [21]. The Dirac equation with different potentials in relativistic quantum mechanics with spin and pseudospin symmetry has been investigated in recent years [22]. Also, other potentials models investigated are given in [20–33]. Recently, Ikhdair and Hamzavi [34] investigated the hyperbolic Rosen–Morse potential (RM) with Coulomb tensor interaction. The RM potential is one of the useful potentials in describing the interatomic interaction of the linear molecules and also in describing the polyatomic vibration energies of NH_3 molecules [34, 35]. It had been shown that the RM and its PT-symmetric version are special cases of the five-parameter exponential-type potential [36, 37]. Compean et al. [38] and Ma et al. [1] obtained the energy spectrum of the trigonometric RM potential using SUSYQM and improved quantization rule methods, respectively.

In the present paper, we intend to investigate the symmetric and asymmetric trigonometric RM potential with the Coulomb-like tensor interaction for arbitrary spin-orbit quantum number κ, using the SUSYQM formalism.

1. SUPERSYMMETRY

In the SUSYQM we normally deal with the partner Hamiltonians [16–18]

$$H_{\pm} = \frac{p^2}{2m} + V_{\pm}(x), \tag{1}$$

where

$$V_{\pm}(x) = \Phi^2(x) \pm \Phi'(x). \tag{2}$$

In the case of good SUSY, i.e., $E_0 = 0$, the ground state of the system is obtained via

$$\phi_0^-(x) = C\,\mathrm{e}^{-U}, \tag{3}$$

where C is a normalization constant and

$$U(x) = \int_{x_0}^{x} dz\, \Phi(z). \tag{4}$$

Next, if the shape invariant condition

$$V_+(a_0, x) = V_-(a_1, x) + R(a_1), \tag{5}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F : a_0 \mapsto a_1 = F(a_0)$ and $R(a_1)$ does not include x, the higher state solutions are obtained via

$$E_n = \sum_{s=1}^{n} R(a_s), \tag{6a}$$

$$\phi_n^-(a_0, x) = \prod_{s=0}^{n-1} \left(\frac{A^\dagger(a_s)}{[E_n - E_s]^{1/2}} \right) \phi_0^-(a_n, x), \tag{6b}$$

$$\phi_0^-(a_n, x) = C \exp\left\{ -\int_0^x dz\, \Phi(a_n, z) \right\}, \tag{6c}$$

where

$$A_s^\dagger = -\frac{\partial}{\partial x} + \Phi(a_s, x) \tag{7}$$

Therefore, this condition determines the spectrum of the bound states of the Hamiltonian

$$H_s = -\frac{\partial^2}{\partial x^2} + V_-(a_s, x) + E_s \tag{8}$$

and the energy eigenfunctions of

$$H_s \phi_{n-s}^-(a_s, x) = E_n \phi_{n-s}^-(a_s, x), \quad n \geqslant s \tag{9}$$

are related via [1–3]

$$\phi_{n-s}^-(a_s, x) = \frac{A^\dagger}{[E_n - E_s]^{1/2}}\, \phi_{n-(s+1)}^-(a_{s+1}, x). \tag{10}$$

2. DIRAC EQUATION WITH AND WITHOUT TENSOR COUPLING

The Dirac equation for spin-$1/2$ particles moving in an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $U(r)$ in the relativistic unit ($\hbar = c = 1$) is [25]

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta(M + S(r)) - i\beta\boldsymbol{\alpha} \cdot \hat{r} U(r)]\, \psi(r) = [E - V(r)]\, \psi(r), \tag{11}$$

where E is the relativistic energy of the system, $\mathbf{p} = -i\nabla$ is the three-dimensional momentum operator and M is the mass of the fermionic particle. $\boldsymbol{\alpha}$, β are the 4×4 Dirac matrices given as

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{12}$$

where I is 2×2 unitary matrix and $\boldsymbol{\sigma}_i$ are the Pauli three-vector matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

The eigenvalues of the spin-orbit coupling operator are $\kappa = (j + 1/2) \succ 0$, $\kappa = -(j + 1/2) \prec 0$ for unaligned $j = l - 1/2$ and aligned spin $j = l + 1/2$, respectively. The set (H, K, J^2, J_z) forms a complete set of conserved quantities. Thus, we can write the spinors as [26]

$$\psi_{n\kappa}(r) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r)\, Y^l_{jm}(\theta, \varphi) \\ iG_{n\kappa}(r)\, Y^{\tilde{l}}_{jm}(\theta, \varphi) \end{pmatrix}, \tag{14}$$

where $F_{n\kappa}(r)$, $G_{n\kappa}(r)$ represent the upper and lower components of the Dirac spinors; $Y^l_{jm}(\theta, \varphi)$, $Y^{\tilde{l}}_{jm}(\theta, \varphi)$ are the spin and pseudospin spherical harmonics; and m is the projection on the z axis. With other known identities [27],

$$\begin{aligned} (\boldsymbol{\sigma} \cdot \mathbf{A})\,(\boldsymbol{\sigma} \cdot \mathbf{B}) &= \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot (\mathbf{A} x \mathbf{B})\,, \\ \boldsymbol{\sigma} \cdot \mathbf{p} &= \boldsymbol{\sigma} \cdot \hat{r} \left(\hat{r} \cdot \mathbf{p} + i \frac{\boldsymbol{\sigma} \cdot \mathbf{L}}{r} \right), \end{aligned} \tag{15}$$

as well as

$$\begin{aligned} (\boldsymbol{\sigma} \cdot \mathbf{L})\, Y^{\tilde{l}}_{jm}(\theta, \varphi) &= (\kappa - 1) Y^{\tilde{l}}_{jm}(\theta, \varphi), \\ (\boldsymbol{\sigma} \cdot \mathbf{L})\, Y^{l}_{jm}(\theta, \varphi) &= -(\kappa - 1) Y^{l}_{jm}(\theta, \varphi), \\ (\boldsymbol{\sigma} \cdot \hat{r})\, Y^{l}_{jm}(\theta, \varphi) &= -Y^{\tilde{l}}_{jm}(\theta, \varphi), \\ (\boldsymbol{\sigma} \cdot \hat{r})\, Y^{\tilde{l}}_{jm}(\theta, \varphi) &= -Y^{l}_{jm}(\theta, \varphi), \end{aligned} \tag{16}$$

we find the following two coupled first-order Dirac equations [27]:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} \right) F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))\, G_{n\kappa}(r), \tag{17}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} \right) G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))\, F_{n\kappa}(r), \tag{18}$$

and the extension to tensor interaction becomes

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))\, G_{n\kappa}(r), \tag{19}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))\, F_{n\kappa}(r), \tag{20}$$

where

$$\Delta(r) = V(r) - S(r), \tag{21}$$

$$\Sigma(r) = V(r) + S(r). \tag{22}$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ in Eqs. (19) and (20), we obtain the second-order Schrödinger-like equation as

$$\left\{\begin{array}{l}\dfrac{d^2}{dr^2}-\dfrac{\kappa(\kappa+1)}{r^2}+\dfrac{2\kappa U(r)}{r}-\dfrac{dU(r)}{dr}-U^2(r)-(M+E_{n\kappa}-\Delta(r))\times\\ \times(M-E_{n\kappa}+\Sigma(r))+\dfrac{\dfrac{d\Delta(r)}{dr}\left(\dfrac{d}{dr}+\dfrac{\kappa}{r}-U(r)\right)}{(M+E_{n\kappa}-\Delta(r))}\end{array}\right\}F_{n\kappa}(r)=0, \tag{23}$$

$$\left\{\begin{array}{l}\dfrac{d^2}{dr^2}-\dfrac{\kappa(\kappa-1)}{r^2}+\dfrac{2\kappa U(r)}{r}+\dfrac{dU(r)}{dr}-U^2(r)-(M+E_{n\kappa}-\Delta(r))\times\\ \times(M-E_{n\kappa}+\Sigma(r))+\dfrac{\dfrac{d\Sigma(r)}{dr}\left(\dfrac{d}{dr}-\dfrac{\kappa}{r}+U(r)\right)}{(M+E_{n\kappa}-\Sigma(r))}\end{array}\right\}G_{n\kappa}(r)=0, \tag{24}$$

where $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$, $\kappa(\kappa+1)=l(l+1)$. The radial wave functions are required to satisfy the necessary conditions; that is, $F_{n\kappa}$ and $G_{n\kappa}$ vanish at the origin and at infinity. At this stage, we take $\Delta(r)$ or $\Sigma(r)$ as the symmetric and asymmetric potentials. Equations (23) and (24) can be exactly solved for $\kappa=0,-1$ and $\kappa=0,1$, respectively, as the spin-orbit centrifugal term vanishes.

