Письма в ЭЧАЯ. 2014. Т. 11, № 7(191). С. 1394–1402

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

ON BRANE SYMMETRIES

A. A. Zheltukhin [1]

Kharkov Institute of Physics and Technology, Kharkov, Ukraine

The geometric approach to branes is reformulated in terms of gauge vector fields interacting with massless tensor multiplets in gravitational backgrounds.

PACS: 11.25.-w

Study of nonlinear dynamics of p-branes [1–14], as well as their quantization, require new tools. The geometric approach [15–17] originally developed for strings seems to be relevant to the problem. The gauge reformulation [18] of this approach using the ideas of Cartan [19], Volkov [20] and Faddeev [21] has shown that strings in D-dimensional space-time form a closed sector of states of the exactly integrable two-dimensional $SO(1,1) \times SO(D-2)$ gauge model. The geometric approach has turned out to be promising for investigation of integrability of branes PDEs [22–24]. Here, we adopt the string gauge approach to p-branes and constuct new gauge-invariant models which have brane solutions.

1. A time-like $(p+1)$-dimensional hypersurface Σ_{p+1} embedded into the D-dimensional Minkowski space-time with the signature $\eta_{mn} = (+,-,\ldots,-)$ is described by its radius vector $\mathbf{x}(\xi^\mu)$ parameterized by the coordinates $\xi^\mu = (\tau, \sigma^r)$, $(r = 1,2,\ldots,p)$. Using a local orthonormal frame $\mathbf{n}_A(\xi^\mu) = (\mathbf{n}_i, \mathbf{n}_a)$ with $A = (i,a)$, attached to Σ_{p+1}, one can expand the infinitesimal displacements $d\mathbf{x}(\xi^\mu)$ and $d\mathbf{n}_A(\xi^\mu)$ in the local basis $\mathbf{n}_A(\xi^\mu)$ at the point ξ^μ

$$d\mathbf{x}(\xi) = \omega^i(\xi)\,\mathbf{n}_i(\xi), \quad \omega^a(\xi) = 0, \tag{1}$$

$$d\mathbf{n}_A(\xi) = -\omega_A{}^B(\xi)\,\mathbf{n}_B(\xi), \tag{2}$$

with the vectors $\mathbf{n}_i(\xi)$, $(i,k = 0,1,\ldots,p)$ tangent and $\mathbf{n}_a(\xi)$, $(a,b = p+1, p+2, \ldots, D-p-1)$ — normal to the hypersurface. The choice $\omega^a = 0$ of the normal displacement of $\mathbf{x}$ breaks down the local Lorentz group $SO(1,D-1)$ of the moving frame to its subgroup $SO(1,p) \times SO(D-p-1)$. Then, the antisymmetric matrix differential form $\omega_{AB} = -\omega_{BA}$ parameterized by ξ^μ and belonging to the Lie algebra of $SO(1,D-1)$ splits into three blocks

$$\omega_A{}^B \equiv \omega_{\mu A}{}^B d\xi^\mu = \begin{pmatrix} A_{\mu i}{}^k & W_{\mu i}{}^b \\ W_{\mu a}{}^k & B_{\mu a}{}^b \end{pmatrix} d\xi^\mu, \tag{3}$$

[1] E-mails: aaz@physto.se; aaz@nordita.org

where ${A_{\mu i}}^k$ and ${B_{\mu a}}^b$ are transformed as the gauge fields of the $SO(1,p)$ and $SO(D-p-1)$ groups on the base space Σ_{p+1}, respectively, and their field strengths ${F_{\mu\nu i}}^k$ and ${H_{\mu\nu a}}^b$ are

$$ {F_{\mu\nu i}}^k \equiv [D^{||}_\mu,\, D^{||}_\nu]_i{}^k = (\partial_{[\mu}A_{\nu]} + A_{[\mu}A_{\nu]})_i{}^k, \tag{4} $$

$$ {H_{\mu\nu a}}^b \equiv [D^{\perp}_\mu,\, D^{\perp}_\nu]_a{}^b = (\partial_{[\mu}B_{\nu]} + B_{[\mu}B_{\nu]})_a{}^b. \tag{5} $$

The derivative $D^{||}_\mu$ in (4) is covariant with respect to the Lorentz gauge group $SO(1,p)$ of the subspaces tangent to Σ_{p+1}

$$ D^{||}_\mu \phi^i_\nu = \partial_\mu \phi^i_\nu + {A_\mu}^i{}_k\, {\phi_\nu}^k. \tag{6} $$

