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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

$SO(2,3)$ NONCOMMUTATIVE GRAVITY MODEL

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In this paper, the noncommutative gravity is treated as a gauge theory of the noncommutative $SO(2,3)_\star$ group, while the noncommutativity is canonical. The Seiberg–Witten (SW) map is used to express noncommutative fields in terms of the corresponding commutative fields. The commutative limit of the model is the Einstein–Hilbert action plus the cosmological term and the topological Gauss–Bonnet term. We calculate the second-order correction to this model and obtain terms that are the zeroth, first, ... and fourth power of the curvature tensor. Finally, we discuss physical consequences of those correction terms in the limit of big cosmological constant.

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INTRODUCTION

Recently, a lot of attention was given to the Anti-de Sitter (AdS) gauge theory and its application to studies of General Relativity (GR), quantization of gravity, AdS/CFT and its applications [1]. In our previous paper [2], we began the study of NonCommutative (NC) gravity based on the AdS gauge group. We started form the MacDowell–Mansouri action in the commutative space-time and generalized it to the NC MacDowell–Mansouri on the canonically deformed space. In this paper, we briefly describe the NC $SO(2,3)_\star$ gauge theory. More details can be found in [3].

The NC space-time is the canonically deformed space-time with the Moyal–Weyl $\star$-product given by

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x)\, g(y)|_{y\to x}. \tag{1}$$

Here $\theta^{\mu\nu}$ is a constant antisymmetric matrix and is considered to be a small deformation parameter. Indices μ, ν take values $0,1,2,3$, and the four-dimensional Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$. In the next section, we shortly describe the commutative $SO(2,3)$ gravity theory. In Sec. 2, the NC $SO(2,3)_\star$ gauge theory via the SW map is introduced. We expand the NC action to the second order in the deformation parameter $\theta^{\alpha\beta}$ and calculate the correction terms to the commutative action. The first-order correction vanishes and we confirm the results already present in the literature. Namely, it was shown that if

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reality of the NC gravity action is imposed, all odd-order corrections (in the NC parameter) have to vanish. The first nonvanishing correction is then the second-order correction. The correction terms we obtain are of the zeroth, first, ... and fourth power in the curvature tensor and are written in a manifestly covariant way. The term that is the zeroth power in the curvature tensor renormalizes the cosmological constant, i.e., we obtain an x-dependent cosmological constant.

1. COMMUTATIVE GRAVITY AS AN AdS GAUGE THEORY

We assume that the space-time has the structure of the 4-dimensional Minkowski space M_4 and follow the usual steps for constructing a gauge theory on M_4 taking the $SO(2,3)$ group as the gauge group. The gauge field takes values in the $SO(2,3)$ algebra, $\omega_\mu = (1/2)\omega_\mu^{AB} M_{AB}$. Here M_{AB} are the generators of the $SO(2,3)$ group and they fulfill

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}). \tag{2}$$

The $5D$ metric is $\eta_{AB} = \mathrm{diag}\,(+,-,-,-,+)$. Indices $A, B, \ldots$ take values $0,1,2,3,5$, while indices $a, b, \ldots$ take values $0,1,2,3$. A representation of this algebra is given by

$$M_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab}, \quad M_{5a} = \frac{1}{2}\gamma_a, \tag{3}$$

where γ_a are four-dimensional Dirac gamma matrices. Then the gauge potential ω_μ^{AB} decomposes into ω_μ^{ab} and ω_μ^{a5}

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB} M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma_{ab} - \frac{1}{2}\omega_\mu^{a5}\gamma_a. \tag{4}$$

The field strength tensor is defined in a usual way by

$$F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2}F_{\mu\nu}^{AB} M_{AB} = \\ = \left(R_{\mu\nu}^{ab} - \frac{1}{l^2}(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a)\right)\frac{\sigma_{ab}}{4} - F_{\mu\nu}^{\ \ a5}\frac{\gamma_a}{2}, \tag{5}$$

where

$$R_{\mu\nu}^{\ \ ab} = \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + \omega_\mu^{ac}\omega_\nu^{cb} - \omega_\mu^{bc}\omega_\nu^{ca}, \tag{6}$$

$$lF_{\mu\nu}^{\ \ a5} = D_\mu e_\nu^a - D_\nu e_\mu^a = T_{\mu\nu}^a. \tag{7}$$

Equations (4), (5), (6), and (7) suggest that one can identify ω_μ^{ab} with the spin connection of the Poincaré gauge theory; ω_μ^{a5}, with the vielbeins; $R_{\mu\nu}^{ab}$, with the curvature tensor; and $F_{\mu\nu}^{a5}$, with the torsion. It was shown in the seventies that one can really do such an identification and relate AdS gauge theory with GR. Different ways were discussed in the literature, see [4]. One way is to start from the following action:

$$S = \frac{il}{64\pi G_N}\mathrm{Tr}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\phi, \tag{8}$$

where G_N is the Newton gravitational constant. An additional auxiliary field $\phi = \phi^A \Gamma_A$, $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, transforming in the adjoint representation of $SO(2,3)$, is introduced. One can show that the action (8) is invariant under the $SO(2,3)$ gauge symmetry. However, if we restrict the field ϕ to be $\phi^a = 0$, $\phi^5 = l$, with an arbitrary constant l, then the symmetry of the action is reduced to the $SO(1,3)$ gauge symmetry. The constraint on the field ϕ can be introduced via a Lagrange multiplier or dynamically [4]. We are not concerned with that problem here. The action obtained after symmetry breaking is given by

$$S = \frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x \,\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}\gamma_5) = \\ = -\frac{1}{16\pi G_N}\int d^4x\Big[\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{\ ab}R_{\rho\sigma}^{\ cd} + eR + 2e\Lambda\Big], \quad (9)$$

where $\Lambda = -3/l^2$ and $e = \det(e_\mu^a)$. This action is written in the first-order formalism: the spin connection ω_μ and the vielbeins e_μ^a are independent fields. Varying the action with respect to the spin connection, we obtain an equation that relates the spin connection and the vielbeins. After the analysis of the equations of motion, we see that after the symmetry breaking, the action (9) describes GR with the negative cosmological constant and the topological Gauss–Bonnet term.

2. NC $SO(2,3)_\star$ GAUGE THEORY

In order to construct the NC $SO(2,3)_\star$ gauge theory, we use the enveloping algebra approach and the Seiberg–Witten (SW) map [5]. The NC action is given by

$$S_{\mathrm{NC}} = -\frac{il}{16\pi G_N}\mathrm{Tr}\int d^4x \epsilon^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu}\star\hat{F}_{\rho\sigma}\star\hat{\phi}. \quad (10)$$

The $\star$-product is the Moyal–Weyl $\star$-product (1), fields with a «hat» are NC fields, and we will use the SW map to expand them in terms of the corresponding commutative fields. One can show that this action is invariant under the NC $SO(2,3)$ gauge transformations. In the limit $\theta^{\alpha\beta} \to 0$, the action (10) reduces to the commutative action (8). The solutions of the SW map for the field strength tensor and the field $\hat{\phi}$ are given in terms of the recursive relations [6]

$$\hat{F}_{\mu\nu}^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\big(\{\hat{\omega}_\kappa \stackrel{\star}{,} \partial_\lambda\hat{F}_{\mu\nu} + D_\lambda\hat{F}_{\mu\nu}\}\big)^{(n)} + \\ + \frac{1}{2(n+1)}\theta^{\kappa\lambda}\big(\{\hat{F}_{\mu\kappa}, \stackrel{\star}{,} \hat{F}_{\nu\lambda}\}\big)^{(n)}, \quad (11)$$

$$\hat{\phi}^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \stackrel{\star}{,} \partial_\lambda\hat{\phi} + D_\lambda\hat{\phi}\}\Big)^{(n)}. \quad (12)$$

We expand the action (10) in orders of the deformation parameter $\theta^{\alpha\beta}$, using the SW map solutions and expanding the $\star$-products that appear in the action. The first-order correction vanishes, as expected. The second-order correction is given by

$$
\begin{aligned}
S^{(2)}_{\rm NC} = {} & \frac{il}{64\pi G_N}\frac{1}{8}\theta^{\alpha\beta}\theta^{\kappa\lambda}\times \\
& \times \,{\rm Tr}\int d^4x\epsilon^{\mu\nu\rho\sigma}\Big\{-\frac{1}{8}\{F_{\alpha\beta},\{F_{\mu\nu},F_{\rho\sigma}\}\}\{\phi,F_{\kappa\lambda}\} + \frac{1}{2}\{F_{\alpha\beta},\{F_{\rho\sigma},\{F_{\kappa\mu},F_{\lambda\nu}\}\}\}\phi + \\
& + \frac{1}{4}\{\{F_{\mu\nu},F_{\rho\sigma}\},\{F_{\kappa\alpha},F_{\lambda\beta}\}\}\phi + \frac{i}{4}\{F_{\alpha\beta},[D_\kappa F_{\mu\nu},D_\lambda F_{\rho\sigma}]\}\phi + \frac{i}{2}[\{D_\kappa F_{\mu\nu},F_{\rho\sigma},D_\lambda F_{\alpha\beta}\}]\phi - \\
& - \frac{1}{2}\{F_{\rho\sigma},\{F_{\alpha\mu},F_{\beta\nu}\}\}\{\phi,F_{\kappa\lambda}\} + \{F_{\alpha\mu},F_{\beta\nu},\{F_{\kappa\rho},F_{\lambda\sigma}\}\}\phi + \\
& + 2\{F_{\rho\sigma},\{F_{\beta\nu},\{F_{\kappa\alpha},F_{\lambda\mu}\}\}\}\phi + i\{F_{\rho\sigma},[D_\kappa F_{\alpha\mu},D_\lambda F_{\beta\nu}]\}\phi + 2i[\{F_{\beta\nu},D_\kappa F_{\alpha\mu}\},D_\lambda F_{\rho\sigma}]\phi - \\
& - \frac{1}{4}\{\phi,F_{\kappa\lambda}\}[D_\alpha F_{\mu\nu},D_\beta F_{\rho\sigma}] + \frac{i}{2}\{D_\kappa D_\alpha F_{\mu\nu},D_\lambda D_\beta F_{\rho\sigma}\}\phi + \\
& + [\{F_{\kappa\alpha},D_\lambda F_{\mu\nu}\},D_\beta F_{\rho\sigma}]\phi + [\{F_{\lambda\nu},D_\alpha F_{\kappa\mu}\},D_\beta F_{\rho\sigma}]\phi + [\{F_{\kappa\mu},D_\alpha F_{\lambda\nu}\},D_\beta F_{\rho\sigma}]\phi\Big\}.
\end{aligned}
$$

