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SPHERICAL MECHANICS FOR A PARTICLE NEAR THE HORIZON OF EXTREMAL BLACK HOLE

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We describe canonical transformation, which links the Hamiltonian of a massive relativistic particle moving near the horizon of an extremal black hole to the conventional form of the conformal mechanics. Thus, like the nonrelativistic conformal mechanics, the investigation of the particle dynamics reduces to analyzing its "spherical sector" defined by the Casimir element of the conformal algebra. We present a detailed list of such systems originating from various types of black hole configurations.

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Conformal mechanics associated with the near-horizon geometry of an extremal black hole is described by the triple

$$H = r\left(\sqrt{(rp_r)^2 + L(p_a,\varphi^a)} - f(p_a,\varphi^a)\right),$$
$$D = rp_r, \quad K = \frac{1}{r}\left(\sqrt{(rp_r)^2 + L(p_a,\varphi^a)} + f(p_a,\varphi^a)\right), \tag{1}$$

which involves the Hamiltonian H, the generator of dilatations D, and the generator of special conformal transformations K. Under the Poisson brackets they form an $so(2,1)$ algebra

$$\{H,D\} = H, \quad \{H,K\} = 2D, \quad \{D,K\} = K. \tag{2}$$

Recently, we suggested a canonical transformation $(p_r, r, p_a, \varphi^a) \to (p_R, R, \tilde{p}_a, \tilde{\varphi}^a)$, which links the Hamiltonian (1) to the conventional nonrelativistic form [1–3] (for related earlier studies, see [4, 5])

$$p_R = -\frac{2D}{\sqrt{2K}}, \quad R = \sqrt{2K}, \quad \widetilde{\varphi}^a = \varphi^a + \frac{\partial U(p_r, r, \psi_a, \varphi^a)}{\partial p_a}, \quad \widetilde{p}_a = p_a - \frac{\partial U(p_r, r, \psi_a, \varphi^a)}{\partial \varphi^a}, \tag{3}$$

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where

$$U(p_r, r, \psi_a, \varphi^a) = \frac{1}{2} \int\limits_{x=rp_r} \log\left(\sqrt{\frac{x^2}{4} + L(p_a, \varphi^a)} + f(\varphi^a, p_a)\right). \tag{4}$$

In these coordinates, the Hamiltonian and the remaining generators of the conformal algebra take the conventional form

$$H = \frac{1}{2}p_R^2 + \frac{2\mathcal{I}}{R^2}, \quad D = Rp_R, \quad K = \frac{R^2}{2}, \tag{5}$$

with $\mathcal{I}$ being the Casimir element of the conformal algebra

$$\mathcal{I} = HK - D^2 = L(p_a, \varphi^a) - (f(p_a, \varphi^a))^2. \tag{6}$$

Thus, we get a canonical transformation relating the model of a massive relativistic particle moving near the horizon of an extremal black hole (relativistic frame) to the conventional form of the conformal mechanics (conformal frame). In other words, specific information regarding a massive relativistic particle moving near the horizon of an extremal black hole in d dimensions is encoded in a $(d-2)$-dimensional Hamiltonian mechanics, which represents the spherical part of the conformal mechanics associated with the relativistic particle. In what follows, we refer to it as the master mechanics

$$\mathcal{I} = L(p_a, \varphi^a) - f^2(p_a, \varphi^a),\ \omega = dp_a \wedge d\varphi^a. \tag{7}$$

As a result, the study of the initial conformal mechanics reduces to the investigation of the master mechanics.

It is worth mentioning that the investigation of the conformal mechanics in terms of its spherical part turned out to be useful for various applications, in particular, in the study of the celebrated Calogero model [6–10]. Moreover, the extraction of the spherical part seems to be a nontrivial reduction mechanism allowing one to construct new integrable systems (on curved backgrounds) starting from the known ones. For example, the spherical part of the rational Calogero model defines a multicenter generalization of the spherical (Higgs) oscillator [6]. All the known conformal mechanics describing the near-horizon particle dynamics are also integrable models. Their spherical sectors are thus integrable models as well.