The covariant derivative $D^{\perp}_\mu$ corresponds to the gauge group $SO(D-p-1)$ of rotations of the local subspaces orthogonal to Σ_{p+1}

$$ D^{\perp}_\mu \phi^a_\nu = \partial_\mu \phi^a_\nu + {B_\mu}^a{}_b\, {\phi_\nu}^b. \tag{7} $$

The off-diagonal blocks ${W_{\mu i}}^b$ in (3) are transformed like charged vector multiplets of the gauge group $SO(1,p)\times SO(D-p-1)$ with their covariant derivatives

$$ (D_\mu W_\nu)_i{}^a = \partial_\mu {W_{\nu i}}^a + {A_{\mu i}}^k {W_{\nu k}}^a + {B_\mu}^a{}_b {W_{\nu i}}^b \tag{8} $$

including the gauge fields ${A_{\mu i}}^k$ and ${B_{\mu a}}^b$.

The integrability conditions of PDEs (1) and (2) are the Maurer–Cartan (MC) equations

$$ d\wedge\omega_A + {\omega_A}^B \wedge \omega_B = 0, \tag{9} $$
$$ d\wedge{\omega_A}^B + {\omega_A}^C \wedge {\omega_C}^B = 0 \tag{10} $$

of the structure of the ambient D-dimensional space with zero torsion and curvature, where the symbols $\wedge$ and $d\wedge$ mean the wedge product and external differential, respectively.

One can see that Eqs. (10), called the Gauss–Codazzi (GC) equations in the differential geometry of surfaces, contain only the differential form ${\omega_A}^B$. The splitting (3) of the matrix indices $A\to(i,a)$ in (10) results in the field representation of the GC equations

$$ {F_{\mu\nu i}}^k = -(W_{[\mu}W_{\nu]})_i{}^k, \tag{11} $$

$$ {H_{\mu\nu a}}^b = -(W_{[\mu}W_{\nu]})_a{}^b, \tag{12} $$

$$ (D_{[\mu}W_{\nu]})_i{}^a = 0, \tag{13} $$

where $[\mu,\nu]$ means antisymmetrization in μ,ν, e.g., $\hat{W}_{[\mu}\hat{W}_{\nu]} \equiv \hat{W}_\mu\hat{W}_\nu - \hat{W}_\nu\hat{W}_\mu$.

For $p=1$ the above constraints coincide with the ones discussed upon the gauge reformulation of the geometric approach for strings [18]. This reformulation reveals an isomorphism between the Nambu–Goto string in the D-dimensional Minkowski space and the exactly solvable sector of the two-dimensional $SO(1,1)\times SO(D-2)$ gauge model including a massless scalar multiplet.

Our main goal is to generalize the string case to p-branes, which implies construction of a $(p+1)$-dimensional $SO(1,p)\times SO(D-p-1)$ gauge model compatible with the GC equations (11)–(13). This step does not suppose in advance any connections of such a model with the existing models for p-branes, but only takes into account the independence of the constraints (11)–(13) of the induced metrics of hypersurfaces imbedded into flat spaces. A class of new gauge actions compatible with the GC constraints is proposed in the next section.

2. The desired $SO(1,p) \times SO(D-p-1)$ gauge-invariant action has to describe the gauge and vector fields in an external gravitational field in $(p+1)$-dimensional pseudo-Riemannian space with a metric $g_{\mu\nu}(\xi)$ parameterized by the coordinates ξ^{μ} (that will be later identified with the coordinates parameterizing the brane hypersurface Σ_{p+1}). The metric $g_{\mu\nu}$ is not considered as a dynamical field in contrast to the fields presented in the GC constraints (11)–(13). The desired gauge and reparameterization invariant action has the form

$$S = \gamma \int d^{p+1}\xi \sqrt{|g|}\, \mathcal{L}, \tag{14}$$

$$\mathcal{L} = \frac{1}{4}\,\mathrm{Sp}\,(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}\,\mathrm{Sp}\,(H_{\mu\nu}H^{\mu\nu}) + \frac{1}{2}\hat{\nabla}_{\mu}W_{\nu}^{ia}\,\hat{\nabla}^{\{\mu}W_{ia}^{\nu\}} - \hat{\nabla}_{\mu}W^{\mu ia}\,\hat{\nabla}_{\nu}W_{ia}^{\nu} + V, \tag{15}$$

where $\{\mu,\nu\}$ means symmetrization in μ and ν, V encodes nonlinear (self)interactions of the vector multiplet ${W_{\mu}}^{ia}$. The generalized covariant derivative $\hat{\nabla}_{\mu}$ in (15) is