3. SOLUTION OF THE DIRAC EQUATION FOR SYMMETRIC TRIGONOMETRIC POTENTIALS

In this section, we are going to solve the Dirac equation with symmetric trigonometric potential by using the SUSYQM.

3.1. Pseudospin and Spin Symmetry Limits for the Symmetric Trigonometric Potential. The exact pseudospin symmetry is proved by Meng et al. [28]. It occurs in Dirac equation when $\dfrac{d\Sigma(r)}{dr}=0$ or $\Sigma(r)=C_{\rm ps}=\text{const}$ [3–5]. In this limit, we take $\Delta(r)$ as the symmetric trigonometric potential [1] and a Coulomb-like potential [29] for the tensor potential added,

$$\Delta(r)=U_0\cot^2(\alpha r), \tag{25}$$

$$U(r)=-\frac{H_c}{r}, \tag{26}$$

where U_0, H_c and a are three positive parameters. Since Dirac equation with the symmetric trigonometric potential has no exact solution, we use an approximation for the centrifugal term as shown in Fig. 1 [31],

$$\frac{1}{r^2}=\alpha^2\left(d_0+\frac{1}{\sin^2(\alpha r)}\right),\quad \alpha<0.1, \tag{27}$$

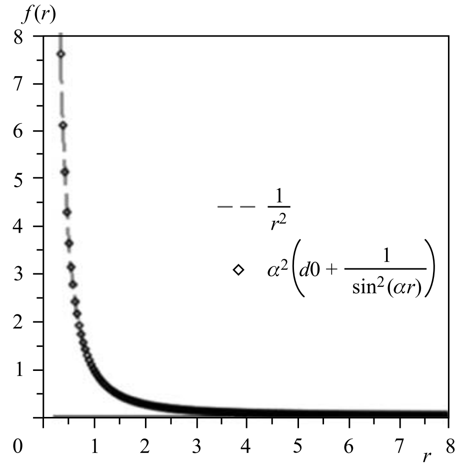

Fig. 1. $\frac{1}{r^2}$ and its approximations for $\alpha = 0.01$

where $d_0 = \dfrac{1}{12}$. Substituting Eqs. (25)–(27) into Eq. (24) yields

$$-\frac{d^2G^{\rm ps}_{n\kappa}}{dr^2} + \begin{bmatrix} \alpha^2\eta_\kappa(\eta_\kappa - 1) - \\ -(M - E^{\rm ps}_{n\kappa} + C_{\rm ps})\,U_0 \end{bmatrix} \operatorname{cosec}^2(\alpha r) G^{\rm ps}_{n\kappa}(r) = \\ = -\begin{bmatrix} (M + E^{\rm ps}_{n\kappa})\,(M - E^{\rm ps}_{n\kappa} + C_{\rm ps}) + \\ + \alpha^2 d_0 \eta_\kappa(\eta_\kappa - 1) + (M - E^{\rm ps}_{n\kappa} + C_{\rm ps})\,U_0 \end{bmatrix} G^{\rm ps}_{n\kappa}(r), \quad (28)$$

where $\kappa = -\tilde{\ell}$ and $\kappa = \tilde{\ell} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively, and

$$\kappa(\kappa - 1) + 2\kappa H_c - H_c + H_c^2 = (\kappa + H_c)\,(\kappa + H_c - 1) = \\ = \eta_\kappa(\eta_\kappa - 1) \to \eta_\kappa = (\kappa + H_c)\,. \quad (29)$$

In the spin symmetric limit $d\Delta(r)/dr = 0$ or $\Delta(r) = C_s = \text{const}$ [3–5]. As in the previous case, substitution of Eqs. (27), (26), (25) into Eq. (23) gives

$$-\frac{d^2F^s_{n\kappa}}{dr^2} + \begin{bmatrix} \alpha^2\Lambda_\kappa(\Lambda_\kappa - 1) + \\ + (M + E^s_{n\kappa} - C_s)\,U_0 \end{bmatrix} \operatorname{cosec}^2(\alpha r) F^s_{n\kappa}(r) = \\ = -\begin{bmatrix} (M + E^s_{n\kappa} - C_s)\,(M - E^s_{n\kappa}) + \\ + \alpha^2\Lambda_\kappa(\Lambda_\kappa - 1) d_0 - (M + E^s_{n\kappa} - C_s)\,U_0 \end{bmatrix} F^s_{n\kappa}(r), \quad (30)$$

where $\kappa = \ell$ and $\kappa = -\ell - 1$ for $\kappa < 0$ and $\kappa > 0$, respectively, and

$$\kappa(\kappa + 1) + 2\kappa H_c + H_c^2 + H_c = (\kappa + H_c)\,(\kappa + H_c + 1) = \\ = \Lambda_\kappa(\Lambda_\kappa - 1) \to \Lambda_\kappa = (\kappa + H_c + 1)\,. \quad (31)$$

3.2. Solution of the Pseudospin Symmetry Limit for the Trigonometric Symmetric Potential. From Eq. (28), we obtained a Schrödinger-like equation of the form

$$-\frac{d^2 G^{\rm ps}_{n\kappa}(r)}{dr^2} + V_{\rm eff}(r) G^{\rm ps}_{n\kappa}(r) = \tilde{E}^{\rm ps}_{n\kappa} G^{\rm ps}_{n\kappa}(r), \tag{32}$$

with the effective potential being

$$V_{\rm eff} = A^{\rm ps} \operatorname{cosec}^2(\alpha r), \tag{33}$$

where

$$A^{\rm ps} = \alpha^2 \eta_\kappa(\eta_\kappa - 1) - MU_0 + E^{\rm ps}_{n\kappa} U_0 - C_{\rm ps} U_0. \tag{34}$$

The corresponding effective energy is given by

$$\tilde{E}^{\rm ps}_{n\kappa} = -M^2 - MC_{\rm ps} + (E^{\rm ps}_{n\kappa})^2 - E^{\rm ps}_{n\kappa} C_{\rm ps} - \eta_\kappa(\eta_\kappa - 1)\alpha^2 d_0 - MU_0 + E^{\rm ps}_{n\kappa} U_0 - C_{\rm ps} U_0. \tag{35}$$

In SUSYQM formalism, the ground-state wave function for the lower component is given as

$$G^{\rm ps}_{0\kappa}(r) = \exp\left(-\int W(r)\, dr\right). \tag{36}$$

Thus, we are dealing with the Riccati equation

$$W^2(r) - W'(r) = V_{\rm eff}(r) - \tilde{E}^{\rm ps}_{0\kappa}, \tag{37}$$

for which we propose a superpotential of the form

$$W(r) = g^{\rm ps} - f^{\rm ps} \cot(\alpha r). \tag{38}$$

Therefore, the exact parameters of our study are obtained via

$$\begin{aligned}(f^{\rm ps})^2 \cot^2(\alpha r) - 2 f^{\rm ps} g^{\rm ps} \cot(\alpha r) + (g^{\rm ps})^2 -{}\\ - \alpha f^{\rm ps} \operatorname{cosec}^2(\alpha r) = A^{\rm ps} \operatorname{cosec}^2(\alpha r) - \tilde{E}^{\rm ps}_{0,\kappa}\end{aligned} \tag{39}$$

and solving Eq. (39) explicitly, we obtain the following set of equations:

$$\tilde{E}^{\rm ps}_{0\kappa} = (f^{\rm ps})^2, \tag{40a}$$

$$f^{\rm ps} = \frac{\alpha \pm \sqrt{\alpha^2 + 4A^{\rm ps}}}{2}, \tag{40b}$$

$$g^{\rm ps} = 0. \tag{40c}$$

The next approach is to construct the partner Hamiltonians as [16–18]

$$V_{\rm eff+}(r) = W^2 + \frac{dW}{dr} = f^{\rm ps}\left(f^{\rm ps} + \alpha\right) \operatorname{cosec}^2(\alpha r) - \left(f^{\rm ps}\right)^2, \tag{41}$$

$$V_{\rm eff-}(r) = W^2 - \frac{dW}{dr} = f^{\rm ps}\left(f^{\rm ps} - \alpha\right) \operatorname{cosec}^2(\alpha r) - \left(f^{\rm ps}\right)^2, \tag{42}$$

