

INTEGRABLE HYDRODYNAMIC CHAINS FOR WZNW MODEL

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The new integrable hydrodynamic equations are obtained for WZNW model with $SU(2)$, $SO(3)$, $SP(2)$ and $SU(\infty)$ constant torsions.

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INTRODUCTION

The integrability of the two-dimensional WZNW is based on the existence of an infinite number of the local and nonlocal currents and on their charges. The n -dimensional WZNW model is described by means of the chiral left $J_A^L = g^{-1}\partial_A g$ or the chiral right $J_A^R = \partial_A g g^{-1}$ currents for arbitrary space-time dimension ($A = 1, \dots, n$), where g is element of the group symmetry of the model. The currents $J_A = J_A^\mu t_\mu$ and t_μ are the generators of the Lie algebra. These chiral currents were related to the left and right multiplication on the group space. The two-dimensional models ($A = 0, 1$) have the following additional chiral currents:

$$J_\mu^L(t, x) = \frac{J_{0\mu} + \delta_{\mu\nu} J_1^\nu}{\sqrt{2}} = U_\mu(x + t), \quad J_\mu^R(t, x) = \frac{J_{0\mu} - \delta_{\mu\nu} J_1^\nu}{\sqrt{2}} = V_\mu(x - t)$$

related to the dynamics on the (t, x) plane. The chiral currents U_μ, V_μ play an important role for the construction and investigation of this type of integrable systems. We cannot separate the movement on the left-moving mode and on the right-moving mode for the σ -model under consideration in order to formulate the movement on only one mode. It was done by the introduction of the Witten term to the Wess–Zumino model. This term introduces a potential for the torsion tensor on the curved space of the group parameters in addition to the metric tensor. It is possible to extract the movement on one mode with the fulfilling of some conditions between the constant torsion tensor and the structure constant of Lie algebra. In this work, Lagrangian and equations of motion in the repere formalism are considered, being the antisymmetric field B_{ab} obtained in terms of the repere. Also,

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the Hamiltonian formalism and the commutations relations are rewritten in new variables. These variables are precisely the chiral currents under the condition that the external torsion coincides (anticoincides) with the structure constants of the $SU(2)$, $SO(3)$, $SP(2)$ algebras. In this manner, the equation of motion for the density of the first Casimir operator is obtained as the inviscid Burgers equation, being its solution expressed as the Lambert function. The integrable infinite dimensional hydrodynamic chains are constructed for WZNW model with the constant $SU(2)$, $SO(3)$, $SP(2)$ torsions and for this model with the $SU(\infty)$, $SO(\infty)$, $SP(\infty)$ constant torsions. Finally, the new equations of motion of hydrodynamic type are explicitly obtained for the initial chiral currents in terms of the symmetric structure constant of the $SU(\infty)$, $SO(\infty)$, $SP(\infty)$ algebras.

LAGRANGIAN AND EQUATION OF MOTION

The conformal invariant two-dimensional nonlinear sigma model is described by WZNW model which is the sigma model [1–4] with Wess–Zumino term [5–8] on the group manifold. To each point of a 2-dimensional world-sheet one associates an element g of a group G . We want to construct an action with the Lagrangian density which is the element of volume of the two-dimensional space invariant under the group transformations:

$$S = \frac{1}{4} \int \frac{\text{Tr}(\omega \wedge dx^\alpha)(\omega \wedge dx^\beta)\eta_{\alpha\beta}}{\epsilon_{\lambda\rho} dx^\lambda \wedge dx^\rho} + \frac{1}{2} \int \text{Tr}(\omega(d) \wedge \omega(d) \wedge \omega(d)). \quad (1)$$

Here $x^\alpha = (t, x)$ are coordinates of the flat two-dimensional space: $\alpha = (0, 1)$ with signature $(-1, 1)$ and $\eta_{\alpha\beta}$ is the diagonal metric of this space. The form $\omega(d) = \omega(d)^\mu t_\mu$ is the differential Cartan one-form which belongs to a simple Lie algebra

$$[t_\mu, t_\nu] = 2iC_{\mu\nu}^\lambda t_\lambda, \quad \text{Tr}(t_\mu t_\nu) = 2g_{\mu\nu} \quad (\mu, \nu = 1, 2, \dots, n). \quad (2)$$

In any parametrization, Cartan forms $\omega(d) = (g^{-1}dg)^\mu t_\mu$ depend on the group parameters ϕ^a : $\omega(d) = \omega(\phi, d\phi)$. The first term of the Lagrangian (1) has the form

$$\frac{\text{Tr}(\omega \wedge dx^\alpha)(\omega \wedge dx^\beta)\eta_{\alpha\beta}}{\epsilon_{\lambda\rho} dx^\lambda \wedge dx^\rho} = 2g_{ab}(\phi) \frac{\partial\phi^a}{\partial x^\alpha} \frac{\partial\phi^b}{\partial x^\beta} \eta^{\alpha\beta} d^2x. \quad (3)$$

