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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

NEWTON–CARTAN SUPERGRAVITY

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We construct a $\mathcal{N} = 2$ supersymmetric extension of the Newton–Cartan gravity by gauging a $\mathcal{N} = 2$ supersymmetric extension of the Bargmann algebra. Due to technical complications, we restrict the construction to the three-dimensional case. We discuss the gauge-fixing of the resulting Newton–Cartan supergravity theory to a Galilean supergravity that contains the Newton potential. An unusual feature is that, in order to realize the supersymmetry on the Newton potential, we need to introduce a dual Newton potential as well. Together, the Newton potential and dual potential, they form a holomorphic function of the two spatial coordinates. We briefly discuss the four-dimensional case.

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INTRODUCTION

It seems natural that, given the nonrelativistic world we live in and assuming that Nature makes use of supersymmetry, it is important to study the nonrelativistic aspects of supersymmetry. So far, in the case of local supersymmetry, basically all efforts have been concentrating on supergravity, the supersymmetric extension of Einstein's relativistic description of gravity. It is well known that nonrelativistically, gravity in a frame with constant acceleration is described by the Newtonian gravity in terms of a Newton potential $\Phi(\mathbf{x})$, which is a function of the spatial coordinates. Given the above motivation, it is natural to ask the following simple question:

What is the supersymmetric extension of the Newton potential?

A related question is: what is the Newtino potential, i.e., the supersymmetric partner of the Newton potential? Given the huge literature on supergravity, it is remarkable that the answer to the above question cannot be found in the literature.

A second motivation to study supersymmetric nonrelativistic gravity is due to the AdS/CMT correspondence, where the boundary field theory exhibits nonrelativistic symmetries. So far, nonrelativistic bulk gravity has not played a prominent role in this context, for an early reference, see [1]. However, it has been argued recently that it does play a role in describing the boundary geometry [2].

The purpose of this paper is to answer the above question, be it so far in three space-time dimensions only. At first sight, one might think that it is enough to simply take the nonrelativistic limit of supergravity to obtain a nonrelativistic one. However, taking such a limit is often nontrivial and messy. A more fruitful approach is to make use of the fact that

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supergravity can often be obtained by the gauging of a space-time algebra [3]. It is much easier to take the nonrelativistic limit of such an algebra. After that we will perform the gauging of the nonrelativistic space-time algebra[1]. This is the strategy that we will follow here. We will first explain the procedure by showing how the Einstein gravity follows from gauging the Poincare algebra. Next, we will derive the so-called Newton–Cartan gravity, which is valid in any frame, by gauging the Bargmann algebra, which is a central extension of the Galilean algebra. Furthermore, we will show how gauge-fixing of the Newton–Cartan theory leads to the Newtonian gravity, which is valid in frames with constant acceleration only, and the Galilean gravity, which is valid in frames with an arbitrary time-dependent acceleration. Finally, we will discuss the gauging of the three-dimensional supersymmetric Bargmann algebra. Upon gauge-fixing this will lead us to the desired supersymmetric extension of the Newton potential.

1. EINSTEIN GRAVITY

The discussion in this section applies to any dimension. One of the noteworthy features of gravity is the absence of the gravitational force in free-falling frames. In the relativistic case, free-falling frames are connected by the Poincare symmetries, i.e., the space-time translations (with parameters ξ^μ) and Lorentz transformations (with parameters $\lambda^\mu{}_\nu$):

- space-time translations: $\delta x^\mu = \xi^\mu$,
- Lorentz transformations: $\delta x^\mu = \lambda^\mu{}_\nu x^\nu$.

In arbitrary, nonfree-falling, frames there is a gravitational force described by a metric tensor $g_{\mu\nu}$. This field can be seen as arising from gauging the Poincare symmetries, whose algebra is given by

$$[M_{ab}, P_c] = -2\eta_{c[a}P_{b]}, \quad [M_{ab}, M_{cd}] = -4M_{[a}{}^{[c}\delta_{b]}{}^{d]}. \tag{1}$$

Here, P_a are the translation generators and M_{ab} are the Lorentz generators.

In gauging this algebra, one associates to each of the generators a gauge field, whose transformation rules, with space-time-dependent parameters, are determined by the structure constants of the algebra. It is straightforward to construct the curvatures that transform covariantly under these transformations, see Table 1.