where $a_0 = f^{\rm ps}$ and a_i is a function of a_0, i.e., $a_1 = f(a_0) = a_0 + \alpha$. Consequently, $a_n = f(a_0) = a_0 + n\alpha$. We see that the shape invariance holds via a mapping of the form $f^{\rm ps} \to f^{\rm ps} + \alpha$. From Eq. (5), we have [16–18]

$$\begin{aligned} &R(a_1) = -a_0^2 + a_1^2, \\ &R(a_2) = -a_1^2 + a_2^2, \\ &R(a_3) = -a_2^2 + a_3^2, \\ &\dots\dots\dots\dots\dots\dots\dots \\ &R(a_n) = -a_{n-1}^2 + a_n^2, \end{aligned} \tag{43}$$

$$\tilde{E}_{0\kappa}^{-} = 0. \tag{44}$$

Therefore, from Eq. (6a) the eigenvalues can be found as

$$\tilde{E}_{n\kappa}^{\rm ps-} = \sum_{k=1}^{n} R(a_\kappa) = -a_0^2 + a_n^2, \tag{45}$$

$$\tilde{E}_{n\kappa}^{\rm ps} = \tilde{E}_{n\kappa}^{\rm ps-} + \tilde{E}_{0\kappa}^{\rm ps} = a_n^2. \tag{46}$$

With the aid of Eqs. (45) and (46), we obtain the energy eigenvalues for the symmetric trigonometric potential model for the pseudospin symmetry limit for any spin-orbit quantum number as

$$\begin{aligned} &-M^2 - MC_{\rm ps} + (E_{n\kappa}^{\rm ps})^2 - E_{n\kappa}^{\rm ps}C_{\rm ps} - \eta_\kappa(\eta_\kappa - 1)\,\alpha^2 d_0 - MU_0 + E_{n\kappa}^{\rm ps}U_0 - C_{\rm ps}U_0 - \\ &\quad - \left[\frac{\alpha \pm \sqrt{\alpha^2 + 4\,(\alpha^2\eta_\kappa(\eta_\kappa - 1) - MU_0 + E_{n\kappa}^{\rm ps}U_0 - C_{\rm ps}U_0)}}{2} + n\alpha\right]^2 = 0. \end{aligned} \tag{47}$$

This is consistent with those in [1, 39, 40] when $H = 0, C_{\rm ps} = 0$.

Thus, the lower component of the wave function is

$$\begin{aligned} G_{n\kappa}(r) = N_{n\kappa}\,(\sin(\alpha r))^{\frac{1}{2}+\sqrt{\frac{1}{2}+\frac{A^{\rm ps}}{\alpha^2}}}\,(1 - \sin(\alpha r))^{-\frac{1}{2}-\left(\sqrt{\frac{A^{\rm ps}}{\alpha^2}-\frac{\tilde{E}_{n,\kappa}^{\rm ps}}{\alpha^2}}+\sqrt{\frac{1}{4}+\frac{A^{\rm ps}}{\alpha^2}}\right)} \times \\ \times P_n^{\left(2\sqrt{\frac{1}{2}+\frac{A^{\rm ps}}{\alpha^2}},\ 2\sqrt{\frac{A^{\rm ps}}{\alpha^2}-\frac{\tilde{E}_{n,\kappa}^{\rm ps}}{\alpha^2}}\right)}(1 - 2\sin\alpha r), \end{aligned} \tag{48}$$

where $N_{n\kappa}$ is the normalization constant. The upper spinor component of the Dirac equation can be calculated as

$$F_{n\kappa}(r) = \frac{1}{M - E_{n\kappa} + C_{\rm ps}}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n\kappa}(r), \tag{49}$$

where $E_{n\kappa} \neq M + C_{\rm ps}$ and when $C_{\rm ps} = 0$ (exact pseudospin symmetry) is obtained which means that only negative energy solutions are possible.

3.3. Solution of the Spin Symmetry Limit for the Symmetric Trigonometric Potential. In this case, using Eq. (31), we get

$$-\frac{d^2F^s_{n\kappa}(r)}{dr^2}+V_{\text{eff}}(r)F^s_{n\kappa}(r)=\tilde{E}^s_{n\kappa}F^s_{n\kappa}(r), \tag{50}$$

with

$$V_{\text{eff}}(r)=B^s\operatorname{cosec}^2(\alpha r), \tag{51}$$

$$B^s=\alpha^2\Lambda_\kappa(\Lambda_\kappa-1)+MU_0+E^s_{n\kappa}U_0-C_sU_0, \tag{52}$$

$$\tilde{E}^s_{n\kappa}=-M^2+MC_s+(E^s_{n\kappa})^2-C_sE^s_{n\kappa}-\alpha^2\Lambda_\kappa(\Lambda_\kappa-1)d_0+MU_0+E^s_{n\kappa}U_0-C_sU_0. \tag{53}$$

By considering the same way of solving Eq. (34), the energy equation for the symmetric trigonometric potential in the presence of tensor interaction in view of the spin symmetry limit is obtained as follows:

$$\begin{aligned}&-M^2+MC_s+(E^s_{n\kappa})^2-C_sE^s_{n\kappa}-\alpha^2\Lambda_\kappa(\Lambda_\kappa-1)d_0+MU_0+E^s_{n\kappa}U_0-C_sU_0-\\&\qquad-\left[\frac{\alpha\pm\sqrt{\alpha^2+4\left(\alpha^2\Lambda_\kappa(\Lambda_\kappa-1)+MU_0+E^s_{n\kappa}U_0-C_sU_0\right)}}{2}+n\alpha\right]^2=0.\end{aligned} \tag{54}$$

The upper component of the wave function is

$$\begin{aligned}F^s_{n\kappa}(r)={}&N_{n\kappa}(\sin(\alpha r))^{\frac{1}{2}+\sqrt{\frac{1}{2}+\frac{B^s}{\alpha^2}}}(1-\sin(\alpha r))^{-\frac{1}{2}-\left(\sqrt{\frac{B^s}{\alpha^2}-\frac{\tilde{E}^s_{n,\kappa}}{\alpha^2}}+\sqrt{\frac{1}{4}+\frac{B^s}{\alpha^2}}\right)}\times\\&\times P_n^{\left(2\sqrt{\frac{1}{2}+\frac{B^s}{\alpha^2}},\,2\sqrt{\frac{B^s}{\alpha^2}-\frac{\tilde{E}^s_{n,\kappa}}{\alpha^2}}\right)}(1-2\sin\alpha r),\end{aligned} \tag{55}$$

and the other component can be simply found via

$$G^s_{n\kappa}(r)=\frac{1}{M+E^s_{n\kappa}-C_s}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F^s_{n\kappa}(r). \tag{56}$$

4. PSEUDOSPIN AND SPIN SYMMETRY LIMITS FOR THE ASYMMETRIC TRIGONOMETRIC POTENTIAL

In this section, we are going to solve the Dirac equation with asymmetric trigonometric potential with Coulomb tensor potential coupling using the SUSYQM.

4.1. Pseudospin and Spin Symmetry Limit. The pseudospin symmetry occurs when $d\Sigma(r)/dr=0$ or equivalently $\Sigma(r)=C_{\text{ps}}=\text{const}$. In this limit, we take $\Delta(r)$ as the asymmetric trigonometric potential and pick up a Coulomb-like tensor potential:

$$\Delta(r)=U_0\cot^2(\alpha r)+U_1\cot(\alpha r),\quad U_0>0,\quad r\in\ [0,a], \tag{57}$$

$$U(r)=-\frac{H_c}{r},\quad H_c=\frac{z_az_be^2}{4\pi\varepsilon_0},\quad r\geqslant R_e. \tag{58}$$