$$\hat{\nabla}_{\mu}W_{\nu ia} := \partial_{\mu}W_{\nu ia} - \Gamma^{\rho}_{\mu\nu}W_{\rho ia} + {A_{\mu i}}^{k}W_{\nu ka} + {B_{\mu a}}^{b}W_{\nu ib} \tag{16}$$

and extends the general covariant derivative including only the Levi–Chivita connection

$$\bigtriangledown_{\mu}W_{\nu ia} = \partial_{\mu}W_{\nu ia} - \Gamma^{\rho}_{\mu\nu}W_{\rho ia}, \quad \bigtriangledown_{\mu}g_{\nu\rho} = 0, \tag{17}$$

where $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu} = (1/2)g^{\rho\gamma}(\partial_{\mu}g_{\nu\gamma} + \partial_{\nu}g_{\mu\gamma} - \partial_{\gamma}g_{\mu\nu})$ are the Cristoffel symbols.

The variation of S (14) in the gauge and vector fields results in the following EOM:

$$\hat{\nabla}_{\mu}F^{\mu\nu}_{ik} = -\hat{\nabla}_{\mu}(W^{[\mu}_{ia}{W^{\nu]a}}_{k}) - \frac{1}{2}W_{\mu[i|a}\hat{\nabla}^{[\nu}{W^{\mu]a}}_{|k]}, \tag{18}$$

$$\hat{\nabla}_{\mu}H^{\mu\nu}_{ab} = -\hat{\nabla}_{\mu}(W^{[\mu}_{ai}{W^{\nu]i}}_{b}) - \frac{1}{2}W_{\mu[a|i}\hat{\nabla}^{[\nu}{W^{\mu]i}}_{|b]}, \tag{19}$$

$$\hat{\nabla}_{\mu}\hat{\nabla}^{\{\mu}W^{\nu\}ia} = 2\hat{\nabla}^{\nu}\hat{\nabla}_{\mu}W^{\mu ia} + \frac{\partial V}{\partial W_{\nu ia}}. \tag{20}$$

With the help of shifted gauge field strengths $\mathcal{F}^{\mu\nu}_{ik}$ and $\mathcal{H}^{\mu\nu}_{ab}$,

$$\mathcal{F}^{ik}_{\mu\nu} = (F_{\mu\nu} + W_{[\mu}W_{\nu]})^{ik}, \tag{21}$$

$$\mathcal{H}^{ab}_{\mu\nu} = (H_{\mu\nu} + W_{[\mu}W_{\nu]})^{ab}, \tag{22}$$

one can present Eqs. (18)–(20) in the compact form

$$\hat{\nabla}_{\mu}\mathcal{F}^{\mu\nu}_{ik} = -\frac{1}{2}W_{\mu[i|a}\hat{\nabla}^{[\nu}{W^{\mu]a}}_{|k]}, \tag{23}$$

$$\hat{\nabla}_{\mu}\mathcal{H}^{\mu\nu}_{ab} = -\frac{1}{2}W_{\mu[a|i}\hat{\nabla}^{[\nu}{W^{\mu]i}}_{|b]}, \tag{24}$$

$$\hat{\nabla}_{\mu}\hat{\nabla}^{[\mu}W^{\nu]ia} = -2[\hat{\nabla}^{\mu},\, \hat{\nabla}^{\nu}]W^{ia}_{\mu} + \frac{\partial V}{\partial W_{\nu ia}}. \tag{25}$$

Further we take into account the generalized first Bianchi identity

$$[\hat{\nabla}_\mu,\, \hat{\nabla}_\nu] = \hat{R}_{\mu\nu} + \hat{F}_{\mu\nu} + \hat{H}_{\mu\nu}, \tag{26}$$

where the Riemann–Cristoffel tensor $\hat{R}_{\mu\nu} \equiv R_{\mu\nu}{}^{\gamma}{}_{\lambda}$ is defined as

$$[\bigtriangledown_\mu,\, \bigtriangledown_\nu]V^\gamma = R_{\mu\nu}{}^{\gamma}{}_{\lambda}V^\lambda =: (\partial_{[\mu}\Gamma^{\gamma}_{\nu]\lambda} + \Gamma^{\gamma}_{[\mu|\rho}\Gamma^{\rho}_{|\nu]\lambda})V^\lambda. \tag{27}$$