Table 1. **Symmetries, generators, gauge fields, gauge parameters, and covariant curvatures corresponding to the Poincare algebra given in Eq.** (1)

Symmetry	Generator	Gauge field	Parameter	Curvature
Space-time translation	P_a	$e_\mu{}^a$	$\zeta^a(x^\mu)$	$R_{\mu\nu}{}^a(P)$
Lorentz transformation	M_{ab}	$\omega_\mu{}^{ab}$	$\lambda^{ab}(x^\mu)$	$R_{\mu\nu}{}^{ab}(M)$

At this stage, the gauge fields do not yet transform in the correct way under general coordinate transformations, neither is the Lorentz gauge field a dependent one as it should be. To achieve this, we must impose the following so-called conventional constraints:

$$R_{\mu\nu}{}^a(P) \equiv 2\partial_{[\mu}e_{\nu]}{}^a - \omega_{[\mu}{}^{ab}e_{\nu]}{}^b = 0: \quad \text{conventional constraints.}$$

[1]It is not a priori clear that the nonrelativistic theory thus obtained can alternatively be obtained by taking the nonrelativistic limit of a supergravity theory.

Assuming now that the so-called Vierbein gauge fields are invertable, we can use the above conventional constraints for two purposes:

- the spin-connection field $\omega_\mu{}^{ab}$ becomes dependent: $\omega_\mu{}^{ab} \to \omega_\mu{}^{ab}(e)$,
- the "local translations", with parameters $\zeta^a(x^\nu)$, become equivalent to general coordinate transformations with parameters $\xi^\mu(x^\nu)$.

The gauge field $e_\mu{}^a$ can now be identified as the Vierbein field and the (dependent) gauge field $\omega_\mu{}^{ab}$ as the spin-connection field.

It is now straightforward to make the connection to the metric formulation of the Einstein gravity. A metric is introduced via $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ and the Christoffel symbol is defined via the Vierbein postulate that relates it to the spin connection:

$$\nabla_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - \omega_\mu{}^{ab} e_\nu{}^b - \Gamma_{\mu\nu}{}^\rho e_\rho{}^a = 0. \tag{2}$$

Finally, the Einstein equation of motion is imposed by hand.

This concludes our brief discussion of the relativistic case. We next turn our attention to the nonrelativistic case, but first without supersymmetry.

2. NEWTON–CARTAN GRAVITY

For simplicity, we consider in this section four space-time dimensions only. In the nonrelativistic case, free-falling frames are connected via the Galilean symmetries. These symmetries consist of time translations (with parameters $\xi^\emptyset$[1]), space translations (with parameters ξ^i $(i = 1, 2, 3)$), spatial rotations (with parameters $\lambda^i{}_j$) and boosts (with parameters λ^i):

- time translations: $\delta t = \xi^\emptyset$,
- space translations: $\delta x^i = \xi^i$,
- spatial rotations: $\delta x^i = \lambda^i{}_j x^j$,
- boosts: $\delta x^i = \lambda^i t$.

Depending on which class of frames one wishes to consider, we distinguish three different cases:

(1) *Newtonian gravity.* The Newtonian gravity is valid in frames with constant acceleration ($\delta x^i = (1/2)\rho^i t^2$). The gravitational force is described by a Newton potential $\Phi(\mathbf{x})$.

(2) *Galilean gravity.* The Galilean gravity describes nonrelativistic gravity in frames with time-dependent acceleration ($\delta x^i = \xi^i(t)$). The gravitational force is described by a time-dependent Newton potential $\Phi(t, \mathbf{x})$.

(3) *Newton–Cartan gravity.* The Newton–Cartan (NC) gravity is valid in the most general frame. The gravitational force is described by a temporal Vierbein τ_μ and a spatial Vierbein $e_\mu{}^a$ with projective inverses τ^μ and $e^\mu{}_a$ ($\mu = \emptyset, 1, 2, 3$; $a = 1, 2, 3$).