Now, substitution of the proper approximation (see Fig. 1)

$$\frac{1}{r^2} = \lim_{\alpha \to 0} \alpha^2 \left(d_0 + \frac{1}{\sin^2(\alpha r)} \right) \tag{59}$$

in Eq. (22) yields

$$\begin{aligned} &\left\{ -\frac{d^2}{dr^2} + \left[\alpha^2 \eta_\kappa (\eta_\kappa - 1) - U_0 M + U_0 E_{n\kappa}^{\rm ps} - U_0 C_{\rm ps} \right] \operatorname{cosec}^2(\alpha r) + \right. \\ &\left. + \left[-U_1 M + U_1 E_{n\kappa}^{\rm ps} - U_1 C_{\rm ps} \right] \cot(\alpha r) \right\} G_{n\kappa}^{\rm ps}(r) = \left\{ E_{n\kappa}^2 - M^2 - (M + E_{n\kappa}) C_{\rm ps} - \right. \\ &\left. - \alpha^2 \eta_\kappa (\eta_\kappa - 1) d_0 - M U_0 + U_0 E_{n\kappa}^{\rm ps} - U_0 C_{\rm ps} \right\} G_{n\kappa}^{\rm ps}(r), \end{aligned} \tag{60}$$

where $\kappa = -\tilde{\ell}$ and $\kappa = \tilde{\ell} + 1$ for $\kappa < 0$ and $\kappa > 0$.
By considering Eq. (23) for the spin symmetry case, we have

$$\begin{aligned} &\left\{ -\frac{d^2}{dr^2} + \left[\alpha^2 \Lambda_\kappa (\Lambda_\kappa - 1) + \left(M + E_{n,\kappa}^s - C_s \right) U_0 \right] \operatorname{cosec}^2(\alpha r) + \right. \\ &\left. + \left(M + E_{n,\kappa}^s - C_s \right) U_1 \cot(\alpha r) \right\} F_{n,\kappa}^s(r) = - \left\{ \left(M + E_{n,\kappa}^s - C_s \right) \times \right. \\ &\left. \times \left(M - E_{n,\kappa}^s \right) - \left(M + E_{n,\kappa}^s - C_s \right) U_0 + \alpha^2 \Lambda_\kappa (\Lambda_\kappa - 1) d_0 \right\} F_{n,\kappa}^s(r), \end{aligned} \tag{61}$$

where $\kappa = \ell$ and $\kappa = -\ell - 1$ for $\kappa < 0$ and $\kappa > 0$, respectively, and

$$\begin{aligned} &\kappa(\kappa + 1) + 2\kappa H_c + H_c^2 + H_c = \\ &= (\kappa + H_c)(\kappa + H_c + 1) = \Lambda_\kappa (\Lambda_\kappa - 1) \to \Lambda_\kappa = (\kappa + H_c + 1). \end{aligned} \tag{62}$$

4.2. Solution of Pseudospin Symmetry Limit with Tensor Interaction. We obtain a Schrodinger-like equation of the form

$$-\frac{d^2 G_{n\kappa}^{\rm ps}(r)}{dr^2} + V_{\rm eff}(r) G_{n\kappa}^{\rm ps}(r) = \tilde{E}_{n\kappa}^{\rm ps} G_{n\kappa}^{\rm ps}(r), \tag{63}$$

with the effective potential being

$$V_{\rm eff} = \tilde{V}_{1\rm ps} \operatorname{cosec}^2(\alpha r) + \tilde{V}_{2\rm ps} \cot(\alpha r), \tag{64}$$

where

$$\begin{aligned} \tilde{V}_{1\rm ps} &= \alpha^2 \eta_\kappa (\eta_\kappa - 1) - U_0 M + U_0 E_{n\kappa}^{\rm ps} - U_0 C_{\rm ps}, \\ \tilde{V}_{2\rm ps} &= -U_1 M + U_1 E_{n\kappa}^{\rm ps} - U_1 C_{\rm ps}. \end{aligned} \tag{65}$$

The corresponding effective energy is given by

$$\tilde{E}_{n\kappa}^{\rm ps} = E_{n\kappa}^2 - M^2 - (M + E_{n\kappa}) C_{\rm ps} - \alpha^2 \eta_\kappa (\eta_\kappa - 1) d_0 - M U_0 + U_0 E_{n\kappa}^{\rm ps} - U_0 C_{\rm ps}. \tag{66}$$

Thus, we are dealing with the Riccati equation

$$\phi^2(r) - \phi'(r) = V_{\text{eff}}(r) - \tilde{E}_{0\kappa}^{\text{ps}}, \tag{67}$$

for which we propose a superpotential of the form

$$\phi(r) = q^{\text{ps}} - p^{\text{ps}} \cot(\alpha r). \tag{68}$$

Therefore, the exact parameters of our study are obtained via

$$\begin{aligned}(p^{\text{ps}})^2 \cot^2(\alpha r) - 2p^{\text{ps}} q^{\text{ps}} \cot(\alpha r) + (q^{\text{ps}})^2 - \alpha p^{\text{ps}} \operatorname{cosec}^2(\alpha r) = \\ = \tilde{V}_{1\text{ps}} \operatorname{cosec}^2(\alpha r) + \tilde{V}_{2\text{ps}} \cot(\alpha r) - \tilde{E}_{0,\kappa}^{\text{ps}}.\end{aligned} \tag{69}$$

And solving Eq. (69) explicitly, we obtain the following set of equations:

$$\tilde{E}_{0\kappa}^{\text{ps}} = (p^{\text{ps}})^2 - (q^{\text{ps}})^2, \tag{70a}$$

$$p^{\text{ps}} = \frac{\alpha \pm \sqrt{\alpha^2 + 4\tilde{V}_{1\text{ps}}}}{2}, \tag{70b}$$

$$q^{\text{ps}} = -\frac{\tilde{V}_{2\text{ps}}}{2p^{\text{ps}}}. \tag{70c}$$

The next approach is to construct the partner Hamiltonians as [16, 22]

$$V_{\text{eff}+}(r) = \phi^2 + \frac{d\phi}{dr} = p^{\text{ps}}(p^{\text{ps}} + \alpha) \operatorname{cosec}^2(\alpha r) + \tilde{V}_{2\text{ps}} \cot(\alpha r) + \frac{\tilde{V}_{2\text{ps}}^2}{4(p^{\text{ps}})^2} - (p^{\text{ps}})^2, \tag{71a}$$

$$V_{\text{eff}-}(r) = \phi^2 - \frac{d\phi}{dr} = p^{\text{ps}}(p^{\text{ps}} - \alpha) \operatorname{cosec}^2(\alpha r) + \tilde{V}_{2\text{ps}} \cot(\alpha r) + \frac{\tilde{V}_{2\text{ps}}^2}{4(p^{\text{ps}})^2} - (p^{\text{ps}})^2, \tag{71b}$$

where $a_0 = p^{\text{ps}}$ and a_i is a function of a_0, i.e., $a_1 = f(a_0) = a_0 + \alpha$. Consequently, $a_n = f(a_0) = a_0 + n\alpha$. We see that the shape invariance holds via a mapping of the form $p^{\text{ps}} \to p^{\text{ps}} + \alpha$. From Eq. (5), we have [29, 30]

$$\begin{aligned}
R(a_1) &= \left(\frac{\tilde{V}_{2\text{ps}}^2}{4a_0^2} - a_0^2\right) - \left(\frac{\tilde{V}_{2\text{ps}}^2}{4a_1^2} - a_1^2\right), \\
R(a_2) &= \left(\frac{\tilde{V}_{2\text{ps}}^2}{4a_1^2} - a_1^2\right) - \left(\frac{\tilde{V}_{2\text{ps}}^2}{4a_2^2} - a_2^2\right), \\
R(a_3) &= \left(\frac{\tilde{V}_{2\text{ps}}^2}{4a_2^2} - a_2^2\right) - \left(\frac{\tilde{V}_{2\text{ps}}^2}{4a_3^2} - a_3^2\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
R(a_n) &= \left(\frac{\tilde{V}_{2\text{ps}}^2}{4a_{n-1}^2} - a_{n-1}^2\right) - \left(\frac{\tilde{V}_{2\text{ps}}^2}{4a_n^2} - a_n^2\right),
\end{aligned} \tag{72}$$

$$\tilde{E}_{0\kappa}^{-} = 0. \tag{73}$$

Therefore, from Eq. (6a) the eigenvalues can be found as

$$\tilde{E}^{\text{ps}-}_{n\kappa} = \sum_{k=1}^{n} R(a_\kappa) = \left(\frac{\tilde{V}^2_{2\text{ps}}}{4a_0^2} - a_0^2\right) - \left(\frac{\tilde{V}^2_{2\text{ps}}}{4a_n^2} - a_n^2\right), \tag{74a}$$

$$\tilde{E}^{\text{ps}}_{n\kappa} = \tilde{E}^{\text{ps}-}_{n\kappa} + \tilde{E}^{\text{ps}}_{0\kappa} = -\left(\frac{\tilde{V}^2_{2\text{ps}}}{4a_n^2} - a_n^2\right). \tag{74b}$$