The identity (26) allows one to present Eq. (25) in the form

$$\begin{aligned}\frac{1}{2}\hat{\nabla}_\mu\hat{\nabla}^{[\mu}W^{\nu]ia} - \mathcal{F}^{\mu\nu i}{}_{k}W^{ka}_{\mu} - \mathcal{H}^{\mu\nu a}{}_{b}W^{ib}_{\mu} = \\ = \frac{1}{2}\frac{\partial V}{\partial W_{\nu ia}} + ([[W^\mu, W^\nu], W_\mu])^{ia} - R^{\mu\nu}W^{ia}_{\mu},\end{aligned} \tag{28}$$

where $R_{\nu\lambda} := R^{\mu}{}_{\nu\mu\lambda}$ is the Ricci tensor. Using the relation

$$\frac{1}{4}\frac{\partial}{\partial W_{\nu ia}}(W_\mu[[W^\mu, W^\rho], W_\rho])^{i}{}_{i} = ([[W^\mu, W^\nu], W_\mu])^{ia} \tag{29}$$

with the commutators of $\hat{W}_\mu$ in the r.h.s., we introduce a shifted potential $\mathcal{V}$

$$\mathcal{V} = V + \frac{1}{2}\mathrm{Sp}\,(W_\mu[[W^\mu, W^\rho], W_\rho]), \tag{30}$$

where the trace $\mathrm{Sp}\,(W_\mu[[W^\mu, W^\rho], W_\rho]) =: (W_\mu[[W^\mu, W^\rho], W_\rho])^{i}{}_{i}$.

As a result, EOM (23), (24) and (28) take the following form:

$$\hat{\nabla}_\mu\mathcal{F}^{\mu\nu}_{ik} = -\frac{1}{2}W_{\mu[i|a}\hat{\nabla}^{[\nu}W^{\mu]a}{}_{|k]}, \tag{31}$$

$$\hat{\nabla}_\mu\mathcal{H}^{\mu\nu}_{ab} = -\frac{1}{2}W_{\mu[a|i}\hat{\nabla}^{[\nu}W^{\mu]i}{}_{|b]}, \tag{32}$$

$$\frac{1}{2}\hat{\nabla}_\mu\hat{\nabla}^{[\mu}W^{\nu]ia} + \mathcal{F}^{\mu\nu i}{}_{k}W^{ka}_{\mu} + \mathcal{H}^{\mu\nu a}{}_{b}W^{ib}_{\mu} = \frac{1}{2}\frac{\partial\mathcal{V}}{\partial W_{\nu ia}} - R^{\mu\nu}W^{ia}_{\mu}. \tag{33}$$

Then, we observe that the first-order PDEs, which coincide with (11)–(13),

$$\mathcal{F}^{ik}_{\mu\nu} = 0, \quad \mathcal{H}^{ab}_{\mu\nu} = 0, \quad \hat{\nabla}^{[\mu}W^{\nu]}_{ia} = 0, \tag{34}$$

form a particular solution of Eqs. (31)–(33) on condition that

$$\frac{1}{2}\frac{\partial\mathcal{V}}{\partial W_{\nu ia}} - R^{\mu\nu}W^{ia}_{\mu} = 0. \tag{35}$$

Due to independence of the Ricci tensor $R^{\mu\nu}$ of $W_{\nu ia}$, Eq. (35) allows one to restore $\mathcal{V}$

$$\mathcal{V} = R^{\mu\nu}W^{ia}_{\mu}W_{\nu ia}. \tag{36}$$

Thus, we find that the action (14) with the Lagrangian density

$$\mathcal{L} = \frac{1}{4}\mathrm{Sp}\,(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}\mathrm{Sp}\,(H_{\mu\nu}H^{\mu\nu}) + \frac{1}{2}\hat{\nabla}_\mu W^{ia}_\nu\,\hat{\nabla}^{\{\mu}W^{\nu\}}_{ia} - \\ - \hat{\nabla}_\mu W^{\mu ia}\,\hat{\nabla}_\nu W^\nu_{ia} + R^{\mu\nu}W^{ia}_\mu W_{\nu ia} - \frac{1}{2}\mathrm{Sp}\,(W_\mu[[W^\mu, W^\rho], W_\rho]) \quad (37)$$

yields the nonlinear Euler–Lagrange equations

$$\hat{\nabla}_\mu \mathcal{F}^{\mu\nu}_{ik} = -\frac{1}{2}W_{\mu[i|a}\hat{\nabla}^{[\nu}W^{\mu]a}{}_{|k]}, \quad (38)$$

$$\hat{\nabla}_\mu \mathcal{H}^{\mu\nu}_{ab} = -\frac{1}{2}W_{\mu[a|i}\hat{\nabla}^{[\nu}W^{\mu]i}{}_{|b]}, \quad (39)$$

$$\frac{1}{2}\hat{\nabla}_\mu\hat{\nabla}^{[\mu}W^{\nu]ia} + \mathcal{F}^{\mu\nu i}{}_k W^{ka}_\mu + \mathcal{H}^{\mu\nu a}{}_b W^{ib}_\mu = 0 \quad (40)$$

for the gauge $A_{\mu i}{}^k$, $B_{\mu a}{}^b$ and vector $W_{\mu ia}$ fields in a given external gravitational field $g_{\mu\nu}(\xi^\rho)$.