We first gauge *all* Galilean symmetries. This will lead us to the most general class of frames with the Newton–Cartan gravity. After that we will restrict the class of frames by gauge-fixing and uncover the Galilean gravity. In fact, it turns out that we have to gauge the *centrally extended* Galilean symmetries, which form the so-called Bargmann algebra:

$$\begin{aligned} &[J_{ab}, P_c] = -2\delta_{c[a}P_{b]}, \quad [J_{ab}, G_c] = -2\delta_{c[a}G_{b]}, \\ &[G_a, H] = -P_a, \quad [G_a, P_b] = -\delta_{ab}Z, \quad a = 1, 2. \end{aligned} \tag{3}$$

[1] We use the symbol $\emptyset$ to indicate the time component $\mu = \emptyset$ of a curved index μ.

Here, H is the time translation, P_a are the translation generators, G_a — the boost generators, J_{ab} — the spatial rotation generators, and Z — the generator corresponding to the central charge transformations. The presence of the central extension can be deduced from the fact that the Lagrangian for a bosonic particle is not invariant under boosts, but transforms with a total time derivative such that the action is invariant. The central extension generator, which is absent in the relativistic case, is related to the nonrelativistic particle number conservation.

Following the relativistic case, we introduce for each generator corresponding gauge fields, gauge parameters and covariant curvatures, see Table 2.

Table 2. **Symmetries, generators, gauge fields, gauge parameters, and covariant curvatures corresponding to the Bargmann algebra given in Eq.** (3)

Symmetry	Generator	Gauge field	Parameter	Curvature
Time translations	H	τ_μ	$\zeta(x^\nu)$	$R_{\mu\nu}(H)$
Space translations	P_a	$e_\mu{}^a$	$\zeta^a(x^\nu)$	$R_{\mu\nu}{}^a(P)$
Boosts	G_a	$\omega_\mu{}^a$	$\lambda^a(x^\nu)$	$R_{\mu\nu}{}^a(G)$
Spatial rotations	J_{ab}	$\omega_\mu{}^{ab}$	$\lambda^{ab}(x^\nu)$	$R_{\mu\nu}{}^{ab}(J)$
Central charge transformations	Z	m_μ	$\sigma(x^\nu)$	$R_{\mu\nu}(Z)$

Our next task is to impose a set of constraints on the curvatures. We first assume that the gauge fields $e_\mu{}^a$ and τ_μ have projective inverses $e^\mu{}_a$ and τ^μ and impose the following conventional constraints:

$$R_{\mu\nu}{}^a(P) = 0, \quad R_{\mu\nu}(Z) = 0. \tag{4}$$

These constraints enable us to solve for the spatial rotation gauge fields $\omega_\mu{}^{ab}$ and the boost gauge fields $\omega_\mu{}^a$. Note, that this is quite different from the relativistic case, where the curvature of time translations is set equal to zero. In the nonrelativistic case, we do set the same curvature equal to zero, but for a rather different reason:

$$R_{\mu\nu}(H) = \partial_{[\mu}\tau_{\nu]} = 0 \rightarrow \tau_\mu = \partial_\mu \tau. \tag{5}$$

Instead of using the above restriction to solve for a gauge field, the constraint is used to define a foliation of the Newtonian space-time. Any choice of the arbitrary function $\tau(x^\mu)$ defines a time direction. It is customary (but not obligatory) to consider the Newtonian gravity in flat space. Following this convention, we impose the additional constraint

$$R_{\mu\nu}{}^{ab}(J) = 0. \tag{6}$$

Combining the above constraints with the Bianchi identities, we find that the only nonzero curvature components at this stage are given by

$$R_{0(a,b)}(G) \neq 0. \tag{7}$$

So far we only discussed the kinematics. The equations of motion of the NC gravity are obtained by imposing the following restriction:

$$R_{0a}{}^a(G) = 0. \tag{8}$$

To go from the NC gravity to the Galilean gravity, which is only valid in frames with arbitrary (time-dependent) acceleration, we need to impose a set of gauge-fixing conditions.

