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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

DEFORMED $\mathcal{N}=4$, $d=1$ SUPERSYMMETRY

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By this contribution we give a brief account of the recently proposed new models of supersymmetric quantum mechanics [1]. They are associated with the nonstandard world-line $SU(2|1)$ supersymmetry which is considered as a deformation of the standard $\mathcal{N}=4$, $d=1$ supersymmetry by a mass parameter m. Employing chiral superfields defined on the cosets of the supergroup $SU(2|1)$, we construct the quantum model on a complex plane and find out interesting interrelations with some previous works on nonstandard $d=1$ supersymmetry.

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1. THE SUPERALGEBRA

We focus on constructing $SU(2|1)$ superspace as a coset superspace of the (centrally extended) superalgebra $su(2|1)$:

$$\begin{gathered}\{Q^i,\bar{Q}_j\}=2m\left(I^i_j-\delta^i_jF\right)+2\delta^i_jH,\quad \left[I^i_j,I^k_l\right]=\delta^k_jI^i_l-\delta^i_lI^k_j,\\ \left[I^i_j,\bar{Q}_l\right]=\frac{1}{2}\delta^i_j\bar{Q}_l-\delta^i_l\bar{Q}_j,\quad \left[I^i_j,Q^k\right]=\delta^k_jQ^i-\frac{1}{2}\delta^i_jQ^k,\\ \left[F,\bar{Q}_l\right]=-\frac{1}{2}\bar{Q}_l,\quad \left[F,Q^k\right]=\frac{1}{2}Q^k.\end{gathered}\tag{1}$$

All other (anti)commutators are vanishing. The parameter m is a contraction parameter deforming the standard $\mathcal{N}=4$, $d=1$ Poincaré supergroup to $SU(2|1)$. Sending $m\to 0$, the $su(2|1)$ superalgebra becomes the standard $\mathcal{N}=4$, $d=1$ Poincaré superalgebra. In the limit $m=0$, the generators I^i_j and F become the $U(2)$ automorphism generators of this $\mathcal{N}=4$, $d=1$ superalgebra. The internal $U(2)$ symmetry is presented by the dimensionless generators I^i_j and F. The mass-dimension generator H commutes with everything and so can be interpreted as the central charge generator. In the quantum-mechanical realization of $SU(2|1)$ we will be interested in, this generator becomes just the canonical Hamiltonian, while in the superspace realization it is interpreted as the time-translation generator.

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2. $SU(2|1)$ SUPERSPACE

The superspace coordinates $\{t, \theta_i, \bar\theta^j\}$ are identified with the parameters associated with the coset of the supergroup $SU(2|1)$:

$$\frac{SU(2|1)}{SU(2)\times U(1)} \sim \frac{\{Q^i, \bar Q_j, H, I^i_j, F\}}{\{I^i_j, F\}}. \tag{2}$$

An element of this supercoset can be conveniently parametrized as

$$g = \exp\left(itH + i\tilde\theta_i Q^i - i\tilde{\bar\theta}^j \bar Q_j\right), \quad \tilde\theta_i = \left[1 - \frac{2m}{3}\left(\bar\theta\cdot\theta\right)\right]\theta_i. \tag{3}$$

All generators of (1) are realized on the superspace coordinates as

$$\begin{aligned} Q^i &= -i\frac{\partial}{\partial\theta_i} + 2im\bar\theta^i\bar\theta^j\frac{\partial}{\partial\bar\theta^j} + \bar\theta^i\frac{\partial}{\partial t}, \\ \bar Q_j &= i\frac{\partial}{\partial\bar\theta^j} + 2im\theta_j\theta_k\frac{\partial}{\partial\theta_k} - \theta_j\frac{\partial}{\partial t}, \quad H = i\partial_t, \\ I^i_j &= \left(\bar\theta^i\frac{\partial}{\partial\bar\theta^j} - \theta_j\frac{\partial}{\partial\theta_i}\right) - \frac{\delta^i_j}{2}\left(\bar\theta^k\frac{\partial}{\partial\bar\theta^k} - \theta_k\frac{\partial}{\partial\theta_k}\right), \quad F = \frac{1}{2}\left(\bar\theta^k\frac{\partial}{\partial\bar\theta^k} - \theta_k\frac{\partial}{\partial\theta_k}\right). \end{aligned} \tag{4}$$

The supercharges $Q^i, \bar Q_j$ generate the following transformation properties of superspace:

$$\delta\theta_i = \epsilon_i + 2m\left(\bar\epsilon\cdot\theta\right)\theta_i, \quad \delta\bar\theta^j = \bar\epsilon^i - 2m\left(\epsilon\cdot\bar\theta\right)\bar\theta^i, \quad \delta t = i\left[\left(\epsilon\cdot\bar\theta\right) + \left(\bar\epsilon\cdot\theta\right)\right]. \tag{5}$$