It is easy to see that Eqs. (38)–(40) have the particular solution (34), which coincides with the GC constraints (11), (12) and (13).

This solves the stated problem of the construction of gauge-invariant model compatible with embedded hypersurfaces using the Gauss mapping. In addition, note that the action (14) with $\mathcal{L}$ (37) looks like a natural generalization of the four-dimensional Dirac scale-invariant gravity theory with the dynamical dilaton and gravitational field $g_{\mu\nu}$ (see, e.g., [26]).

The above-said hints at consideration of the $(p+1)$-dimensional space-time of the gauge model defined by (14), (37) as a $(p+1)$-dimensional world hypersurface swept by a p-brane in the D-dimensional Minkowski space. Our next step is to prove that the conjecture follows from the remaining Maurer–Cartan equations (9) and to find the corresponding modification of the proposed model.

3. To prove the mentioned statement, we come back to the MC equations (9) and split their matrix indices $A \to (i, a)$. This yields the following equations:

$$D^{||}_{[\mu}\omega^i_{\nu]} = 0, \quad (41)$$

$$\omega^i_{[\mu}W_{\nu]ia} = 0, \quad (42)$$

with the derivative $D^{||}_\mu$ defined by (6). As shows $d\mathbf{x}$ squaring, the object ω^i_μ plays the role of a $(p+1)$-bein for the hypersurface Σ_{p+1}, which connects its orthonormal frame $\mathbf{n}_i$ with the local natural frame $\mathbf{e}_\mu$, and represents the metric $G_{\mu\nu}(\xi^\rho)$ of Σ_{p+1} by the quadratic form

$$\omega^i_\mu\omega^\mu_k = \delta^i_k, \quad \mathbf{e}_\mu = \omega^i_\mu\mathbf{n}_i, \quad G_{\mu\nu} = \omega^i_\mu\eta_{ik}\omega^k_\nu. \quad (43)$$

One can solve the constraints (42) and express $W_{\mu i}{}^a$ in terms of the symmetric components $l_{\mu\nu}{}^a$ of the second fundamental form of Σ_{p+1}:

$$W_{\mu i}{}^a = -l_{\mu\nu}{}^a\omega^\nu_i, \quad l_{\mu\nu}{}^a := \mathbf{n}^a\partial_{\mu\nu}\mathbf{x}. \quad (44)$$

The general solution of the constraints (41) is equivalent to the "tetrade postulate"

$$\nabla^{||}_{\mu}\omega^i_{\nu} \equiv \partial_\mu\omega^i_\nu - \Gamma^\rho_{\mu\nu}\omega^i_\rho + A_\mu{}^i{}_k\omega^k_\nu = 0, \tag{45}$$

which identifies the gauge connection $A_\mu{}^i{}_k$ with the background metric connection $\Gamma^\rho_{\mu\nu}$ by means of the gauge transformation

$$\Gamma^\rho_{\nu\lambda} = \omega^\rho_i A_{\nu ik}\omega^k_\lambda + \partial_\nu\omega^k_\lambda\omega^\rho_k \equiv \omega^\rho_i D^{||}_\nu\omega^i_\lambda. \tag{46}$$

Therefore, the hypersurface metric $G_{\mu\nu}$ has to be identified with the backgroud metric $g_{\mu\nu}$ introduced *ad hoc* in the gauge-invariant action (14).

Then, the Riemann tensor $R_{\mu\nu}{}^\gamma{}_\lambda$ (27) and the field strength $F_{\mu\nu i}{}^k$ (4) become dependent

$$R_{\mu\nu}{}^\gamma{}_\lambda = \omega^\gamma_i F_{\mu\nu}{}^i{}_k\omega^k_\lambda, \quad R_{\nu\lambda} = \omega^\mu_i F_{\mu\nu}{}^i{}_k\omega^k_\lambda, \tag{47}$$

and the use of the GC constraint (11) for $F_{\mu\nu}{}^i{}_k$ allows one to express the Ricci tensor as

$$R^{\nu\lambda} = -\omega^i_\mu (W^{[\mu}W^{\nu]})_i{}^k\omega^\lambda_k. \tag{48}$$