We first impose the following three gauge conditions, where we have indicated which symmetries are gauge-fixed:

- $\tau_\mu(x^\nu) = \delta_\mu{}^\emptyset \rightarrow$ constant time translations: $\xi^\emptyset(x^\nu) = \xi^\emptyset$,
- $\omega_\mu{}^{ab}(x^\nu) = 0 \rightarrow$ constant spatial rotations: $\lambda^{ab}(x^\nu) = \lambda^{ab}$,
- $e_i{}^a(x^\nu) = \delta_i{}^a \rightarrow$ time-dependent spatial translations: $\xi^a(x^\nu) = \xi^a(t) + \ldots$

The dots in the third item mean compensating transformations, whose specific form we do not give here. At this point, the only independent gauge fields left are the central charge gauge field m_μ and the following components of the Dreibein field $e_\mu{}^a$:

$$e_\mu{}^a(x^\nu) = (-\tau^a(x^\nu), \delta_i{}^a). \tag{9}$$

From now on, we will not distinguish between curved indices i and flat indices a anymore. There is one subtlety. We have used the gauge field $\omega_\mu{}^{ab}$ to restrict the spatial rotations. However, this gauge field is not independent. Using its explicit solution in the gauge condition $\omega_\mu{}^{ab} = 0$ leads to the following additional relation:

$$\tau_i(x^\nu) + m_i(x^\nu) = \partial_i m(x^\nu). \tag{10}$$

We next continue and impose two further gauge conditions:

- $m(x^\nu) = 0 \rightarrow$ time-dependent central charge: $\sigma(x^\nu) = \sigma(t) + \ldots$
- $\tau^a(x^\nu) = 0 \rightarrow$ no boost transformations: $\lambda^a(x^\nu) = 0 + \ldots$

Combining this with the relation (10), we deduce that $m_i(x^\mu) = 0$.

At this point, the only nonzero gauge field component left is the time component $m_\emptyset(x^\mu)$ of the central charge gauge field, which plays the role of the Newton potential $\Phi(x^\mu)$: $m_\emptyset(x^\nu) \equiv \Phi(x^\nu)$. Taking all compensating transformations into account, we find that the Newton potential transforms under the acceleration extended Galilean symmetries as follows:

$$\delta\Phi(x^\nu) = \xi^\emptyset\partial_\emptyset\Phi(x^\nu) + \xi^i(t)\partial_i\Phi(x^\nu) - \lambda^i{}_j x^j\partial_i\Phi(x^\nu) + \ddot{\xi}^k(t)x^k + \dot{\sigma}(t). \tag{11}$$

One may verify that indeed the acceleration extended Galilean symmetries with these transformation rules form a closed algebra. Using that the only nonzero component of the boost gauge field is given by $\omega_\emptyset{}^a(x^\nu) = -\partial^a\Phi(x^\nu)$, the NC equation of motion (8) leads to the expected Galilean equation of motion for the Newton potential Φ, i.e., $\Delta\Phi = 0$.

This finishes our discussion of the bosonic case. We next consider what happens if we include supersymmetry.

3. NEWTON–CARTAN SUPERGRAVITY

In this section, we restrict to three space-time dimensions. All spinors are (2-component) Majorana spinors. To start with, we consider a $\mathcal{N} = 2$ supersymmetric extension of the Bargmann algebra [4]. The reason that we consider two supersymmetries can be found in the bosonic discussion of the previous section. Although the NC gravity theory is formally invariant under local time translations, it is a fake so-called Stueckelberg symmetry. Fixing the Stueckelberg symmetry leads to an invariance under global time translations only. The only translations that are truly gauged are the spatial translations. After gauge-fixing the Galilean gravity theory is invariant under the remaining time-dependent spatial translations.

Since supersymmetry is, roughly speaking, the "square root" of a time or space translation, we expect a similar phenomenon to happen for the supersymmetries. One supersymmetry squares to a time translation and is only gauged à la Stueckelberg. The other symmetry, as it turns out, squares to a time-dependent central charge transformation and is truly gauged. Schematically, we have the following situation [5]:

$$\begin{aligned} \{Q^+, Q^+\} &\sim \text{constant time translations}, \\ \{Q^+, Q^-(t)\} &\sim \text{time-dependent spatial translations}, \\ \{Q^-(t), Q^-(t)\} &\sim \text{time-dependent central charge transformations}. \end{aligned} \tag{12}$$

Here, Q^+ and Q^- are the generators of the constant and time-dependent supersymmetry, respectively.

Since the gauging procedure is very similar, but more involved than the bosonic case, we will be brief. For more details, we refer to the original paper [6]. In Table 3, we have indicated all symmetries, generators, gauge fields, parameters, and supercovariant curvatures (indicated with a hat) like we did in the previous two sections.