Then we have the invariant integration measure $\mu = dt\, d^2\theta\, d^2\bar\theta(1 + 2m\,\bar\theta\cdot\theta)$, $\delta\mu = 0$. The deformed covariant derivatives $\mathcal{D}^i, \bar{\mathcal{D}}_j$ are written as

$$\begin{aligned} \mathcal{D}^i &= \left[1 + m\left(\bar\theta\cdot\theta\right) - \frac{3m^2}{4}\left(\bar\theta\cdot\theta\right)^2\right]\frac{\partial}{\partial\theta_i} - m\bar\theta^i\theta_j\frac{\partial}{\partial\theta_j} - i\bar\theta^i\frac{\partial}{\partial t} + \\ &\qquad + m\bar\theta^i\tilde F - m\bar\theta^j\tilde I^i_j + \frac{m^2}{2}\left(\bar\theta\cdot\theta\right)\bar\theta^j\tilde I^i_j - \frac{m^2}{2}\bar\theta^i\bar\theta^j\theta_k\tilde I^k_j, \\ \bar{\mathcal{D}}_j &= -\left[1 + m\left(\bar\theta\cdot\theta\right) - \frac{3m^2}{4}\left(\bar\theta\cdot\theta\right)^2\right]\frac{\partial}{\partial\bar\theta^j} + m\bar\theta^k\theta_j\frac{\partial}{\partial\bar\theta^k} + i\theta_j\frac{\partial}{\partial t} - \\ &\qquad - m\theta_j\tilde F + m\theta_k\tilde I^k_j - \frac{m^2}{2}\left(\bar\theta\cdot\theta\right)\theta_k\tilde I^k_j + \frac{m^2}{2}\theta_j\bar\theta^l\theta_k\tilde I^k_l. \end{aligned} \tag{6}$$

3. CHIRAL MULTIPLET

One can define $SU(2|1)$ counterpart of the $\mathcal{N} = 4$, $d = 1$ chiral multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. This is due to the existence of the invariant chiral coset $SU(2|1)$ superspace

$$t_L = t + \frac{i}{2m}\ln(1 + 2m\,\bar\theta\cdot\theta), \quad \delta\theta_i = \epsilon_i + 2m\left(\bar\epsilon\cdot\theta\right)\theta_i, \quad \delta t_L = 2i\left(\bar\epsilon\cdot\theta\right). \tag{7}$$

The multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is described by the chiral superfield Φ:

$$\bar{\mathcal{D}}_j\Phi = 0, \quad \tilde{I}^i_j\Phi = 0, \quad \tilde{F}\Phi = 2\kappa\Phi, \tag{8}$$

where κ is a fixed external $U(1)$ charge. The constraints give the solution

$$\Phi(t,\theta,\bar\theta\,) = \left[1 + 2m\left(\bar\theta\cdot\theta\right)\right]^{-\kappa}\left(z + \sqrt{2}\,\theta_i\xi^i + \varepsilon^{ij}\theta_i\theta_j B\right). \tag{9}$$

General superfield Lagrangian is constructed as

$$\mathcal{L}_k = \frac{1}{4}\int d^2\theta\, d^2\bar\theta\,(1 + 2m\,\bar\theta\cdot\theta)\, f(\Phi,\Phi^\dagger). \tag{10}$$

Eliminating auxiliary fields by their equations of motion, we obtain the on-shell Lagrangian where bosonic part is

$$\mathcal{L} = g\dot{\bar z}\dot z + 2i\kappa m\,(\dot{\bar z}z - \dot z\bar z)\, g - \frac{im}{2}\,(\dot{\bar z}f_{\bar z} - \dot z f_z) - m^2 V, \tag{11}$$

where

$$V = \kappa\,(\bar z\partial_{\bar z} + z\partial_z)\, f - \kappa^2\,(\bar z\partial_{\bar z} + z\partial_z)^2\, f, \quad g = g(z,\bar z) = \partial_z\partial_{\bar z}f(z,\bar z). \tag{12}$$

Thus, the standard $\mathcal{N}=4, d=1$ kinetic term is deformed to nontrivial Lagrangian with WZ term, and potential term. The latter vanishes for $\kappa=0$; however, the WZ term vanishes only in the limit $m=0$. So, the basic novel point compared to the standard $\mathcal{N}=4$ Kähler sigma model for the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is the necessary presence of the WZ term with the strength m, together with the Kähler kinetic term.

