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PERTURBATIVE SERIES AND THE $1/N$ EXPANSION FOR THE QED β FUNCTION

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A comparison of the 5-loop perturbative series and the $1/N$ expansion for the QED renormalization-group β function in the Minimal Subtraction (MS) scheme is performed. The good agreement between two expansions is found which proves that the MS β function is adequately described by both series.

Проведен сравнительный анализ 5-петлевого ряда теории возмущений и ряда $1/N$-разложения для ренормгрупповой β-функции КЭД в схеме минимальных вычитаний. Получено хорошее согласие между двумя видами разложений. Это указывает на то, что β-функция адекватно описывается обоими рядами.

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INTRODUCTION

The concept of the β function, which depends on the choice of the renormalization scheme, is the cornerstone of the Quantum Field Theory renormalization group approach, see [1] and references therein.

At the same time, the nature of perturbative series for the β function in Quantum Electrodynamics still remains an unresolved question, although one can believe that it is an asymptotic sign-alternating series. Then one can hope that the error of the truncated series of this type is estimated by the value of the first discarded (or the last included) term of the expansion. For the recent discussion of the behavior of asymptotic series of both sign-alternating and sign-constant types, see [2].

It is known from the work [3] that the renormalization-group β function in the $g\phi^4$-theory is indeed expanded into the sign-alternating asymptotic series with factorially growing coefficients. But in the QED case the situation is more complicated. The asymptotic estimates of [4] and [5] analogous to those of [3] were obtained only for the gauge-invariant subclasses of diagrams with fixed number of fermion loops. As discussed in [4] and [5], in the case of complete QED the strong cancellations between coefficients of subsets of diagrams with different fixed numbers of fermion loops are expected. Thus, the asymptotic behavior of perturbative coefficients of the β function in QED is in fact unknown.

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Since QED is the basis of modern Quantum Field Theory it is rather important to obtain definite conclusions concerning the behavior of its pertubative expansions.

Quite recently the 5-loop approximation for the QED renormalization-group β function in different renormalization schemes was obtained, first for one active lepton [6] and then for an arbitrary number N_F of flavors [7,8]. These results are obtained after more than twenty years since the calculation of the 4-loop order [9]. On the other hand, there is a calculation [10] of the first nontrivial leading term of the $1/N_F$ expansion for the QED β function in the MS scheme. It is quite interesting to compare the available 5-loop perturbative series and the nonperturbative $1/N_F$ series to analyze their consistency. This is the purpose of the present letter.

MAIN PART

Let us first cite the $1/N_F$ result for the β function from the work [10]:

$$\beta(K) = \frac{2}{3}K + \frac{1}{N_F}\frac{1}{2}K \int\limits_{-K/3}^{0} dx \frac{\Gamma(4+2x)(1+2x)(1+2x/3)(1-x)}{[\Gamma(2+x)]^2\Gamma(3+x)\Gamma(1-x)} + O(1/N_F^2) =$$
$$= \frac{2}{3}K + \frac{K^2}{2N_F}\left[1 - \frac{11}{2\cdot 3}\frac{1}{2}\left(\frac{K}{3}\right) - \frac{7\cdot 11}{2^2\cdot 3^2}\frac{1}{3}\left(\frac{K}{3}\right)^2 + \right.$$
$$+ \left(\frac{107}{2^3\cdot 3^2} + 2\zeta(3)\right)\frac{1}{4}\left(\frac{K}{3}\right)^3 + \left(\frac{251}{2^4\cdot 3^2} - \frac{11}{3}\zeta(3) + 3\zeta(4)\right)\frac{1}{5}\left(\frac{K}{3}\right)^4 +$$
$$+ \left(\frac{67}{2^5} - \frac{7\cdot 11}{2\cdot 3^2}\zeta(3) - \frac{11}{2}\zeta(4) + 2\cdot 3\zeta(5)\right)\frac{1}{6}\left(\frac{K}{3}\right)^5 +$$
$$+ \left(\frac{5\cdot 7\cdot 41}{2^6\cdot 3^2} + \frac{107}{2^2\cdot 3^2}\zeta(3) - \frac{7\cdot 11}{2^2\cdot 3}\zeta(4) - 11\zeta(5) + \right.$$
$$\left.\left. + 2\cdot 5\zeta(6) - 2\zeta^2(3)\right)\frac{1}{7}\left(\frac{K}{3}\right)^6 + \ldots\right] + O(1/N_F^2), \quad (1)$$

here $K \equiv \alpha N_F/\pi$ is the coupling which has to be held fixed in the large N_F limit, α being the fine structure constant.