Here we introduce the notation

$$g_{ab}(\phi) = g_{\mu\nu} \omega_a^\mu \omega_b^\nu, \quad dx^\gamma \wedge dx^\alpha = \epsilon^{\gamma\alpha} d^2x. \quad (4)$$

One can see that $g_{ab}(\phi)$ is metric tensor on the curved space of local fields ϕ^a ($a = 1, 2, \dots, n$). The $\omega_a^\mu(\phi)$ forms a repere basis on the tangent space with the metric $g_{\mu\nu}$ in arbitrary point of the curved space ϕ^a . Here we want to rewrite a WZNW model as sigma model of string type, equipped with an antisymmetric field $B_{ab}(\phi)$, in terms of the repere for the arbitrary metric $g_{ab}(\phi)$ and for any dimension n :

$$\text{Tr}(\omega(d) \wedge \omega(d) \wedge \omega(d)) = g_{\mu\nu} \left(\Omega_a^\mu \frac{\partial\Omega_b^\nu}{\partial x^C} - \Omega_b^\mu \frac{\partial\Omega_a^\nu}{\partial x^C} \right) \frac{\partial\Phi^a}{\partial x^A} \frac{\partial\Phi^b}{\partial x^B} \epsilon^{ABC} d^3x. \quad (5)$$

The integrability condition $\partial_A g = g \Omega_A^\mu t_\mu$ was used:

$$\partial_A \Omega_B^\mu - \partial_B \Omega_A^\mu + 2i C^{\mu\nu\lambda} \Omega_A^\nu \Omega_B^\lambda = 0.$$

Here x^A ($A = 0, 1, 2$) are coordinates of the three-dimensional space-time, $\Omega^\mu(d)$ is a one-form on this space. Let us separate the last component of index A ($A = \alpha, 2; \alpha = 0, 1$) in Eq. (5). Then the second term of the action has the following form:

$$\int g_{\mu\nu} \epsilon^{\alpha\beta 2} (\Omega_a^\mu \partial_2 \Omega_b^\nu - \partial_2 \Omega_a^\mu \Omega_b^\nu) \frac{\partial \Phi^a}{\partial x^\alpha} \frac{\partial \Phi^b}{\partial x^\beta} d^3 x = \int d^2 x \int_0^M \epsilon^{\alpha\beta 2} B_{ab2} \frac{\partial \Phi^a}{\partial x^\alpha} \frac{\partial \Phi^b}{\partial x^\beta} dx^2. \quad (6)$$

Here $B_{ab2} = g_{\mu\nu} (\Omega_a^\mu \partial_2 \Omega_b^\nu - \partial_2 \Omega_a^\mu \Omega_b^\nu) = -B_{ba2}$. We will integrate on the coordinate x^2 in the limits $(0, M)$ with the following boundary conditions:

$$\Phi^a(x^\alpha, x^2) |_{x^2=0} = \phi^a(x^\alpha), B_{ab2}(x^\alpha, x^2) |_{x^2=0} = B_{ab}(x^\alpha).$$

The integral in x^2 on the lower limit of integration equals zero, which is easily seen by using the expansion of the integrand into the Taylor series. Consequently, the total action is

$$S = \frac{1}{2} \int d^2 x [g_{ab}(\phi) \eta^{\alpha\beta} + B_{ab}(\phi) \epsilon^{\alpha\beta}] \frac{\partial \phi^a}{\partial x^\alpha} \frac{\partial \phi^b}{\partial x^\beta}. \quad (7)$$

Here $g_{ab}(\phi) = g_{ba}(\phi)$ is the metric tensor of the group space G and $\phi^a(x)$ are the group parameters, $a, b = 1, 2, \dots, n$. The background field $B_{ab}(\phi)$ on the group space G is the antisymmetric tensor field $B_{ab}(\phi(x)) = -B_{ba}(\phi(x))$. The coordinates $x^\alpha = (t, x), \alpha = 0, 1$ belong to the 2-dimensional world-sheet with the constant metric tensor $\eta_{\alpha\beta}$ and the signature $(-1, 1)$. Let us introduce a repere $e_\mu^a(\phi) = \omega_\mu^a$ on the compact group space G and its inverse $e_a^\mu(\phi) = \omega_a^\mu$ such that the metric tensor can be explicitly written as

$$g_{ab}(\phi) = e_a^\mu(\phi) e_b^\nu(\phi) \delta_{\mu\nu}, \quad \delta_{\mu\nu} = e_\mu^a(\phi) e_\nu^b(\phi) g_{ab}(\phi). \quad (8)$$