4. THE MODEL ON A PLANE

The model on a plane corresponds to the Lagrangian

$$\mathcal{L} = \frac{1}{4}\int d^2\theta\, d^2\bar\theta\,(1 + 2m\,\bar\theta\cdot\theta)\,\Phi\Phi^\dagger. \tag{13}$$

Then, the explicit on-shell Lagrangian takes the form

$$\mathcal{L} = \dot{\bar z}\dot z + im\left(2\kappa - \frac{1}{2}\right)(\dot{\bar z}z - \dot z\bar z) + \frac{i}{2}\left(\bar\eta_i\dot\eta^i - \dot{\bar\eta}_i\eta^i\right) + \\ + 2\kappa\,(2\kappa - 1)\, m^2\bar z z + (1 - 2\kappa)\, m\,(\bar\eta\cdot\eta)\,. \tag{14}$$

It is invariant under the following on-shell transformations:

$$\delta z = -\sqrt{2}\,\epsilon_i\eta^i, \qquad \delta\eta^i = \sqrt{2}\, i\bar\epsilon^i\dot z - 2\sqrt{2}\kappa m\bar\epsilon^i z. \tag{15}$$

Performing Legendre transformations, we obtain the corresponding canonical Hamiltonian

$$H = \left[p_z - \frac{i}{2}\,(1 - 4\kappa)\, m\bar z\right]\left[p_{\bar z} + \frac{i}{2}\,(1 - 4\kappa)\, mz\right] + \\ + 2\kappa\,(1 - 2\kappa)\, m^2\bar z z + (1 - 2\kappa)\, m\,\eta^k\bar\eta_k. \tag{16}$$

The full set of eigenfunctions reads

$$\Psi^{(\ell;n)} = a^{(\ell;n)}\Omega^{(\ell;n)} + b_i^{(\ell;n)}\eta^i\,\Omega^{(\ell-1;n+1)} + c^{(\ell;n)}\,\varepsilon_{ij}\eta^i\eta^j\,\Omega^{(\ell-2;n+2)} \tag{17}$$

for $\ell \geqslant 2$, and

$$\begin{aligned} \Psi^{(1;n)} &= a^{(1;n)}\,\Omega^{(1;n)} + b_i^{(1;n)}\eta^i\,\Omega^{(0;n+1)}, \\ \Psi^{(0;n)} &= a^{(0;n)}\,\Omega^{(0;n)}. \end{aligned} \tag{18}$$

These superfunctions are constructed in terms of the bosonic functions

$$\Omega^{(\ell;n)} = \bar{z}^n \exp\left(-\frac{mz\bar{z}}{2}\right) L_\ell^{(n)}(mz\bar{z}) = \frac{z^{-n}}{\ell!}\exp\left(\frac{mz\bar{z}}{2}\right)\frac{d^\ell}{dw^\ell}\left(\mathrm{e}^{-mw}w^{n+\ell}\right)\Big|_{w=z\bar{z}}, \tag{19}$$

where $L_\ell^{(n)}$ are generalized Laguerre polynomials. The numbers ℓ and n are integers satisfying $\ell \geqslant 0$, $n \geqslant -\ell$. The spectrum depends on ℓ and n as

$$\hat{H}\,\Psi^{(\ell;n)} = m(\ell + 2\kappa n)\,\Psi^{(\ell;n)}. \tag{20}$$

We observe that the ground state ($\ell = 0$) and the first excited states ($\ell = 1$) are special, in the sense that they encompass nonequal numbers of bosonic and fermionic states. Indeed, $\Omega^{(0;n)}$ is a singlet of $SU(2|1)$ for any n. The wave functions for $\ell = 1$ form the fundamental representation of $SU(2|1)$ (one bosonic and two fermionic states), while those for $\ell \geqslant 2$ form the typical $(2|2)$ representations. It can be explained by the eigenvalues of the Casimir operators [2]

$$C_2(\ell) = (\ell - 1)\,\ell, \quad C_3(\ell) = \left(\ell - \frac{1}{2}\right)(\ell - 1)\,\ell. \tag{21}$$

Thus, they are vanishing for the wave functions with $\ell = 0, 1$, confirming the interpretation of the corresponding representations as atypical, and are nonvanishing on the wave functions with $\ell \geqslant 2$, implying them to form typical representations of $SU(2|1)$.

The same deviations from the standard rule of equality of the bosonic and fermionic states were observed in [3]. As it turned out, these $d = 1$ supersymmetric models with $\mathcal{N} = 4$ "weak supersymmetry" are easily reproduced from our superfield approach based on the $SU(2|1)$ multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ [1].

CONCLUSIONS

We constructed the $SU(2|1)$ superspace and used it for constructing deformed models associated with the chiral multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. We systematically studied the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ model on a plane. The relevant Hilbert space of wave superfunctions were constructed. We found the spectrum of wave functions and analyzed the structure of wave functions in the framework of the $SU(2|1)$ representation theory [2].

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