The function $\beta(K)$ is defined as

$$\alpha\beta(K) = \mu\frac{d}{d\mu}\alpha(\mu), \quad (2)$$

where μ is the renormalization scale.

In the numerical form the result of Eq. (1) reads

$$\beta(K) = \frac{2}{3}K + \frac{K^2}{2N_F}\big(1 - 3.055555556\cdot 10^{-1}K - 7.921810700\cdot 10^{-2}K^2 +$$
$$+ 3.602060109\cdot 10^{-2}K^3 + 1.438230317\cdot 10^{-3}K^4 - 1.906442773\cdot 10^{-3}K^5 +$$
$$+ 1.521260392\cdot 10^{-4}K^6 + 3.588903124\cdot 10^{-5}K^7 + \ldots\big) + O(1/N_F^2).$$

One can see from the above equation the fast decrease of the coefficients of the expansion of the $1/N_F$ term which demonstrates the very good convergence of the series. The authors of the work [10] established that the radius of convergence of this $\beta(K)$ expansion is $K = 15/2$. They checked numerically that the $1/N_F$ term has the only zero at $K = 0$ and is positive in the convergence region. The authors of the work [10] also found that for the physical value $N_F = 3$ the $1/N_F$ term is never larger than 15% of the leading term $2K/3$.

Let us now cite the 5-loop result for the β function in the MS scheme from the work [7]. In the normalization of Eq. (2) it is

$$\beta = 8\pi\frac{1}{\alpha}\left\{N_F\left[\frac{4A^2}{3}\right] + 4N_F A^3 - A^4\left[2N_F + \frac{44}{9}N_F^2\right] + \right.$$
$$+ A^5\left[-46N_F + \frac{760}{27}N_F^2 - \frac{832}{9}\zeta(3)N_F^2 - \frac{1232}{243}N_F^3\right] +$$
$$+ A^6\left(N_F\left[\frac{4157}{6} + 128\zeta(3)\right] + N_F^2\left[-\frac{7462}{9} - 992\zeta(3) + 2720\zeta(5)\right] + \right.$$
$$\left.\left. +N_F^3\left[-\frac{21758}{81} + \frac{16000}{27}\zeta(3) - \frac{416}{3}\zeta(4) - \frac{1280}{3}\zeta(5)\right] + N_F^4\left[\frac{856}{243} + \frac{128}{27}\zeta(3)\right]\right)\right\}, \quad (3)$$

where $A \equiv \alpha/4\pi$.

Let us also present this formula in terms of K to have it closer to the form of the $1/N_F$ expansion:

$$\beta = \frac{2}{3}K + \frac{K^2}{2N_F}\left[1 - \frac{11}{2^2\cdot 3}\left(\frac{K}{3}\right) - \frac{7\cdot 11}{2^2\cdot 3^3}\left(\frac{K}{3}\right)^2 + \left(\frac{K}{3}\right)^3\left(\frac{107}{2^5\cdot 3^2} + \frac{1}{2}\zeta(3)\right)\right] +$$
$$+ \frac{K^3}{N_F^2}\left[-\frac{1}{2^4} + K\left(\frac{5\cdot 19}{2^4\cdot 3^3} - \frac{13}{2\cdot 3^2}\zeta(3)\right) + \right.$$
$$\left. + K^2\left(-\frac{11\cdot 13\cdot 43}{2^8\cdot 3^4} + \frac{5^3}{2^2\cdot 3^3}\zeta(3) - \frac{13}{2^4\cdot 3}\zeta(4) - \frac{5}{2\cdot 3}\zeta(5)\right)\right] +$$
$$+ \frac{K^4}{N_F^3}\left[-\frac{23}{2^6} + K\left(-\frac{7\cdot 13\cdot 41}{2^8\cdot 3^2} - \frac{31}{2^4}\zeta(3) + \frac{5\cdot 17}{2^4}\zeta(5)\right)\right] + \frac{K^5}{N_F^4}\left(\frac{4157}{2^{10}\cdot 3} + \frac{1}{2^2}\zeta(3)\right).$$

The numerical form of the above equation for the value $N_F = 3$ is

$$\beta = 0.63662\alpha + 0.151982\alpha^2 - 0.050393\alpha^3 - 0.0819407\alpha^4 + 0.0412278\alpha^5, \quad (4)$$

this is the monotonically increasing function for $\alpha > 0$.