Here $\delta_{\mu\nu}$ ($\mu, \nu = 1, 2, \dots, n$) is a constant tensor on the tangent space of the compact group space G at some point $\phi^a(x)$ with the same signature as $g_{ab}(\phi)$. To introduce the Hamiltonian, we rewrite the Lagrangian density and the equation of motion in terms the world-sheet coordinates (t, x) :

$$L = \frac{1}{2} g_{ab}(\phi) \left[\frac{\partial \phi^a}{\partial t} \frac{\partial \phi^b}{\partial t} - \frac{\partial \phi^a}{\partial x} \frac{\partial \phi^b}{\partial x} \right] + B_{ab}(\phi) \frac{\partial \phi^a}{\partial t} \frac{\partial \phi^b}{\partial x}. \quad (9)$$

Then the equation of motion takes the form

$$g_{ab}(\phi) \left[\frac{\partial^2 \phi^a}{\partial t \partial t} - \frac{\partial^2 \phi^a}{\partial x \partial x} \right] + \Gamma_{abc}(\phi) \left[\frac{\partial \phi^b}{\partial t} \frac{\partial \phi^c}{\partial t} - \frac{\partial \phi^b}{\partial x} \frac{\partial \phi^c}{\partial x} \right] + 2H_{abc}(\phi) \frac{\partial \phi^b}{\partial t} \frac{\partial \phi^c}{\partial x} = 0, \quad (10)$$

$$\Gamma_{abc} = \frac{1}{2} \left(\frac{\partial g_{ab}}{\partial \phi^c} + \frac{\partial g_{ac}}{\partial \phi^b} - \frac{\partial g_{bc}}{\partial \phi^a} \right), \quad H_{abc} = \frac{\partial B_{ab}}{\partial \phi^c} + \frac{\partial B_{ca}}{\partial \phi^b} + \frac{\partial B_{bc}}{\partial \phi^a}, \quad (11)$$

where $\Gamma_{abc}(\phi)$ are the Christoffel symbols. It is a symmetric function in b, c . The canonical momentum is as follows:

$$p_a(\phi(t, x)) = \frac{\delta L}{\delta \left(\frac{\partial \phi^a}{\partial t} \right)} = g_{ab}(\phi) \frac{\partial \phi^b}{\partial t} + B_{ab}(\phi) \frac{\partial \phi^b}{\partial x}. \quad (12)$$

By definition, the Hamiltonian is

$$H(\phi, p) = \frac{1}{2} g^{ab}(\phi) \left[p_a - B_{ac}(\phi) \frac{\partial \phi^c}{\partial x} \right] \left[p_b - B_{bd}(\phi) \frac{\partial \phi^d}{\partial x} \right] + \frac{1}{2} g_{ab}(\phi) \frac{\partial \phi^a}{\partial x} \frac{\partial \phi^b}{\partial x}. \quad (13)$$

Now let us introduce new dynamical variables as follows:

$$J_{0\mu}(\phi) = e_a^\mu(\phi) \left[p_a - B_{ab}(\phi) \frac{\partial \phi^b}{\partial x} \right], \quad J_{1\mu}(\phi) = e_a^\mu(\phi) \frac{\partial \phi^a}{\partial x}. \quad (14)$$

We see that the Hamiltonian (13) is factorized in these variables:

$$H = \frac{1}{2} [\delta^{\mu\nu} J_{0\mu}(\phi) J_{0\nu}(\phi) + \delta_{\mu\nu} J_1^\mu(\phi) J_1^\nu(\phi)]. \quad (15)$$

The equations of motion in terms of these variables are of first order:

$$\begin{aligned} \partial_0 J_1^\mu(\phi) - \partial_1 J_0^\mu(\phi) &= C_{\nu\lambda}^\mu J_0^\nu(\phi) J_1^\lambda(\phi), \\ \partial_0 J_0^\mu(\phi) - \partial_1 J_1^\mu(\phi) &= -H_{\nu\lambda}^\mu(\phi) J_0^\nu(\phi) J_1^\lambda(\phi). \end{aligned} \quad (16)$$