Thus, we obtain the energy spectrum for the asymmetric trigonometric Rosen–Morse potential in view of pseudospin symmetric as

$$\begin{aligned} &E^2_{n\kappa} - M^2 - (M + E_{n\kappa})\, C_{\text{ps}} - \alpha^2 \eta_\kappa(\eta_\kappa - 1) d_0 - M U_0 + U_0 E^{\text{ps}}_{n\kappa} - U_0 C_{\text{ps}} + \\ &\quad + \frac{\left(-U_1 M + U_1 E^{\text{ps}}_{n\kappa} - U_1 C_{\text{ps}}\right)^2}{4\left(\dfrac{\alpha \pm \sqrt{\alpha^2 + 4[\alpha^2\eta_\kappa(\eta_\kappa - 1) - U_0 M + U_0 E^{\text{ps}}_{n\kappa} - U_0 C_{\text{ps}}]}}{2} + n\alpha\right)^2} - \\ &\quad - \left(\frac{\alpha \pm \sqrt{\alpha^2 + 4[\alpha^2\eta_\kappa(\eta_\kappa - 1) - U_0 M + U_0 E^{\text{ps}}_{n\kappa} - U_0 C_{\text{ps}}]}}{2} + n\alpha\right)^2 = 0. \end{aligned} \tag{75}$$

This is consistent with the result of [1] when $H = 0$, $C_{\text{ps}} = 0$ and that of [39, 40] when $U_1 = 0$.

In what follows, we find the lower component of the wave function as

$$\begin{aligned} G^{\text{ps}}_{n\kappa}(r) = N_{n\kappa} (\cot(\alpha r))^{\frac{1}{2} + \sqrt{\frac{1}{4} - Q^{\text{ps}}_2}} (1 - \cot(\alpha r))^{1 - \left(\sqrt{\frac{1}{4} - Q^{\text{ps}}_1 - Q^{\text{ps}}_2} - \sqrt{\frac{1}{4} - Q^{\text{ps}}_2}\right)} \times \\ \times P_n^{(2\sqrt{\frac{1}{4} - Q^{\text{ps}}_2},\ 2\sqrt{\frac{1}{4} - Q^{\text{ps}}_1 - Q^{\text{ps}}_2})}(1 + 2\cot(\alpha r)), \end{aligned} \tag{76}$$

where

$$Q^{\text{ps}}_1 = \frac{\tilde{V}_{1\text{ps}} + \tilde{V}_{2\text{ps}}}{\alpha^2}, \tag{77}$$

$$Q^{\text{ps}}_2 = \frac{\tilde{V}_{1\text{ps}} - \tilde{E}^{\text{ps}}_{n,\kappa}}{\alpha^2}, \tag{78}$$

and $N_{n\kappa}$ is the normalization constant. The upper spinor component of the Dirac equation can be calculated as

$$F^{\text{ps}}_{n\kappa}(r) = \frac{1}{M - E^{\text{ps}}_{n\kappa} + C_{\text{ps}}}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right) G^{\text{ps}}_{n\kappa}(r), \tag{79}$$

where $E_{n\kappa} \neq M + C_{\text{ps}}$ and when $C_{\text{ps}} = 0$ (exact pseudospin symmetry) is obtained which means that only negative energy solutions are possible.

4.3. Solution of the Spin Symmetry Limit with Tensor Interaction. In this case,

$$-\frac{d^2 F^s_{n\kappa}(r)}{dr^2} + V_{\text{eff}}(r) F^s_{n\kappa}(r) = \tilde{E}^s_{n\kappa} F^s_{n\kappa}(r), \tag{80}$$

with

$$V_{\text{eff}} = \tilde{V}_{1s}\,\text{cosec}^2\,(\alpha r) + \tilde{V}_{2s}\cot(\alpha r), \tag{81}$$

where

$$\tilde{V}_{1s} = \alpha^2\Lambda_\kappa(\Lambda_\kappa - 1) + \left(M + E^s_{n,\kappa} - C_s\right)U_0, \tag{82}$$

$$\tilde{V}_{2s} = MU_1 + E^s_{n,\kappa}U_1 - C_sU_1 \tag{83}$$

and

$$\tilde{E}^s_{n\kappa} = (E^s_{n\kappa})^2 - M^2 - E^s_{n\kappa}C_s + MC_s + MU_0 + E^s_{n,\kappa}U_0 - C_sU_0 - \alpha^2\Lambda_\kappa(\Lambda_\kappa - 1)d_0. \tag{84}$$

Table 1. **Energies in the pseudospin symmetry limit for the symmetric trigonometric potential $M = 1$, $C_{\text{ps}} = -5$, $U_0 = -0.5$, $\alpha = 0.01$, $d_0 = 1/12$**

$\tilde{\ell}$	n, $k < 0$	(ℓ, j)	$E^{\text{ps}}_{n\kappa}$, fm^{-1} $(H = 0)$	$E^{\text{ps}}_{n\kappa}$, fm^{-1} $(H = 0.75)$	$n-1$, $\kappa > 0$	$(\ell+2$, $j+1)$	$E^{\text{ps}}_{n\kappa}$, fm^{-1} $(H = 0)$	$E^{\text{ps}}_{n\kappa}$, fm^{-1} $(H = 0.75)$
1	1, –1	$1S_{1/2}$	–4.000356299	–4.000224387	0, 2	$0d_{3/2}$	–4.000356299	–4.000535205
2	1, –2	$1P_{3/2}$	–4.000604376	–4.000411066	0, 3	$0f_{5/2}$	–4.000604376	–4.000839694
3	1, –3	$1d_{5/2}$	–4.000927313	–4.000678204	0, 4	$0g_{7/2}$	–4.000927313	–4.001217495
4	1, –4	$1f_{7/2}$	–4.001323301	–4.001019492	0, 5	$0h_{9/2}$	–4.001323301	–4.001667872
1	2, –1	$2S_{1/2}$	–4.000685933	–4.000525546	1, 2	$1d_{3/2}$	–4.000685933	–4.000906432
2	2, –2	$2P_{3/2}$	–4.000990890	–4.000753632	1, 3	$1f_{5/2}$	–4.000990890	–4.001274378
3	2, –3	$2d_{5/2}$	–4.001378576	–4.001080450	1, 4	$1g_{7/2}$	–4.001378576	–4.001719463
4	2, –4	$2f_{7/2}$	–4.001842407	–4.001487517	1, 5	$1h_{9/2}$	–4.001842407	–4.002238863
1	3, –1	$3S_{1/2}$	–4.001148004	–4.000970643	2, 2	$2d_{3/2}$	–4.001148004	–4.001400229
2	3, –2	$3P_{3/2}$	–4.001497400	–4.001224979	2, 3	$2f_{5/2}$	–4.001497400	–4.001823007
3	3, –3	$3d_{5/2}$	–4.001942203	–4.001600422	2, 4	$2g_{7/2}$	–4.001942203	–4.002329963
4	3, –4	$3f_{7/2}$	–4.002469005	–4.002066490	2, 5	$2h_{9/2}$	–4.002469005	–4.002914742

Table 2. **Energies in the spin symmetry limit for the symmetric trigonometric potential $M = 1$, $C_s = 5$, $U_0 = 0.5$, $\alpha = 0.01$, $d_0 = 1/12$**

ℓ	n, $k < 0$	(ℓ, j)	$E^s_{n\kappa}$, fm^{-1} $(H = 0)$	$E^s_{n\kappa}$, fm^{-1} $(H = 0.75)$	n, $\kappa > 0$	(ℓ, j)	$E^s_{n\kappa}$, fm^{-1} $(H = 0)$	$E^s_{n\kappa}$, fm^{-1} $(H = 0.75)$
1	0, –2	$0P_{3/2}$	4.000146446	4.000058799	0, 1	$0P_{1/2}$	4.000146446	4.000275000
2	0, –3	$0d_{5/2}$	4.000326895	4.000184771	0, 2	$0d_{3/2}$	4.000326895	4.000509672
3	0, –4	$0f_{7/2}$	4.000579623	4.000383307	0, 3	$0f_{5/2}$	4.000579623	4.000816536
4	0, –5	$0g_{9/2}$	4.000904523	4.000654085	0, 4	$0g_{7/2}$	4.000904523	4.001195521
1	1, –2	$1P_{3/2}$	4.000356299	4.000224387	1, 1	$1P_{1/2}$	4.000356299	4.000535205
2	1, –3	$1d_{5/2}$	4.000604376	4.000411066	1, 2	$1d_{3/2}$	4.000604376	4.000839694
3	1, –4	$1f_{7/2}$	4.000927313	4.000678204	1, 3	$1f_{5/2}$	4.000927313	4.001217495
4	1, –5	$1g_{9/2}$	4.001323301	4.001019492	1, 4	$1g_{7/2}$	4.001323301	4.001667872
1	2, –2	$2P_{3/2}$	4.000685933	4.000525546	2, 1	$2P_{1/2}$	4.000685933	4.000906432
2	2, –3	$2d_{5/2}$	4.00099089	4.000753632	2, 2	$2d_{3/2}$	4.000990890	4.001274378
3	2, –4	$2f_{7/2}$	4.001378576	4.001080450	2, 3	$2f_{5/2}$	4.001378576	4.001719463
4	2, –5	$2g_{9/2}$	4.001842407	4.001487517	2, 4	$2g_{7/2}$	4.001842407	4.002238863