The compact part of the near-horizon conformal particle allows one to investigate it in terms of the action-angle variables. There are several reasons to be concerned about the action-angle variables:

• A formulation of an integrable system in action-angle variables gives a comprehensive geometric description of its dynamics and is a useful tool for developing the perturbation theory [11].

• The use of the action-angle variables provides a criterion for the (non)equivalence of two integrable systems because a system in the action-angle variables has two main characteristics:

i) the functional dependence of the Hamiltonian on the action variables;

ii) the domain of the action variables.

• The action-angle formulation suggests the only systematic way to reveal hidden symmetries. Moreover, using this formulation one can construct new nontrivial examples of maximally superintegrable systems [12].

• The action-angle variables yield the semiclassical quantization straightforward via imposing the Bohr–Sommerfeld quantization conditions.

In this way we analyzed the spherical sector of the conformal mechanics on the Reissner–Nordström, dilaton–axion and Kerr backgrounds [13, 14], as well as on the Myers–Perry background with equal rotation parameters [3]. Spherical mechanics describing these systems looks as follows:

Reissner–Nordström BH. The spherical mechanics associated with the near-horizon Reissner–Nordström black hole is governed by the Hamiltonian

$$\mathcal{I} = p_\theta^2 + \frac{(p_\varphi + ep\cos\theta)^2}{\sin^2\theta} + (mM)^2 - (eq)^2, \quad \omega = dp_\theta \wedge d\theta + dp_\varphi \wedge d\varphi, \tag{8}$$

where m and e are the mass and the electric charge of a particle, while M, q and g are the mass, the electric and magnetic charges of the black hole, respectively. This is precisely the spherical Landau problem (a particle on a two-dimensional sphere in the presence of a constant magnetic field generated by the Dirac monopole) shifted by the constant $\mathcal{I}_0 = (mM)^2 - (eq)^2$. A link between the two systems was discussed in [4, 5].

Clément–Gal'tsov BH. This solution of the Einstein–Maxwell–dilaton–axion theory can be viewed as interpolating between the near-horizon extremal Reissner–Nordström black hole and the near-horizon extremal Kerr black hole [15]. The corresponding spherical mechanics reads

$$\mathcal{I} = p_\theta^2 + \frac{(p_\varphi \cos\theta - e)^2}{\sin^2\theta} + m^2. \tag{9}$$

Here m and e are the mass and the electric charge of a particle. This system coincides with the planar rotator [1, 14].

Kerr BH. Spherical mechanics associated with the near-horizon Kerr geometry is defined by the integrable but not exactly solvable system [13]:

$$\mathcal{I} = p_\theta^2 + \left[\left(\frac{1+\cos^2\theta}{2\sin\theta}\right)^2 - 1\right] p_\varphi^2 + \left(\frac{1+\cos^2\theta}{2}\right)(mr_0)^2, \tag{10}$$

where m is the mass of a particle and r_0 is the horizon radius.

Kerr–Newman–AdS–dS BH. The Kerr–Newman–AdS–dS black hole is a solution of the Einstein–Maxwell equations with a nonvanishing cosmological constant [16]. Its near-horizon limit has been constructed in [17], while the conformal mechanics on this background was built in [18]. The Hamiltonian of the corresponding spherical mechanics reads

$$\mathcal{I} = \frac{p_\theta^2}{\alpha(\theta)} + \left(\frac{\Gamma(\theta)}{\gamma(\theta)} - k^2\right)[p_\varphi + e\lambda(\theta)]^2 + U(\theta). \tag{11}$$

It describes a particle probe on a two-dimensional curved space with the metric

$$ds^2 = \alpha(\theta)\, d\theta^2 + \frac{d\varphi^2}{\Gamma(\theta)/\gamma(\theta) - k^2}, \tag{12}$$

which moves in the potential and magnetic fields defined by the expressions

$$U(\theta) = m^2\Gamma(\theta) - \frac{e^2k^2f^2(\theta)}{\Gamma(\theta)/\gamma(\theta) - k^2}, \qquad \lambda(\theta)d\varphi = \frac{\Gamma(\theta)f(\theta)}{\Gamma(\theta) - k^2\gamma(\theta)}d\varphi. \tag{13}$$