Here $C^{\mu\nu\lambda}$ is the structure constant tensor which can be obtained from the Maurer–Cartan equation:

$$C_{\nu\lambda}^\mu = \frac{\partial e_a^\mu(\phi)}{\partial x^b} [e_\nu^b(\phi) e_\lambda^a(\phi) - e_\nu^a(\phi) e_\lambda^b(\phi)] = \left[\frac{\partial e_a^\mu(\phi)}{\partial x^b} - \frac{\partial e_b^\mu(\phi)}{\partial x^a} \right] e_\nu^b(\phi) e_\lambda^a(\phi) \quad (17)$$

and the canonical Poisson bracket (PB) is

$$\{\phi^a(x), p_b(y)\} = \delta_b^a \delta(x - y). \quad (18)$$

Now we consider the commutation relations for the functions $J_{0\mu}(\phi(x)), J_{1\mu}(\phi(x)) = \delta_{\mu\nu} J_1^\nu(\phi(x))$ on the phase space under the PB (18):

$$\begin{aligned} \{J_{0\mu}(\phi(x)), J_{0\nu}(\phi(y))\} &= C_{\mu\nu}^\lambda J_{0\lambda}(\phi(x)) \delta(x - y) + H_{\mu\nu}^\lambda(\phi(x)) J_{1\lambda}(\phi(x)) \delta(x - y), \\ \{J_{0\mu}(\phi(x)), J_{1\nu}(\phi(y))\} &= C_{\mu\nu}^\lambda J_{1\lambda}(\phi(x)) \delta(x - y) + g_{\mu\nu} \frac{\partial}{\partial x} \delta(x - y), \\ \{J_{1\mu}(\phi(x)), J_{1\nu}(\phi(y))\} &= 0. \end{aligned} \quad (19)$$

Let us introduce the chiral variables

$$U_\mu = \frac{J_{0\mu} + \delta_{\mu\nu} J_1^\nu}{\sqrt{2}}, \quad V_\mu = \frac{J_{0\mu} - \delta_{\mu\nu} J_1^\nu}{\sqrt{2}}. \quad (20)$$

The commutation relations for the chiral currents $U^\mu(\phi), V^\mu(\phi)$ are not Poisson brackets because the torsion $H_{\mu\nu}^\lambda(\phi)$ is not a smooth function. These commutation relations form an algebra, if $H_{\mu\nu}^\lambda(\phi)$ is a constant tensor. The interesting cases arise if $H_{\mu\nu}^\lambda = \pm C_{\mu\nu}^\lambda$. In the case $H_{\mu\nu}^\lambda = -C_{\mu\nu}^\lambda$ the variables $U_\mu(\phi)$ form the closed Kac–Moody algebra [9, 10] for the right chiral currents:

$$\{U_\mu(\phi(x)), U_\nu(\phi(y))\}_2 = C_{\mu\nu}^\lambda U_\lambda(\phi(x))\delta(x-y) + \delta_{\mu\nu}\partial_x\delta(x-y). \quad (21)$$

Here we have noted the PB (21) as PB_2 . The last relations are not essential. In the case of $H_{\mu\nu}^\lambda = C_{\mu\nu}^\lambda$, the variables $V_\mu(\phi)$ form the closed Kac–Moody algebra for the left chiral currents:

$$\{V_\mu(\phi(x)), V_\nu(\phi(y))\} = C_{\mu\nu}^\lambda V_\lambda(\phi(x)) - \delta_{\mu\nu}\partial_x\delta(x-y). \quad (22)$$

Notice that the Kac–Moody algebra [9, 10] has been considered as a hidden symmetry of the two-dimensional chiral models [11]. In 1983 one of the authors (VDG) with Volkov and Tkach [12] considered the algebra of the nonlocal charges in σ -model in the framework of the integrability of this model. We have shown in this previous reference that the nonlocal charges form the enveloped algebra over the Kac–Moody algebra. If $C_{\mu\nu}^\lambda = H_{\mu\nu}^\lambda$ the equation of motion is

$$\partial_+ V_\mu(\phi(t, x)) = 0, \quad \partial_- U_\mu(\phi(t, x)) = C_{\mu}^{\nu\lambda} V_\nu(\phi) U_\lambda(\phi). \quad (23)$$

We see from Eqs. (21) and (23) that the chiral currents U_μ form the closed system in the first case and, from Eqs. (22) and (23), that the chiral currents V_μ also form the closed system in the second case. Precisely, the chiral currents are the generators of group transformations with the structure constants $C_\lambda^{\mu\nu}$ in the tangent space.

INTEGRABLE WZNW MODEL WITH CONSTANT TORSION

The components of the torsion C_{abc} are the structure constants of the Lie algebra. In the bi-Hamiltonian approach to the integrable string models with the constant torsion, we have considered the conserved primitive chiral invariant currents (densities of the dynamical Casimir operators) $C_n(U(x))$, as the local fields of a Riemann manifold [13, 14]. The primitive and nonprimitive local charges of the invariant chiral currents form the hierarchy of the new Hamiltonians. The primitive invariant currents are the densities of the Casimir operators; in contrast, the nonprimitive currents are functions of the primitive ones. The commutation relations (21) show that the currents U^μ form the closed algebra. Therefore, we will consider PBs of the right chiral currents U^μ and the Hamiltonians constructed only from the right currents. The constant torsion does not contribute to the equations of motion, but it gives the possibility to introduce the group structure and the symmetric structure constants. This paper was stimulated by the papers [16, 17] concerning the local conserved charges in two-dimensional models. In [16] the local invariant chiral currents, as polynomials of the initial chiral currents of the $SU(n)$, $SO(n)$, $SP(n)$, were constructed for principal chiral models. Their paper [16] was based on [17] involving the invariant tensors for the simple Lie algebras. Let us take t_μ the generators of the $SU(n)$, $SO(n)$, $SP(n)$ Lie algebras (2). There are additional relations for the generators of the Lie algebra in the defining matrix