Table 3. **Energies in the pseudospin limit for the asymmetric trigonometric potential** $M = 1$**,** $C_{\text{ps}} = -5$**,** $U_0 = -0.5$**,** $U_1 = 1$**,** $\alpha = 0.01$**,** $d_0 = 1/12$

$\tilde{\ell}$	n, $k < 0$	(ℓ, j)	$E^{\text{ps}}_{n\kappa}$, fm^{-1} $(H = 0)$	$E^{\text{ps}}_{n\kappa}$, fm^{-1} $(H = 0.75)$	$n - 1$, $\kappa > 0$	$(\ell + 2, j + 1)$	$E^{\text{ps}}_{n\kappa}$, fm^{-1} $(H = 0)$	$E^{\text{ps}}_{n\kappa}$, fm^{-1} $(H = 0.75)$
1	1, –1	$1S_{1/2}$	–4.000346925	–4.000218056	0, 2	$0d_{3/2}$	–4.000346925	–4.000521544
2	1, –2	$1P_{3/2}$	–4.000589045	–4.000400388	0, 3	$0f_{5/2}$	–4.000589045	–4.000818663
3	1, –3	$1d_{5/2}$	–4.000904155	–4.000661087	0, 4	$0g_{7/2}$	–4.000904155	–4.001187290
4	1, –4	$1f_{7/2}$	–4.001290526	–4.000994097	0, 5	$0h_{9/2}$	–4.001290526	–4.001626728
1	2, –1	$2S_{1/2}$	–4.000666933	–4.000509978	1, 2	$1d_{3/2}$	–4.000666933	–4.000882344
2	2, –2	$2P_{3/2}$	–4.000964811	–4.000733096	1, 3	$1f_{5/2}$	–4.000964811	–4.001241541
3	2, –3	$2d_{5/2}$	–4.001343238	–4.001052246	1, 4	$1g_{7/2}$	–4.001343238	–4.001675914
4	2, –4	$2f_{7/2}$	–4.00179589	–4.001449559	1, 5	$1h_{9/2}$	–4.001795890	–4.002182764
1	3, –1	$3S_{1/2}$	–4.001115069	–4.000941265	2, 2	$2d_{3/2}$	–4.001115069	–4.001361741
2	3, –2	$3P_{3/2}$	–4.001456697	–4.00119039	2, 3	$2f_{5/2}$	–4.001456697	–4.001774733
3	3, –3	$3d_{5/2}$	–4.001891120	–4.001557345	2, 4	$2g_{7/2}$	–4.001891120	–4.002269673
4	3, –4	$3f_{7/2}$	–4.002405394	–4.002012467	2, 5	$2h_{9/2}$	–4.002405394	–4.002840453

Table 4. **Energies in the spin symmetry limit for the asymmetric trigonometric potential** $M = 1$**,** $C_s = 5$**,** $U_0 = 0.5$**,** $U_1 = 1$**,** $\alpha = 0.01$**,** $d_0 = 1/12$

ℓ	n, $k < 0$	(ℓ, j)	$E^{s}_{n\kappa}$, fm^{-1} $(H = 0)$	$E^{s}_{n\kappa}$, fm^{-1} $(H = 0.75)$	n, $\kappa > 0$	(ℓ, j)	$E^{s}_{n\kappa}$, fm^{-1} $(H = 0)$	$E^{s}_{n\kappa}$, fm^{-1} $(H = 0.75)$
1	0, –2	$0P_{3/2}$	4.000142827	4.000057310	0, 1	$0P_{1/2}$	4.000142827	4.000268250
2	0, –3	$0d_{5/2}$	4.000318881	4.000180219	0, 2	$0d_{3/2}$	4.000318881	4.000497207
3	0, –4	$0f_{7/2}$	4.000565455	4.000373919	0, 3	$0f_{5/2}$	4.000565455	4.000796601
4	0, –5	$0g_{9/2}$	4.000882447	4.000638104	0, 4	$0g_{7/2}$	4.000882447	4.001166367
1	1, –2	$1P_{3/2}$	4.000346925	4.000218056	1, 1	$1P_{1/2}$	4.000346925	4.000521544
2	1, –3	$1d_{5/2}$	4.000589045	4.000400388	1, 2	$1d_{3/2}$	4.000589045	4.000818663
3	1, –4	$1f_{7/2}$	4.000904155	4.000661087	1, 3	$1f_{5/2}$	4.000904155	4.00118729
4	1, –5	$1g_{9/2}$	4.001290526	4.000994097	1, 4	$1g_{7/2}$	4.001290526	4.001626728
1	2, –2	$2P_{3/2}$	4.000666933	4.000509978	2, 1	$2P_{1/2}$	4.000666933	4.000882344
2	2, –3	$2d_{5/2}$	4.000964811	4.000733096	2, 2	$2d_{3/2}$	4.000964811	4.001241541
3	2, –4	$2f_{7/2}$	4.001343238	4.001052246	2, 3	$2f_{5/2}$	4.001343238	4.001675914
4	2, –5	$2g_{9/2}$	4.001795890	4.001449559	2, 4	$2g_{7/2}$	4.001795890	4.002182764

Table 5. **Energies in the pseudospin symmetry limit for the symmetric trigonometric potential** $M = 1$**,** $H = 0.75$**,** $U_0 = -0.5$**,** $\alpha = 0.01$**,** $d_0 = 1/12$

C_p	$E^{\text{ps}}_{n\kappa}$, fm^{-1}				
	$1S_{1/2}$	$1P_{3/2}$	$1d_{5/2}$	$0d_{3/2}$	$0f_{5/2}$
–5	–4.000224387	–4.000411066	–4.000678204	–4.000535205	–4.000839694
–4.5	–3.500282057	–3.500505490	–3.500827023	–3.500654851	–3.501021481
–4	–3.000376373	–3.000654701	–3.001058324	–3.000842096	–3.001302582
–3.5	–2.500554185	–2.500923475	–2.501465055	–2.501174710	–2.501793128
–3	–2.000985948	–2.001535485	–2.002356049	–2.001915565	–2.002854011
–2.5	–1.502867170	–1.503941328	–1.505599628	–1.504706993	–1.506609220
–2	–1.079078753	–1.081429631	–1.085421330	–1.083238928	–1.087936875
–1.5	–1.029686192	–1.030196200	–1.031111755	–1.030603502	–1.031720384

Table 6. **Energies in the spin symmetry limit for the symmetric trigonometric potential** $M = 1$, $H = 0.75$, $U_0 = 0.5$, $\alpha = 0.01$, $d_0 = 1/12$

C_s	$E^s_{n\kappa}$, fm^{-1}				
	$0P_{3/2}$	$1d_{5/2}$	$2f_{7/2}$	$1P_{1/2}$	$2f_{5/2}$
1.5	1.010066322	1.030196200	1.050344183	1.030603502	1.052394969
2	1.038348287	1.081429631	1.116585548	1.083238928	1.123158042
2.5	1.500523931	1.503941328	1.510380111	1.504706993	1.514191650
3	2.000210988	2.001535485	2.004066419	2.001915565	2.006012076
3.5	2.500129176	2.500923475	2.502439171	2.501174710	2.503730244
4	3.000092550	3.000654701	3.001725530	3.000842096	3.002689580
4.5	3.500071953	3.500505490	3.501330181	3.500654851	3.502098887
5	4.000058799	4.000411066	4.001080450	4.000535205	4.001719463