Here we denoted

$$\Gamma(\theta) = \frac{r_0^2}{1+\nu_+^2}\left(1+\nu_+^2\cos^2\theta\right), \quad \alpha(\theta) = \left(\frac{r_+^2}{r_0^2}\right)\frac{1+\nu_+^2}{1-\nu_0^2\cos^2\theta},$$
$$\gamma(\theta) = \left[\frac{r_+(1+\nu_+^2)}{1-\nu_0^2}\right]\frac{(1-\nu_0^2\cos^2\theta)\sin^2\theta}{1+\nu_+^2\cos^2\theta}, \tag{14}$$
$$f(\theta) = \frac{1+\nu_+^2}{\nu_+(1-\nu_0^2)}\frac{\frac{q_e}{2}(1-\nu_+^2\cos^2\theta)+q_m\cos\theta}{1+\nu_+^2\cos^2\theta}$$

and used the following notation for the constant parameters:

$$\nu_+ \equiv \frac{a}{r_+}, \quad \nu_0 \equiv \frac{a}{l}, \quad k \equiv 2\left(\frac{r_0}{r_+}\right)^2\frac{1+\nu_0^2}{\nu_+(1+\nu_+^2)^2},$$
$$r_0^2 = r_+^2\frac{(1+\nu_+^2)(1-r_+^2/l^2)}{1+6r_+^2/l^2-3r_+^4/l^4-q^2/l^2}. \tag{15}$$

Above m and e are the mass and the electric charge of a particle, r_+ is the horizon radius and l^2 is linked to the cosmological constant via $\Lambda = -3/l^2$. The parameters M, a, q_e and q_m are related to the mass, angular momentum, electric and magnetic charges of the black hole, respectively (for explicit relations, see, e.g., [17])

$$a^2 = \frac{r_+^2(1+3r_+^2/l^2)-q^2}{1-r_+^2/l^2}, \qquad M = \frac{r_+[(1+r_+^2/l^2)^2-q^2/l^2]}{1-r_+^2/l^2}. \tag{16}$$

This system reduces to the near-horizon Kerr particle when $q_e = q_m = 0$ and $l^2 \to \infty$.

$5d$ **Myers–Perry BH.** In the case of the five-dimensional near-horizon Myers–Perry black hole, one reveals a three-dimensional integrable system governed by the Hamiltonian

$$\mathcal{I} = \frac{1}{4}p_\theta^2 + \frac{\rho_0^4}{4(a+b)^2}\left[\frac{p_{\tilde\phi}^2}{a^2\sin^2\theta}+\frac{p_\psi^2}{b^2\cos^2\theta}-\frac{1}{\rho_0^2}\left(\frac{b}{a}p_\phi+\frac{a}{b}p_\psi\right)^2\right] -$$
$$-\frac{1}{4}\left(\sqrt{\frac{b}{a}}p_\phi+\sqrt{\frac{a}{b}}p_\psi\right)^2 + m^2\rho_0^2, \quad \rho_0^2 = ab + a^2\cos^2\theta + b^2\sin^2\theta. \tag{17}$$

This system is integrable but not exactly solvable for arbitrary values of rotation parameters a, b. For the special case that the rotation parameters are equal to each other $a = b$, it becomes exactly solvable and maximally superintegrable

$$\mathcal{I} = \frac{1}{4}\left[p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} + \frac{p_\psi^2}{\cos^2\theta} - \frac{3}{2}(p_\phi+p_\psi)^2 + 8(mr_0)^2\right]. \tag{18}$$

Fixing the momenta p_ϕ, p_ψ, we arrive at the one-dimensional system on the circle given by the modified Pöshle–Teller potential.