representation. There is the following relation for the symmetric double product of the generators of $SU(n)$ algebra:

$$\{t_\mu, t_\nu\} = \frac{4}{n}\delta_{\mu\nu} + 2d_{\mu\nu\lambda}t_\lambda, \quad \mu = 1, \dots, n^2 - 1, \tag{24}$$

where $d_{\mu\nu\lambda}$ is a totally symmetric structure constant tensor. The Killing tensor $g_{\mu\nu}$ equals $\delta_{\mu\nu}$ for the compact Lie algebras. Similar relation for the totally symmetric triple product of the $SO(n)$ and $SP(n)$ algebras has the form

$$t_{(\mu}t_\nu t_{\lambda)} = v_{\mu\nu\lambda}^\rho t_\rho, \tag{25}$$

where $v_{\mu\nu\lambda\rho}$ is a totally symmetric structure constant tensor. The invariant chiral currents are the Liouville coordinates and they can be constructed as the product of the invariant symmetric tensors:

$$d_{(\mu_1 \dots \mu_n)} = d_{\mu_1 \mu_2}^{k_1} d_{\mu_3 k_1}^{k_2} \dots d_{\mu_{n-1} \mu_n}^{k_{n-3}}, \quad d_{\mu_1 \mu_2} = \delta_{\mu_1 \mu_2}.$$

For the $SU(n)$ group and the initial chiral currents $U^\mu(\phi(x))$, we have

$$C_n(U(\phi(x))) = d_{(\mu_1 \dots \mu_n)} U_{\mu_1} U_{\mu_2} \dots U_{\mu_n}, \quad C_2(U(\phi(x))) = \delta_{\mu\nu} U^\mu U^\nu. \tag{26}$$

Analogously, a similar construction can be used for $SO(n)$, $SP(n)$ groups. The invariant chiral currents can be constructed as product of the invariant symmetric constant tensors:

$$v_{(\mu_1 \dots \mu_{2n})} = v_{\mu_1 \mu_2 \mu_3}^{\nu_1} v_{\mu_4 \mu_5}^{\nu_2} \dots v_{\mu_{2n-2} \mu_{2n-1} \mu_{2n}}^{\nu_{n-1}}, \quad v_{\mu_1 \mu_2} = \delta_{\mu_1 \mu_2},$$

and the corresponding initial chiral currents U^μ

$$C_{2n}(U(\phi(x))) = v_{\mu_1 \dots \mu_{2n}} U^{\mu_1} \dots U^{\mu_{2n}}, \quad C_2(U(\phi(x))) = \delta_{\mu_1 \mu_2} U^{\mu_1} U^{\mu_2}. \tag{27}$$

The invariant chiral currents for $SU(2)$, $SO(3)$, $SP(2)$ have the form

$$C_{2n} = (C_2)^n. \tag{28}$$

Another family of the invariant symmetric currents J_n based on the invariant symmetric chiral currents of simple Lie groups are realized as the symmetric trace of the n product chiral currents $U(x) = t_\mu U^\mu$, $\mu = 1, \dots, n^2 - 1$:

$$J_n(U(\phi(x))) = \text{Sym Tr}(U \dots U). \tag{29}$$

These invariant currents are the polynomials of the product of the basic chiral currents C_k , $k = 2, 3, \dots, k$ [13, 14]. Let us introduce the PB of hydrodynamic type for the chiral currents in the Liouville form [18]:

$$\{C_m(\phi)(x), C_n(\phi)(y)\} = -W_{mn}(\phi(y)) \frac{\partial}{\partial y} \delta(y-x) + W_{nm}(\phi(x)) \frac{\partial}{\partial x} \delta(x-y). \tag{30}$$

The asymmetric Hamiltonian function $W_{mn}(U(\phi(x)))$ for the finite dimensional $SU(n)$, $SO(n)$, $SP(n)$ group has the following form:

$$W_{mn}(C(U(x))) = \frac{n-1}{m+n-2} \sum_k a_k C_{m+n-2,k}(U(x)), \quad \sum_{k=0} a_k = mn. \tag{31}$$