Taking into account (43)–(47) permits one to transit from the gauge $A_{\nu ik}$ and vector $W_{\mu i}{}^a$ fields to the Cristoffel symbols and $l_{\mu\nu}{}^a = -\omega^i_\nu W_{\mu i}{}^a$, respectively, that transforms (11)–(13) into

$$R_{\mu\nu}{}^\gamma{}_\lambda = l_{[\mu}{}^{\gamma a}l_{\nu]\lambda a}, \tag{49}$$

$$H_{\mu\nu}{}^{ab} = l_{[\mu}{}^{\gamma a}l_{\nu]\gamma}{}^b, \tag{50}$$

$$\nabla^\perp_{[\mu}l_{\nu]\rho a} = 0, \tag{51}$$

where $\nabla^\perp_\mu l_{\nu\rho}{}^a := \partial_\mu l_{\nu\rho}{}^a - \Gamma^\lambda_{\mu\nu}l_{\lambda\rho}{}^a - \Gamma^\lambda_{\mu\rho}l_{\nu\lambda}{}^a + B^{ab}_\mu l_{\nu\rho b}$.

As is seen, exclusion of $F_{\mu\nu i}{}^k$ transforms the constraint (11) into (49), which generalizes the *Gauss Theorema Egregium* for a $(p+1)$-dimensional hypersurface embedded into the D-dimensional Minkowski space. The absence of $F_{\mu\nu i}{}^k$ allows one not to consider the group $SO(1,p)$ as an explicit symmetry of the desired action. As a result, we obtain the following $SO(D-p-1)$ gauge-invariant action in a gravitational background possessing the solution (49)–(51):

$$S = \gamma\int d^{p+1}\xi\sqrt{|g|}\,\mathcal{L},$$

$$\mathcal{L} = -\frac{1}{4}\mathrm{Sp}\,(H_{\mu\nu}H^{\mu\nu}) + \frac{1}{2}\nabla^\perp_\mu l_{\nu\rho a}\nabla^{\perp\{\mu}l^{\nu\}\rho a} - \nabla^\perp_\mu l^\mu_{\rho a}\nabla^\perp_\nu l^{\nu\rho a} - \\ - \frac{1}{2}\mathrm{Sp}\,(l_a l_b)\mathrm{Sp}\,(l^a l^b) + \mathrm{Sp}\,(l_a l_b l^a l^b) - \mathrm{Sp}\,(l_a l^a l_b l^b). \tag{52}$$

To prove this, let us consider the following action:

$$S = \gamma\int d^{p+1}\xi\sqrt{|g|}\left\{-\frac{1}{4}\,\mathrm{Sp}\,(H_{\mu\nu}H^{\mu\nu}) + \right. \\ \left. + \frac{1}{2}\nabla^\perp_\mu l_{\nu\rho a}\nabla^{\perp\{\mu}l^{\nu\}\rho a} - \nabla^\perp_\mu l^\mu_{\rho a}\nabla^\perp_\nu l^{\nu\rho a} + V\right\}. \tag{53}$$

Variation of (53) in the dynamical fields ${l_{\mu\nu}}^a$, ${B_\mu}^{ab}$ gives their EOM

$$\nabla^\perp_\nu \mathcal{H}^{\nu\mu}_{ab} = \frac{1}{2} l_{\nu\rho[a} \nabla^{\perp[\mu} l^{\nu]\rho}{}_{b]}, \tag{54}$$

$$\frac{1}{2}\nabla^\perp_\mu \nabla^{\perp[\mu} l^{\nu]\rho a} = -[\nabla^{\perp\mu}, \nabla^{\perp\nu}] {l_\mu}^{\rho a} + \frac{1}{2}\frac{\partial V}{\partial l_{\nu\rho a}}, \tag{55}$$

where $\mathcal{H}^{ab}_{\mu\nu} := H^{ab}_{\mu\nu} - l_{[\mu}{}^{\gamma a} l_{\nu]\gamma}{}^b$. Equations (54), (55) have the GC constraints (50), (51)

$$\mathcal{H}^{ab}_{\mu\nu} = 0, \quad \nabla^\perp_{[\mu} l_{\nu]\rho a} = 0, \tag{56}$$

as their particular solution provided that

$$\frac{1}{2}\frac{\partial V}{\partial l_{\nu\rho a}} = [\nabla^{\perp\mu}, \nabla^{\perp\nu}] {l_\mu}^{\rho a}. \tag{57}$$