We will compare Eq. (1) and Eq. (3) for $N_F = 3$. The results of the comparison are presented in the table.

The values of the β function calculated in the $1/N_f$ expansion and within perturbation theory for different values of α

β function	$\alpha = 1/137$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 1$
In the $1/N_F$ expansion	0.00465494	0.0651364	0.133032	0.737883
In perturbation theory	0.00465494	0.0651236	0.132882	0.697496

We see that even for $\alpha = 1$ when the convergence of the series (4) is quite questionable both results agree within 5%. Thus, two different expansions (the usual perturbative series and the $1/N_F$ series) give numerically very close values for the QED β function in the wide interval of α. It definitely indicates that both expansions give good approximations for $\beta(\alpha)$.

CONCLUSIONS

We have got a proof that the perturbative QED series adequately describes the renormalization-group β function. Presently one has no reliable information about the structure of perturbative series in Quantum Electrodynamics (except the common words that they can be asymptotic series), so the obtained result seems to be quite illuminating.

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REFERENCES

1. *Bogolyubov N. N., Shirkov D. V.* Introduction to the Theory of Quantized Fields // Intersci. Monogr. Phys. Astron. 1959. V. 3. P. 1–720.
2. *Shirkov D. V.* A Few Lessons from Perturbative pQCD Analysis at Low Energies // Nucl. Phys. Proc. Suppl. 2012. V. 225–227. P. 5–9.
3. *Lipatov L. N.* Divergence of the Perturbation Theory Series and the Quasiclassical Theory // Sov. Phys. JETP. 1977. V. 45. P. 216–223 (Zh. Eksp. Teor. Fiz. 1977. V. 72. P. 411–427).
4. *Itzykson C., Parisi G., Zuber J.-B.* Asymptotic Estimates in Quantum Electrodynamics // Phys. Rev. D. 1977. V. 16. P. 996–1013.
5. *Bogomolny E. B., Kubyshin Y. A.* Asymptotical Estimates for Graphs with a Fixed Number of Fermionic Loops in Quantum Electrodynamics. The Extremal Configurations with the Symmetry Group $O(2) \times O(3)$ // Sov. J. Nucl. Phys. 1982. V. 35. P. 114–199 (Yad. Fiz. 1982. V. 35. P. 202–212).
6. *Kataev A. L., Larin S. A.* Analytical Five-Loop Expressions for the Renormalization Group QED β-Function in Different Renormalization Schemes // Pisma Zh. Eksp. Teor. Fiz. 2012. V. 96. P. 64–67 (JETP Lett. 2012. V. 96. P. 61–65; arXiv:1205.2810[hep-ph]).
7. *Baikov P. A. et al.* Vector Correlator in Massless QCD at Order $O(\alpha_s^4)$ and the QED β-Function at Five Loop // JHEP. 2012. V. 1207. P. 017; arXiv:1206.1284[hep-ph].
8. *Baikov P. A. et al.* The Relation between the QED Charge Renormalized in $\overline{\rm MS}$ and On-Shell Schemes at Four Loops, the QED On-Shell β-Function at Five Loops and Asymptotic Contributions to the Muon Anomaly at Five and Six Loops // Nucl. Phys. B. 2013. V. 867. P. 182–202; arXiv:1207.2199[hep-ph].
9. *Gorishny S. G., Kataev A. L., Larin S. A.* Analytical Four Loop Result for Beta Function in QED in MS and MOM Schemes // Phys. Lett. B. 1987. V. 194. P. 429–434;
   Gorishny S. G. et al. The Analytical Four Loop Corrections to the QED β-Function in the MS-Scheme and to the QED ψ-Function: Total Reevaluation // Phys. Lett. B. 1991. V. 256. P. 81–86.
10. *Palanques-Mestre A., Pascual P.* The $1/N_F$ Expansion of the γ and β Functions in QED // Commun. Math. Phys. 1984. V. 95. P. 277–287.

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