This PB can be rewritten as the PB of the hydrodynamic type by using the following equalities:

$$B(y)A(x)\frac{\partial}{\partial x}\delta(x-y) = B(y)A(y)\frac{\partial}{\partial x}\delta(x-y) - B(y)\frac{\partial A(y)}{\partial y}\delta(x-y),$$

$$\frac{\partial A(y)}{\partial y}\delta(x-y) + A(x)\frac{\partial}{\partial x}\delta(x-y) = A(y)\frac{\partial}{\partial x}\delta(x-y), \quad \frac{\partial}{\partial x}\delta(x-y) = -\frac{\partial}{\partial y}\delta(y-x).$$

Above, the invariant total symmetric currents $C_{n,k}$, $k = 1, 2 \dots$ are new currents, polynomials of the product of the basic invariant currents $C_{n_1}C_{n_2} \dots C_{n_n}$, $n_1 + \dots + n_n = n$. They can be obtained by means of the explicit computation of the total symmetric invariant currents J_n using the different replacements of the double product (24) for the $SU(n)$ group and of the triple product (25) for the $SO(n)$, $SP(n)$ groups into the expressions for the invariant currents J_n [13]. Here are only $l = n - 1$ primitive invariant tensors for $SU(n)$ algebra, $l = (n - 1/2)$ for $SO(n)$ algebra and $l = (n/2)$ for $SP(n)$ algebra. Higher invariant currents C_n for $n \geq l + 1$ are nonprimitive currents and they are polynomials of primitive currents. By using formula (30) we can obtain the expression for these polynomials within the condition $J_k = 0$ for $k > l$ for the generating function:

$$\det(1 - \lambda t_\mu U^\mu) = \exp \operatorname{Tr}(\ln(1 - \lambda U)) = \exp\left(-\sum_{k=2}^{\infty} \frac{\lambda^k}{k} J_k\right).$$

The corresponding charges for nonprimitive chiral currents C_n are not Casimir operators. Consequently, the WZNW model is not an integrable system for the group symmetry of the finite rank $l \geq 1$.

INTEGRABLE WZNW MODELS WITH $SU(2)$, $SO(3)$, $SP(2)$ CONSTANT TORSIONS

There is one primitive invariant tensor for the algebras of $SU(2)$, $SO(3)$, $SP(2)$. As we have pointed out, the invariant nonprimitive tensors for $n \geq 2$ are functions of the primitive tensors. Let us introduce the local chiral currents based on the invariant symmetric polynomials on the $SU(2)$, $SO(3)$, $SP(2)$ Lie groups:

$$C_2(U) = \delta_{\mu\nu}U^\mu U^\nu, \quad C_{2n}(U) = (\delta_{\mu\nu}U^\mu U^\nu)^n,$$

where $n = 1, 2, \dots$ and $\mu, \nu = 1, 2, 3$. The PB of Liouville coordinate $C_2(U(x))$ has the following form:

$$\{C_2(U(x)), C_2(U(y))\} = -2C_2(U(y))\partial_y\delta(y-x) + 2C_2(U(x))\partial_x\delta(x-y).$$

We will consider the invariant chiral $C_2(U(x))$ as a local field on the Riemann space of the chiral currents. As the Hamiltonians we choose the following functions:

$$H_{2(n+1)} = \frac{1}{2(n+1)} \int_0^{2\pi} C_2^{n+1}(U(y))dy, \quad n = 0, 1, \dots, \infty. \tag{32}$$

The equation of motion for the density of the first Casimir operator is as follows:

$$\frac{\partial C_2}{\partial t_{2(n+1)}} - (2n+1)(C_2)^n \frac{dC_2}{dx} = 0, \quad (33)$$

and the equation for the currents $C_2^n = C_{2n}$ is

$$\frac{\partial C_2^n}{\partial \tau_n} + (C_2)^n \frac{dC_2^n}{dx} = 0, \quad \tau_n = -(2n+1)t_{2(n+1)}. \quad (34)$$

The above equation is precisely inviscid Burgers equation. We will find the solution in the form

$$C_2^n(\tau_n, x) = \exp(a + i(x - \tau_n C_2^n(\tau_n, x))). \quad (35)$$

To obtain the solution of Eq. (34), it is convenient to rewrite this equation of motion as

$$Y_n = Z_n e^{Z_n}, \quad Y_n = i\tau_n e^{(a+ix)}, \quad Z_n = i\tau_n C_2^n. \quad (36)$$

Then the inverse transformation $Z_n = Z_n(Y_n)$ is defined by means of the periodical Lambert function [14]:

$$C_2^n(\tau_n, x) = \frac{1}{i\tau_n} W(i\tau_n e^{a+ix}). \quad (37)$$

Consequently, the solution for the first Casimir operator is

$$C_2(t_{2(n+1)}, x) = \left[\frac{i}{(2n+1)t_{2(n+1)}} W(-i(2n+1)t_{2(n+1)} e^{a+ix}) \right]^{1/n}. \quad (38)$$