This expanded action is manifestly invariant under the commutative $SO(2,3)$ gauge transformations. This result is guaranteed by the SW map. After the symmetry breaking, which is obtained by taking the field ϕ to be $\phi^a = 0$ and $\phi^5 = l$, we obtain

$$
\begin{aligned}
S^{(2)}_{\rm NC} = {} & -\frac{l^2}{64\pi G_N}\theta^{\alpha\beta}\theta^{\kappa\lambda}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\int d^4x\Big\{\frac{1}{256}\Big(F^{cd}_{\mu\nu}F^{ab}_{\rho\sigma}F^{mn}_{\alpha\beta}F_{\kappa\lambda mn} - \\
& - 8F^{ab}_{\mu\nu}F^{c5}_{\rho\sigma}F^{de}_{\kappa\lambda}F^{5}_{\alpha\beta e} + F^{ab}_{\alpha\beta}F^{cd}_{\kappa\lambda}(F^{mn}_{\mu\nu}F_{\rho\sigma mn} + 2F^{m5}_{\mu\nu}F^{5b}_{\rho\sigma m})\Big) - \\
& - \frac{1}{32}\Big(F^{ab}_{\kappa\lambda}F^{cd}_{\mu\nu}F^{mn}_{\alpha\rho}F_{\beta\sigma mn} + 2F^{ab}_{\alpha\beta}F^{cd}_{\rho\sigma}F^{m5}_{\kappa\mu}F^{5}_{\lambda\nu m} + F^{ab}_{\kappa\mu}F^{cd}_{\lambda\nu}F^{mn}_{\alpha\beta}F_{\rho\sigma mn}\Big) - \\
& - \frac{1}{128}\Big(F^{ab}_{\kappa\alpha}F^{cd}_{\lambda\beta}(F^{mn}_{\mu\nu}F_{\rho\sigma mn} + 2F^{m5}_{\mu\nu}F^{5}_{\rho\sigma m}) + F^{ab}_{\mu\nu}F^{cd}_{\rho\sigma}(F^{mn}_{\kappa\alpha}F_{\lambda\beta mn} + 2F^{m5}_{\kappa\alpha}F^{\ 5}_{\lambda\beta m})\Big) + \\
& + \frac{1}{16}F^{ab}_{\alpha\beta}\Big((D_\kappa F_{\mu\nu})^{cm})(D_\lambda F_{\rho\sigma})_m^{\ d} + (D_\kappa F_{\mu\nu})^{c5}(D_\lambda F_{\rho\sigma})_5^{\ d}\Big) - \\
& - \frac{1}{16}\Big((D_\kappa F_{\mu\nu})^{ab}(D_\lambda F_{\alpha\beta})^{d5}F^{c5}_{\rho\sigma} + (D_\kappa F_{\mu\nu})^{a5}(D_\lambda F_{\alpha\beta})^{b5}F^{cd}_{\rho\sigma}\Big) + \\
& + \frac{1}{16}F^{ab}_{\alpha\mu}F^{cd}_{\beta\nu}\Big(F^{mn}_{\kappa\rho}F_{\lambda\sigma mn} + 2F^{m5}_{\kappa\rho}F_{\lambda\sigma m5}\Big) + \\
& + \frac{1}{16}\Big(F^{ab}_{\rho\sigma}F^{cd}_{\beta\nu}(F^{mn}_{\kappa\alpha}F_{\lambda\mu mn} + 2F^{m5}_{\kappa\alpha}F_{\lambda\mu m5}) + F^{ab}_{\kappa\alpha}F^{cd}_{\lambda\mu}F^{mn}_{\rho\sigma}F_{\beta\nu mn} - \\
& - 4(F^{ab}_{\kappa\alpha}F^{c5}_{\lambda\mu} + F^{a5}_{\kappa\alpha}F^{bc}_{\lambda\mu})F^{de}_{\rho\sigma}F_{\beta\nu e5}\Big) - \\
& - \frac{1}{8}F^{ab}_{\rho\sigma}\Big((D_\kappa F_{\alpha\mu})^{cm}(D_\lambda F_{\beta\nu})^{d}_{m} + (D_\kappa F_{\alpha\mu})^{c5}(D_\lambda F_{\beta\nu})_5^{\ d}\Big) + \\
& + \frac{1}{2}\Big(F^{ab}_{\kappa\mu}(D_\alpha F_{\lambda\nu})^{c5}(D_\beta F_{\rho\sigma})^{d5} + F^{a5}_{\kappa\mu}(D_\alpha F_{\lambda\nu})^{bc}\Big)(D_\beta F_{\rho\sigma})^{d5} + \\
& + \frac{1}{8}\Big(F^{ab}_{\kappa\alpha}(D_\lambda F_{\mu\nu})^{c5} + F^{a5}_{\kappa\alpha}(D_\lambda F_{\mu\nu})^{bc}\Big)(D_\beta F_{\rho\sigma})^{d5} - \frac{1}{32}(D_\kappa D_\alpha F_{\mu\nu})^{ab}(D_\lambda D_\beta F_{\rho\sigma})^{cd}\Big\}.
\end{aligned}
\tag{13}
$$