With the help of the GC equations (49)–(51) and the Bianchi identity

$$[\nabla^\perp_\gamma, \nabla^\perp_\nu] l^{\mu\rho a} = R_{\gamma\nu}{}^\mu{}_\lambda l^{\lambda\rho a} + R_{\gamma\nu}{}^\rho{}_\lambda l^{\mu\lambda a} + H_{\gamma\nu}{}^a{}_b l^{\mu\rho b}, \tag{58}$$

one can transform (57) into solvable equation for the self-interaction potential V

$$\frac{1}{2}\frac{\partial V}{\partial l_{\nu\rho a}} = (l^a l^b)^{\rho\nu}\,\mathrm{Sp}\,(l_b) + (2 l_b l^a l^b - l^a l_b l^b - l_b l^b l^a)^{\rho\nu} - l^{\rho\nu b}\,\mathrm{Sp}\,(l_b l^a). \tag{59}$$

Equation (59) has the following solution for V accompanied by the trace constraints:

$$V = -\frac{1}{2}\,\mathrm{Sp}\,(l_a l_b)\mathrm{Sp}\,(l^a l^b) + \mathrm{Sp}\,(l_a l_b l^a l^b) - \mathrm{Sp}\,(l_a l^a l_b l^b), \quad \mathrm{Sp}\,(l_a) = 0. \tag{60}$$

The constraints $\mathrm{Sp}\,(l_a) = 0$ express the well-known algebraic conditions of minimality for a $(p+1)$-dimensional hypersurface embedded into the Minkowski spaces. These conditions are equivalent to the nonlinear equations of motion of p-branes

$$\Box^{(p+1)}\mathbf{x} = 0, \tag{61}$$

where $\Box^{(p+1)} := \frac{1}{\sqrt{|G|}}\partial_\alpha \sqrt{|G|} G^{\alpha\beta}\partial_\beta$ is the reparameterization invariant Laplace–Beltrami operator on Σ_{p+1} [24].

Equation (61) follows from the Dirac action for p-branes with the minimal hypersurfaces in the Minkowski space-time

$$S = T\int d^{p+1}\xi\sqrt{|G|}, \tag{62}$$

where G is the determinant of the induced metric $G_{\alpha\beta} := \partial_\alpha\mathbf{x}\partial_\beta\mathbf{x}$.

It proves that the $SO(D-p-1)$ gauge-invariant action (52) for the interacting gauge and tensor fields B^{ab}_μ and $l^a_{\mu\nu}$, respectively, in a gravitational background has the particular solution presented by the first-order Gauss–Codazzi PDEs (49)–(51). The solution describes minimal $(p+1)$-dimensional hypersurfaces embedded into the D-dimensional Minkowski space-time.

To sum up, the gauge reformulation of the geometric approach to $(p+1)$-dimensional hypersurfaces embedded into the D-dimensional Minkowski space was proposed. The new set of $SO(1,p) \times SO(D-p-1)$ gauge-invariant models possessing exact solutions for gauge fields and vector multiplets in gravitational backgrounds, was constructed. The Dirac p-branes were shown to be the solutions of $(p+1)$-dimensional gauge model presented by the Gauss–Codazzi constraints for the $SO(D-p-1)$ gauge vector fields and massless tensor multiplets in curved backgrounds.

Acknowledgements. I am grateful to E. Ivanov and V. Pervushin for interesting remarks, A. Rosly for valuable comments and sending the paper [27], to Physics Department of Stockholm University and Nordic Institute for Theoretical Physics NORDITA for kind hospitality and support.