With these results, the equation of motion for the initial chiral current U^μ defined by the PB (21) and the Hamiltonian (32) is

$$\frac{\partial U_\mu}{\partial t_{2(n+1)}} = \frac{\partial}{\partial x} [U_\mu (UU)^n] = nU_\mu C_2^{n-1} \frac{\partial}{\partial x} C_2 + C_2^n \frac{\partial}{\partial x} U_\mu, \quad \mu = 1, 2, 3. \quad (39)$$

It is easy to test that equation of motion (33) is in full agreement with Eq. (39) simply by multiplication with the chiral current U_μ on the both sides of Eq. (39). It is possible to rewrite this equation as a linear equation by using the solution (37) which diagonalizes Eq. (39):

$$\frac{\partial U_\mu}{\partial t_{2(n+1)}} = \frac{\partial U_\mu}{\partial x} f_n + U_\mu \frac{\partial}{\partial x} f_n$$

or as the linear nonhomogeneous equation

$$\begin{aligned} \frac{\partial z^\mu}{\partial t_{2(n+1)}} &= f_n(t_n, x) \frac{\partial z^\mu}{\partial x} + \frac{\partial}{\partial x} f_n(t_n, x), \quad z^\mu = \ln U^\mu, \\ f_n &= C_2^n, \quad \frac{\partial z^\mu}{\partial x} = \frac{1}{U^\mu} \frac{\partial U^\mu}{\partial x} \quad (\text{not sum}). \end{aligned} \quad (40)$$

INFINITE DIMENSIONAL HYDRODYNAMIC CHAINS

The first example of the infinite dimensional hydrodynamic chains is based on the invariant chiral currents $C_{2n} = (C_2)^n$, $n = 1, 2, \dots, \infty$ of the WZNW model with the $SU(2)$, $SO(3)$, $SP(2)$ constant torsions. The PB of the different degrees of the invariant chiral currents $C_2^n(x)$, $C_2^m(x)$ has the form

$$\{C_2^m(x), C_2^n(y)\} = \frac{2nm(2m-1)}{n+m-1} C_2^{m+m-1}(x) \frac{\partial \delta(x-y)}{\partial x} - \frac{2nm(2n-1)}{n+m-1} C_2^{n+m-1}(y) \frac{\partial \delta(y-x)}{\partial y}. \quad (41)$$

The equation of motion for invariant current C_2^m with Hamiltonian

$$H_{2n} = \frac{1}{2n} \int_0^{2\pi} C_{2n}(y) dy$$

has the form

$$\frac{\partial C_2^m}{\partial t_{2n}} = \frac{m(2n-1)}{m+n-1} \frac{\partial C_2^{m+n-1}}{\partial x}.$$

After the redefinition $C_2^n = C_{2n} = C_p$ we can obtain the standard form of the hydrodynamic chain:

$$\{C_p(x), C_q(y)\} = \frac{pq(p-1)}{p+q-2} C_{p+q-2}(x) \frac{\partial \delta(x-y)}{\partial x} - \frac{pq(q-1)}{p+q-2} C_{p+q-2}(y) \frac{\partial \delta(y-x)}{\partial y}. \quad (42)$$

The second example of the infinite dimensional chain is based on the invariant chiral currents of the WZNW model with the $SU(\infty)$, $SO(\infty)$, $SP(\infty)$ constant torsions. If dimension of matrix representation n is not ended ($n \rightarrow \infty$), all the chiral currents are the primitive currents. This is easy to see from the expression for the new chiral currents $C_{m,k}$ (see, e.g., [13, 14]). The PB in Liouville coordinates $C_m(x)$, $m = 2, 3, \dots, \infty$ takes the form

$$\{C_m(x), C_n(y)\} = -W_{mn}(C(y)) \frac{\partial}{\partial y} \delta(y-x) + W_{nm}(C(x)) \frac{\partial}{\partial x} \delta(x-y), \quad (43)$$

$$W_{mn}(C(x)) = \frac{mn(n-1)}{m+n-2} C_{m+n-2}(x). \quad (44)$$

This PB obeys the skew-symmetric condition: $\{C_m(x), C_n(y)\} = -\{C_n(y), C_m(x)\}$. However, the Jacobi identity imposes conditions on the Hamiltonian function $W_{mn}(C(x))$ [18]:

$$(W_{kp} + W_{pk}) \frac{\partial W_{mn}}{\partial C_k} = (W_{km} + W_{mk}) \frac{\partial W_{pn}}{\partial C_k}, \quad \frac{dW_{kp}}{dx} \frac{\partial W_{nm}}{\partial C_k} = \frac{dW_{km}}{dx} \frac{\partial W_{np}}{\partial C_k}. \quad (45)$$