Table 3. **Symmetries, generators, gauge fields, gauge parameters, and covariant curvatures corresponding to the $\mathcal{N} = 2$ supersymmetric Bargmann algebra, whose schematic form is partly given in Eq.** (12)

Symmetry	Generator	Gauge field	Parameter	Curvature
Time translations	H	τ_μ	$\zeta(x^\nu)$	$\hat{R}_{\mu\nu}(H)$
Space translations	P_a	${e_\mu}^a$	$\zeta^a(x^\nu)$	${\hat{R}_{\mu\nu}}{}^a(P)$
Boosts	G_a	${\omega_\mu}^a$	$\lambda^a(x^\nu)$	${\hat{R}_{\mu\nu}}{}^a(G)$
Spatial rotations	J_{ab}	${\omega_\mu}^{ab}$	$\lambda^{ab}(x^\nu)$	${\hat{R}_{\mu\nu}}{}^{ab}(J)$
Central charge transformations	Z	m_μ	$\sigma(x^\nu)$	$\hat{R}_{\mu\nu}(Z)$
Two supersymmetries	$Q^\pm$	$\psi_{\mu\pm}$	$\epsilon_\pm(x^\nu)$	$\hat{\psi}_{\mu\nu\pm}$

We next impose a number of conventional and additional constraints on the curvatures and obtain the NC supergravity [6]. One finds that the supersymmetry algebra closes provided the following restrictions hold:

$$\gamma^\mu \tau^\nu \hat{\psi}_{\mu\nu-} = 0, \quad {e^\mu}_a {e^\nu}_b \hat{\psi}_{\mu\nu-} = 0. \tag{13}$$

The supersymmetry of the first equation, which can be considered a fermionic equation of motion, leads to the bosonic NC equation of motion given in Eq. (8). Following the bosonic case, the gauge-fixing to a Galilean supergravity theory is straightforward except for two subtleties. After gauge-fixing the only nonzero fermionic gauge-field component left is the time-component of one of the gravitini:

$$\Psi(x^\nu) \equiv \psi_{\emptyset-}(x^\nu) \quad \text{with} \quad \gamma^i \partial_i \Psi(x^\nu) = 0, \quad i = 1, 2. \tag{14}$$

However, it turns out that this field is not the Newtino potential. The true Newtino potential is a spinor χ that is related to Ψ as follows:

$$\gamma_i \Psi = \partial_i \chi \quad \text{with} \quad \gamma^1 \partial_1 \chi = \gamma^2 \partial_2 \chi. \tag{15}$$

This is the first subtlety. The second one is that we find that the Newtino potential transforms both to the Newton potential Φ as well as to the *dual* Newton potential Ξ: $\partial_i \Xi = -\varepsilon_{ij}\, \partial^j \Phi$. We thus find the following supersymmetric extension of the 3D Newton potential:

$$\begin{aligned}
\delta\Phi &= \frac{1}{2}\bar{\epsilon}_-(t)\gamma^{0i}\,\partial_i\chi + \frac{1}{2}\bar{\epsilon}_+\dot{\chi}, \quad \text{Newton potential,}\\
\delta\Xi &= \frac{1}{2}\bar{\epsilon}_-(t)\gamma^{i}\,\partial_i\chi - \frac{1}{2}\bar{\epsilon}_+\gamma_0\dot{\chi}, \quad \text{dual Newton potential,}\\
\delta\chi &= x^i\gamma_i\dot{\epsilon}_-(t) + \frac{1}{2}\Xi\epsilon_+ - \frac{1}{2}\Phi\gamma_0\epsilon_+, \quad \text{Newtino potential.}
\end{aligned}$$

The resulting algebra is a superversion of the algebra of accelerated extended Galilean symmetries. For more details, we refer to [6].

CONCLUSIONS

In this paper, we have shown how to obtain a supersymmetric extension of the Newton potential in three space-time dimensions. We expect that a similar construction is possible in four space-time dimensions, but for that purpose extra fields are needed, just like in the relativistic case.

Clearly, much more work remains to be done. A first goal would be to give a proper superspace description of the above results and develop a nonrelativistic tensor calculus. We hope to report about these interesting issues in the nearby future.

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