Table 7. **Energies in the pseudospin symmetry limit for the asymmetric trigonometric potential** $M = 1$, $H = 0.75$, $U_0 = -0.5$, $U_1 = 1$, $\alpha = 0.01$, $d_0 = 1/12$

C_p	$E^{\rm ps}_{n\kappa}$, fm^{-1}				
	$1S_{1/2}$	$1P_{3/2}$	$1d_{5/2}$	$0d_{3/2}$	$0f_{5/2}$
–5	–4.000218056	–4.000400388	–4.000661087	–4.000521544	–4.000818663
–4.5	–3.500270828	–3.500487104	–3.500797952	–3.500631519	–3.500985912
–4	–3.000353831	–3.000619164	–3.001003188	–3.000797492	–3.001235519
–3.5	–2.500499548	–2.500841559	–2.501341419	–2.501073514	–2.501644077
–3	–2.000803415	–2.001280288	–2.001987488	–2.001608074	–2.002416281
–2.5	–1.501660921	–1.502426204	–1.50358821	–1.5029636	–1.504294637
–2	–1.006333788	–1.00799218	–1.010613475	–1.00919894	–1.012214729
–1.5	–0.620041578	–0.622756458	–0.627416228	–0.624861339	–0.630380407

Table 8. **Energies in the spin symmetry limit for the asymmetric trigonometric potential** $M = 1$, $H = 0.75$, $U_0 = 0.5$, $U_1 = 1$, $\alpha = 0.01$, $d_0 = 1/12$

C_s	$E^s_{n\kappa}$, fm^{-1}				
	$0P_{3/2}$	$1d_{5/2}$	$2f_{7/2}$	$1P_{1/2}$	$2f_{5/2}$
1.5	0.559212359	0.622756458	0.672587696	0.624861339	0.680135075
2	1.001044968	1.00799218	1.020640054	1.00919894	1.026397006
2.5	1.500327734	1.502426204	1.506434164	1.5029636	1.509168641
3	2.000177129	2.001280288	2.003388689	2.001608074	2.005070433
3.5	2.500118093	2.500841559	2.502221682	2.501073514	2.50341425
4	3.000087677	3.000619164	3.001631354	3.000797492	3.002548888
4.5	3.500069408	3.500487104	3.501281543	3.500631519	3.502024825
5	4.000057310	4.000400388	4.001052246	4.000521544	4.001675914

Energy eigenvalues equation for the asymmetric trigonometric potential in the presence of tensor interaction in view of the spin symmetry limit is obtained as follows:

$$(E^s_{n\kappa})^2 - M^2 - E^s_{n\kappa}C_s + MC_s + MU_0 + E^s_{n,\kappa}U_0 - C_sU_0 - \alpha^2\Lambda_\kappa(\Lambda_\kappa - 1) + \\ + \frac{\left(MU_1 + E^s_{n,\kappa}U_1 - C_sU_1\right)^2}{4\left(\dfrac{\alpha \pm \sqrt{\alpha^2 + 4[\alpha^2\Lambda_\kappa(\Lambda_\kappa - 1) + \left(M + E^s_{n,\kappa} - C_s\right)U_0]}}{2} + n\alpha\right)^2} - \\ - \left(\frac{\alpha \pm \sqrt{\alpha^2 + 4[\alpha^2\Lambda_\kappa(\Lambda_\kappa - 1) + \left(M + E^s_{n,\kappa} - C_s\right)U_0]}}{2} + n\alpha\right)^2 = 0. \quad (85)$$

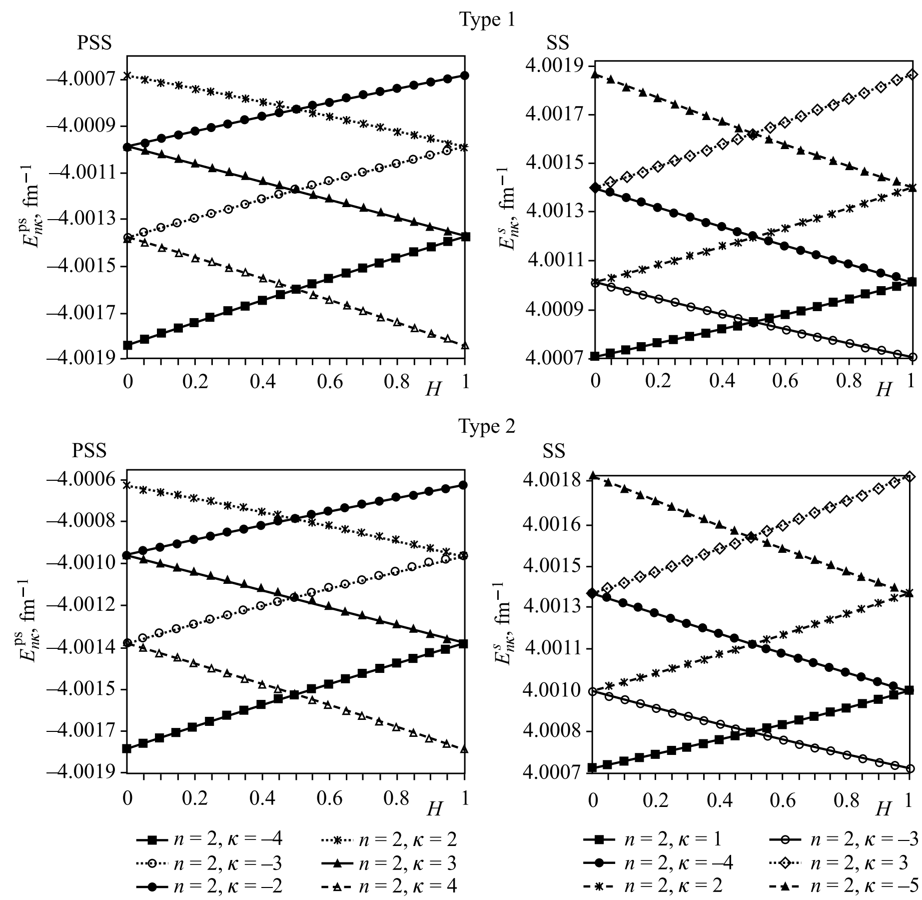

Fig. 2. Type 1: Energy vs. H for pseudospin and spin symmetries limit for the symmetric trigonometric potential. PSS: $M = 1$, $C_{\rm ps} = -5$, $U_0 = -0.5$, $\alpha = 0.01$, $d_0 = 1/12$; SS: $M = 1$, $C_s = 5$, $U_0 = 0.5$, $\alpha = 0.01$, $d_0 = 1/12$. Type 2: Energy vs. H for pseudospin and spin symmetries limit for for the asymmetric trigonometric potential. PSS: $M = 1$, $C_{\rm ps} = -5$, $U_0 = -0.5$, $U_1 = 1$, $\alpha = 0.01$, $d_0 = 1/12$; SS: $M = 1$, $C_s = 5$, $U_0 = 0.5$, $U_1 = 1$, $\alpha = 0.01$, $d_0 = 1/12$

Thus, the upper wave function becomes

$$F_{n\kappa}^{s}(r) = N_{n\kappa}\left(\cot(\alpha r)\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}-Q_2^s}}\left(1-\cot(\alpha r)\right)^{1-\left(\sqrt{\frac{1}{4}-Q_1^s-Q_2^{\mathrm{ps}}}-\sqrt{\frac{1}{4}-Q_2^s}\right)} \times \\ \times P_n^{\left(2\sqrt{\frac{1}{4}-Q_2^s},2\sqrt{\frac{1}{4}-Q_1^s-Q_2^s}\right)}(1+2\cot(\alpha r)), \quad (86)$$