The Jacobi identity is satisfied by the metric tensor $W_{mn}(C(x))$ (44). The algebra of charges $\int_0^{2\pi} C_n(x) dx$ is the Abelian algebra. Now let us choose the Casimir operators C_n as the Hamiltonians:

$$H_n = \frac{1}{n} \int_0^{2\pi} C_n(x) dx, \quad n = 2, 3, \dots \quad (46)$$

Then the equations of motion for the densities of Casimir operators are the following:

$$\frac{\partial C_m(x)}{\partial t_n} = \frac{m(n-1)}{m+n-2} \frac{\partial C_{m+n-2}}{\partial x}. \quad (47)$$

Thus, the invariant chiral currents with the $SU(2)$, $SO(3)$, $SP(2)$ constant torsion and the invariant chiral currents with the $SU(\infty)$, $SO(\infty)$, $SP(\infty)$ constant torsion form the same infinite hydrodynamic chain (42), (43), (44). This PB (43) is particular case of the M -bracket given by Dorfman [19] and Kupershmidt [20] for $M = 2$ and describes the hydrodynamic chains. We can construct new nonlinear equations of motion for the initial chiral currents U^μ using the flat PB_2 (21) and the Hamiltonians H_n (46), where $C_n(x)$ is defined by Eq. (26) for the $SU(\infty)$ group:

$$\begin{aligned} \frac{\partial U_\mu(x)}{\partial t_n} &= \frac{1}{n} \int_0^{2\pi} dy \{U_\mu(x), C_n(U(y))\}_2, \\ \frac{\partial U_\mu(x)}{\partial t_n} &= \frac{\partial}{\partial x} [d_{\nu_1 \nu_2}^{k_1} d_{k_1 \nu_3}^{k_2} \dots d_{\nu_{n-1} \mu}^{k_{n-3}} U^{\nu_1}(x) \dots U^{\nu_{n-1}}(x)]. \end{aligned} \quad (48)$$

As an example, we consider $n = 3$:

$$\frac{\partial U_\mu}{\partial t_3} = \frac{\partial}{\partial x} (d_{\mu\nu\lambda} U^\nu U^\lambda), \quad \mu = 1, 2, \dots, \infty. \quad (49)$$

It is easy to see that this dynamical system is a bi-Hamiltonian one:

$$\frac{\partial U_\mu(x)}{\partial t_3} = \frac{1}{3} \int_0^{2\pi} dy \{U_\mu(x), C_3(U(y))\}_2 = \frac{1}{2} \int_0^{2\pi} dy \{U_\mu(x), C_2(U(y))\}_3. \quad (50)$$

Above, the PB_3 has the form

$$\{U_\mu(x), U_\nu(y)\}_3 = 2d_{\mu\nu\lambda} U^\lambda. \quad (51)$$

Let us remind that $d_{\mu\nu\lambda}$ are the symmetric structure constants of the $SU(\infty)$ algebra in a matrix representation. This PB satisfies Jacobi identity for $(n \rightarrow \infty)$:

$$d_{\sigma\mu\nu} d_{\sigma\lambda\rho} + d_{\sigma\mu\lambda} d_{\sigma\nu\rho} + d_{\sigma\mu\rho} d_{\sigma\nu\lambda} = \frac{1}{n} (\delta_{\mu\nu} \delta_{\lambda\rho} + \delta_{\mu\lambda} \delta_{\nu\rho} + \delta_{\nu\rho} \delta_{\nu\lambda}).$$

Analogously, we can obtain the equation of motion for the chiral currents of $SO(\infty)$ and $SP(\infty)$:

$$\frac{\partial U_\mu(x)}{\partial t_n} = \frac{\partial}{\partial x} [v_{\nu_1 \nu_2 \nu_3}^{k_1} \dots v_{\nu_{2n-2} \nu_{2n-1} \mu}^{k_{2n-3}} U^{\nu_1} \dots U^{\nu_{2n-1}}]. \quad (52)$$

To see how it works, for example, let us consider $n = 4$:

$$\frac{\partial U_\mu}{\partial t_4} = \frac{\partial}{\partial x} (v_{\mu\nu\lambda\rho} U^\nu U^\lambda U^\rho), \quad \mu = 1, 2, \dots, \infty. \quad (53)$$

Also we can obtain a solution for the metric function $W_{mn}(C(x))$ which is analog of the Dubrovin–Novikov metric tensor $W_{\mu\nu} = \frac{\partial^2 F}{\partial U^\mu \partial U^\nu}$:

$$C_m(U(x)) = mF(U(x)), \quad F(x, t_n) = g\left(t_n + \frac{x}{n-1}\right),$$

and $g\left(t_n + \frac{x}{n-1}\right)$ is an arbitrary function of its argument.

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