$5d$ **Myers–Perry–AdS–dS BH.** A generalization of the five-dimensional rotating black hole solution by Myers and Perry to include a cosmological constant was constructed in [19]. Its near-horizon limit was built in [20]. The corresponding spherical mechanics reads

$$\mathcal{I} = \frac{1}{2}\Delta_\theta p_\theta^2 + \frac{1}{2}\left(\frac{\rho_0^4}{\Delta_\theta \sin^2\theta} - \frac{(1+r_0^2/l^2)b^2\rho_0^2}{\Delta_\theta} - \frac{4a^2(r_0^2+b^2)^2}{4r_0^2}\right)p_\phi^2 + \\ + \frac{1}{2}\left(\frac{\rho_0^4}{\Delta_\theta \cos^2\theta} - \frac{(1+r_0^2/l^2)a^2\rho_0^2}{\Delta_\theta} - \frac{4b^2(r_0^2+a^2)^2}{4r_0^2}\right)p_\psi^2 - \\ - \left(\frac{(1+r_0^2/l^2)ab\rho_0^2}{\Delta_\theta} + \frac{4ab(r_0^2+a^2)(r_0^2+b^2)}{4r_0^2}\right)p_\phi p_\psi + g^2\cos^2\theta. \quad (19)$$

Here g^2 is a coupling constant which vanishes for $a = b$, m is the particle mass, and we denoted

$$\Delta = \frac{1}{r^2}(r^2+a^2)(r^2+b^2)\left(1+\frac{r^2}{l^2}\right) - 2M, \quad \Delta_\theta = 1 - \frac{a^2\cos^2\theta}{l^2} - \frac{b^2\sin^2\theta}{l^2}, \\ \rho_0^2 = ab + a^2\cos^2\theta + b^2\sin^2\theta, \quad \Xi_a = 1 - \frac{a^2}{l^2}, \quad \Xi_b = 1 - \frac{b^2}{l^2}. \quad (20)$$

The parameters M, a, and b are linked to the mass and the angular momenta (for explicit relations, see, e.g., [20]). l^2 is taken to be positive for AdS and negative for dS and is related to the cosmological constant via $\Lambda = -6/l^2$.

Higher-Dimensional Rotating BH [2,3,21]. The Hamiltonian of the spherical mechanics associated with a rotating extremal near-horizon black hole in $d = 2n+1$ dimensions, which has equal rotation parameters, reads

$$\mathcal{I} = \sum_{i,j=1}^{n-1}(\delta_{ij} - \mu_i\mu_j)p_{\mu_i}p_{\mu_j} + \sum_{i=1}^{n}\frac{p_{\phi_i}^2}{\mu_i^2} - \frac{(n+1)}{n}\left(\sum_{i=1}^{n}p_{\phi_i}\right)^2, \quad (21)$$

where (μ_i, p_{μ_i}) with $i = 1, \ldots, n-1$, and (ϕ_j, p_{ϕ_j}) with $j = 1, \ldots, n$ form canonical pairs obeying the conventional Poisson brackets. μ_n^2 entering the second sum in (21) is found from the unit sphere equation $\sum\limits_{i=1}^{n}\mu_i^2 = 1$. For $d = 2n$ the Hamiltonian is modified to take the form

$$\mathcal{I} = \sum_{i,j=1}^{n-1}((2n-3)\rho_0^2\delta_{ij} - \mu_i\mu_j)p_{\mu_i}p_{\mu_j} + \sum_{i,j=1}^{n-1}\left(\frac{(2n-3)\rho_0^2}{\mu_i^2}\delta_{ij} - \\ - \frac{(2n-3)^2\rho_0^2}{2(n-1)} - \frac{2}{n-1}\right)p_{\phi_i}p_{\phi_j} + m^2\rho_0^2, \quad \rho_0^2 = \frac{2(n-1)}{2n-3} - \sum_{i=1}^{n-1}\mu_i^2, \quad (22)$$

where (μ_i, p_{μ_i}) and (ϕ_j, p_{ϕ_j}) with $j = 1, \ldots, n-1$ form canonical pairs, and m^2 is a coupling constant. Note that, as compared to the previous case, the number of the azimuthal coordinates is decreased by one.

Considering further reduction over the cyclic variables and investigating the integrability, we established in [2] that the spherical mechanics corresponding to the $(2n+1)$-dimensional black hole is a maximally superintegrable and exactly solvable system, i.e., it is completely similar to the five-dimensional black hole with the coinciding rotation parameters. In contrast with this case, the spherical mechanics corresponding to the $2n$-dimensional black hole lacks only one constant of the motion to become maximally superintegrability and is not exactly solvable. The solution of its equations of motion is given by elliptic integrals and is similar to that derived for the Kerr background in [13].

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