REFERENCES

1. *Floratos E., Illipoulos J.* A Note on the Classical Symmetries of the Closed Bosonic Membranes // Phys. Lett. B. 1988. V. 201. P. 237.
2. *de Witt B., Hoppe J., Nicolai G.* On the Quantum Mechanics of Supermembranes // Nucl. Phys. B. [FS23]. 1988. V. 305. P. 545.
3. *de Witt B., Lusher M., Nicolai G.* The Supermembrane is Unstable // Nucl. Phys. B. 1989. V. 320. P. 135.
4. *Bandos I. A., Zheltukhin A. A.* Null Super p-Branes Quantum Theory in Four-Dimensional Space-Time // Fortschr. Phys. 1993. V. 41. P. 619;
   $N=1$ Super p-Branes in Twistor-Like Lorentz Harmonic Formulation // Class. Quant. Grav. 1995. V. 12. P. 609.
5. *Bordemann M., Hoppe J.* The Dynamics of Relativistic Membranes I: Reduction to 2-Dimensional Fluid Dynamics // Phys. Lett. B. 1993. V. 317. P. 315;
   The Dynamics of Relativistic Membranes II: Nonlinear Waves and Covariantly Reduced Membrane Equations // Ibid. V. 325. P. 359.
6. *Polchinski J.* Dirichlet Branes and Ramond–Ramond Charges // Phys. Rev. Lett. 1995. V.75. P. 4724.
7. *Collins P. A., Tucker R. W.* Transversity of a Massless Relativistic Membrane // Nucl. Phys. B. 1976. V. 112. P. 150.
8. *Kikkawa K., Yamasaki M.* Can the Membrane be a Unification Model? // Progr. Theor. Phys. 1986. V. 76. P. 1379.
9. *Howe P. S., Sezgin E.* Superbranes // Phys. Lett. B. 1997. V. 390. P. 441.
10. *Hoppe J., Nicolai H.* Relativistic Minimal Surfaces // Phys. Lett. B. 1987. V. 196. P. 451.
11. *Axenides M., Floratos E. G., Perivolaropoulos L.* Rotating Toroidal Branes in Supermembrane and Matrix Theory // Phys. Rev. D. 2002. V. 66. P. 085006.
12. *Arnlind J., Hoppe J., Theisen S.* Spinning Membranes // Phys. Lett. B. 2004. V. 599. P. 118.
13. *Axenides M., Floratos E. G.* Euler Top Dynamics of Nambu–Goto p-Branes // JHEP. 2007. V. 0703. P. 093.
14. *Hoppe J.* $U(1)$-Invariant Membranes and Singularity Formation // Compl. Anal. Oper. Theory. 2009. V. 3. P. 419.
15. *Lund F., Regge T.* Unified Approach to Strings and Vortices with Soliton Solution // Phys. Rev. D. 1976. V. 14. P. 1524.

16. *Omnes R.* A New Geometric Approach to the Relativistic String // Nucl. Phys. B. 1979. V. 149. P. 269.
17. *Barbashov B. M., Nesterenko V. V.* Differential Geometry and Nonlinear Field Models // Fortschr. Phys. 1980. V. 28. P. 427;
    Introduction to the Relativistic String Theory: World Sci., 1990.
18. *Zheltukhin A. A.* Connection between a Relativistic String and Two-Dimensional Field Model // Sov. J. Nucl. Phys. 1981. V. 33. P. 927;
    Classical Relativistic String as an Exactly Solvable Sector of $SO(1,1) \times SO(2)$ Gauge Mode // Phys. Lett. B. 1982. V. 116. P. 147;
    Classical Relativistic String as a Two-Dimensional $SO(1,1) \times SO(2)$ Gauge Model // Theor. Math. Phys. 1982. V. 52. P. 666;
    Gauge Description and Nonlinear String Equations in D-Dimensional Space-Time // Theor. Math. Phys. 1984. V. 56. P. 785.
19. *Cartan E.* Riemannian Geometry in an Orthogonal Frame. Singapore: World Sci., 2001.
20. *Volkov D. V.* // Part. Nucl. 1973. V. 4. P. 3.
21. *Semenov-Tyan-Shansky M. A., Faddeev L. D.* On Theory of Nonlinear Chiral Fields // Vestn. LGU. 1977. V. 13, No. 3 (in Russian).
22. *Trzetrzelewski M., Zheltukhin A. A.* Exact Solutions for $U(1)$ Globally Invariant Membranes // Phys. Lett. B. 2009. V. 679. P. 523.
23. *Zheltukhin A. A., Trzetrzelewski M.* $U(1)$-Invariant Membranes: The Geometric Formulation, Abel and Pendulum Differential Equations // J. Math. Phys. 2010. V. 51. P. 062303.
24. *Zheltukhin A. A.* Toroidal p-Branes, Anharmonic Oscillators and (Hyper)Elliptic Solutions // Nucl. Phys. B. 2012. V. 858. P. 142;
    Laplace–Beltrami Operator and Exact Solutions for Branes // Nucl. Phys. B. 2013. V. 867. P. 763;
    Generalized Hooke Law for Relativistic Membranes and p-Branes // Probl. At. Sci. Technol. 2012. V. 1. P.7; On Nonlinearity of p-Brane Dynamics // Intern. J. Geom. Meth. Mod. Phys. 2012. V. 9. P. 1261017.
25. *Dirac P. A. M.* Long Range Forces and Broken Symmetries // Proc. Roy. Soc. London. A. 1973. V. 333. P. 403.
26. *Pervushin V., Pavlov A.* Principles of Quantum Universe. Saarbrucken. LAP LAMBERT Acad. Publ., 2013.
27. *Hitchin N.* The Self-Duality Equations on a Riemann Surface // Proc. London Math. Soc. 1987. V. 55, No. 3. P. 59.