Here $D_\alpha F_{\mu\nu}$ is the $SO(2,3)$ covariant derivative and its components are

$$(D_\alpha F_{\mu\nu})^{ab} = \nabla_\alpha F_{\mu\nu}^{ab} - \frac{1}{l^2}(e_\alpha^a T_{\mu\nu}^{\ \ b} - e_\alpha^b T_{\mu\nu}^{\ \ a}),$$
$$(D_\alpha F_{\mu\nu})^{a5} = \frac{1}{l}(\nabla_\alpha T_{\mu\nu}^a + e_\alpha^m F_{\mu\nu m}^{\ \ \ a}).$$

3. DISCUSSION

The result (13) is very complicated. In order to see the physical consequences of this model, we have to make some additional requirements. We will first assume that the torsion in the zeroth order vanishes, $F_{\mu\nu}^{a5} = 0$. Since we have no fermionic matter, this assumption is valid. Then we will expand $F_{\mu\nu}^{bc}$ in terms of the curvature tensor and the vielbeins, using (5). Since the torsion vanishes, the curvature tensor will have the usual symmetry properties: $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$. Finally, we discuss different scales in the theory. We have three parameters: the cosmological constant $\Lambda = -3/l^2$ (the length parameter l is related with the radius of the AdS space), the NC parameter $\theta^{\alpha\beta}$, and the powers of the curvature tensor (powers of derivatives). Depending on the values of these three parameters, we can analyze different limits of the model: big cosmological constant and low energies (lower powers of curvature dominate), or big cosmological constant and high energies (higher powers of curvature dominate), and so on.

Let us assume that we are interested in the limit of the big cosmological constant and low energies. In that case, from (13), we include only the term that is zeroth order in the curvature. The resulting action is given by

$$S = -\frac{1}{16\pi G_N}\int d^4x\left[\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{ab}R_{\rho\sigma}^{cd} + \sqrt{-g}R + 2\sqrt{-g}\Lambda\right] + \\ + \frac{3\theta^{\alpha\beta}\theta^{\kappa\lambda}}{16\pi G_N l^6}\int d^4x\sqrt{-g}g_{\alpha\kappa}g_{\beta\lambda}. \quad (14)$$

To obtain the equations of motion, we vary the action with respect to the metric $g_{\rho\sigma}$. The result is

$$R_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}(R + 2\Lambda) + \frac{3}{l^6}\theta^{\alpha\beta}\theta^{\kappa\lambda}\left(\frac{1}{2}g_{\rho\sigma}g_{\alpha\kappa}g_{\beta\lambda} + 2g_{\beta\lambda}g_{\alpha\rho}g_{\kappa\sigma}\right) = 0. \quad (15)$$

A simple analysis shows that the flat space $g_{\rho\sigma} = \eta_{\rho\sigma}$ is not the solution of these equations. Therefore, although in the action (14) the cosmological constant is renormalized with the x-dependent term $\theta^{\alpha\beta}\theta^{\kappa\lambda}g_{\alpha\kappa}g_{\beta\lambda}$, this is not enough to completely cancel its effect and the resulting space-time remains curved. If we are interested in the linearized theory, we would have to expand around the AdS space-time. This problem and further analysis are the subject of the future investigations.

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