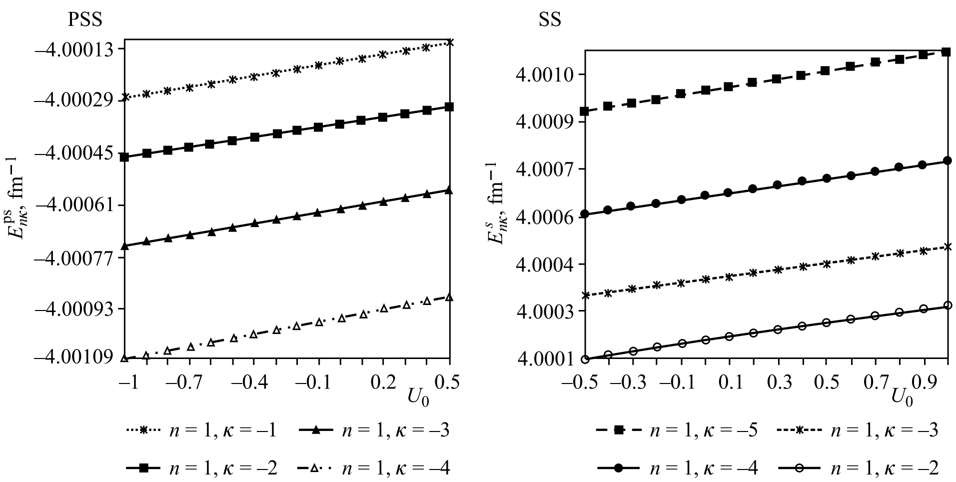

Fig. 3. Energy vs. U_0 for pseudospin and spin symmetries limit for the symmetric trigonometric potential. PSS: $M=1$, $C_{\mathrm{ps}}=-5$, $H=0.75$, $\alpha=0.01$, $d_0=1/12$; SS: $M=1$, $C_s=5$, $H=0.75$, $\alpha=0.01$, $d_0=1/12$

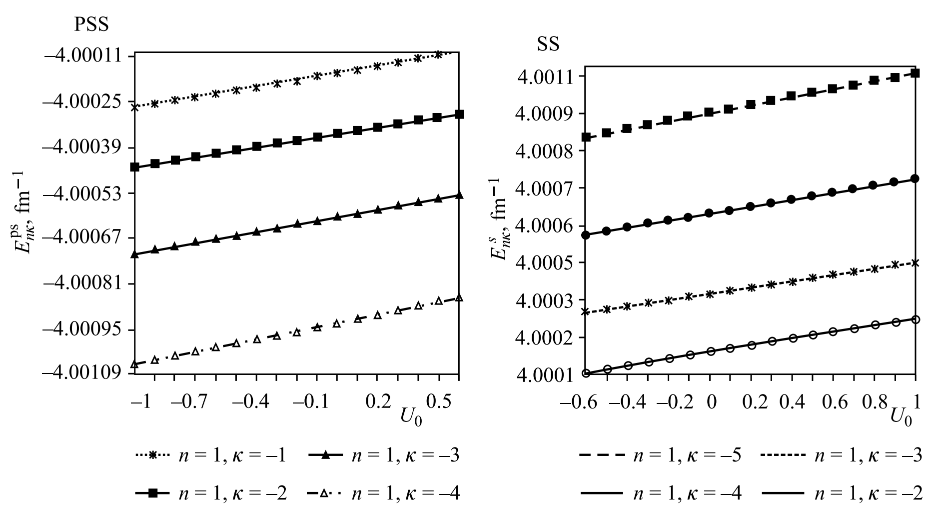

Fig. 4. Energy vs. U_0 for pseudospin and spin symmetries limit for the asymmetric trigonometric potential. PSS: $M=1$, $C_{\mathrm{ps}}=-5$, $H=0.75$, $U_1=1$, $\alpha=0.01$, $d_0=1/12$; SS: $M=1$, $C_s=5$, $H=0.75$, $U_1=1$, $\alpha=0.01$, $d_0=1/12$

where

$$Q_1^s = \frac{\tilde{V}_{1s} + \tilde{V}_{2s}}{\alpha^2}, \tag{87}$$

$$Q_2^s = \frac{\tilde{V}_{1s} - \tilde{E}_{n,\kappa}^s}{\alpha^2}, \tag{88}$$

and the other component can be simply found via

$$G_{n\kappa}^s(r) = \frac{1}{M + E_{n\kappa}^s - C_s}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right) F_{n\kappa}^s(r). \tag{89}$$

We have obtained the energy eigenvalues in the absence ($H = 0$) and the presence ($H = 0.75$) of the Coulomb-like tensor potential for various values of the quantum numbers n and κ. The results are calculated in Tables 1 and 2 for the symmetric trigonometric potential and in Tables 3 and 4 for the asymmetric trigonometric potential under the condition of the pseudospin and spin symmetries, respectively, and we can clearly see that there is the

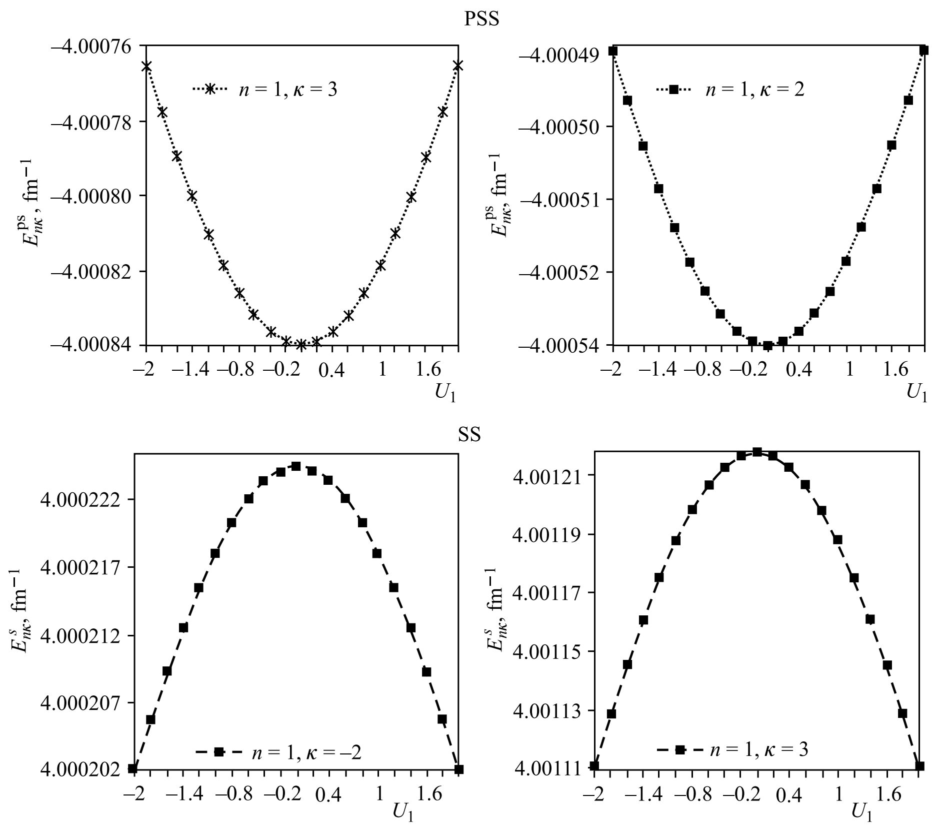

Fig. 5. Energy vs. U_1 for pseudospin and spin symmetries limit for the asymmetric trigonometric potential. PSS: $M = 1$, $C_{ps} = -5$, $H = 0.75$, $U_0 = -0.5$, $\alpha = 0.01$, $d_0 = 1/12$; SS: $M = 1$, $C_s = 5$, $H = 0.75$, $U_0 = 0.5$, $\alpha = 0.01$, $d_0 = 1/12$

degeneracy between the bound states in the absence of the tensor interaction and in the presence of the tensor interaction, these degeneracies are changed. Also in Tables 5–8, we observe the behavior of the energy eigenvalues versus difference values of the $C_{\rm ps}$ and C_s for both symmetries limit. In Fig. 2, we obtain the effects of the tensor interaction on the bound states in view of the pseudospin and spin symmetry limits for the symmetric and asymmetric trigonometric potentials, respectively. Figure 2 shows that the magnitude of the energy difference between the degenerate states increases as H increases. In Figs. 3 and 4, we show the effects of the potential parameter U_0 on the bound states under the conditions of the pseudospin and spin symmetry limits for $H = 0.75$ for the first and second choices of the potential. In Fig. 5, we present the dependence of the bound-state energy levels on potential parameter U_1 for the second choices of the potential.

CONCLUSIONS

In this work, we have obtained the approximate solutions of the Dirac equation for the symmetric and asymmetric trigonometric Rosen–Morse potentials within the Coulomb tensor interaction term in the framework of pseudospin and spin symmetry limits using the SUSYQM method. We have obtained the energies eigenvalues and the corresponding lower and upper wave functions expressed in terms of the Jacobi polynomials. Finally, the results of our work have been compared with the previous work of others authors in the literature.

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