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ANALYTICAL SOLUTIONS OF THE DKP EQUATION UNDER TIETZ–HUA POTENTIAL IN $(1+3)$ DIMENSIONS

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The approximate relativistic Duffin–Kemmer–Petiau equation with Tietz–Hua potential in $(1+3)$ dimensions for spin-one particles is investigated by approximating the centrifugal term by Pekeris approximation and using parametric generalization of the Nikiforov method to obtain the bound state solution. The energy eigenvalues and eigenfunctions are obtained in closed form.

Исследуется приближенное релятивистское уравнение Даффина–Кеммера–Петье с потенциалом Тица–Хуа в измерениях $(1+3)$ для частиц со спином 1. Для этого центробежный член аппроксимируется приближением Пекериса и используется параметрическое обобщение метода Никифорова, чтобы получить решение для связанного состояния. Собственные значения энергии и собственные функции получаются в замкнутой форме.

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INTRODUCTION

The Duffin–Kemmer–Petiau equation (DKP) is the relativistic equation that describes spin-one and spin-zero particles [1–3]. The applications of the DKP formalism have been useful in the study of relativistic interactions of spin-one and spin-zero hadrons in nuclei. It is also used in the deuteron–nucleus scattering [4] and for explaining the quark confinement problems of quantum chromodynamics (QCD) theory in particle physics [5]. In recent times, many authors have devoted their interest to studying the DKP equation with different potential interactions [6–10]. Hassanabadi et al. [11] investigated the DKP equation for hyperbolical potential in $(1+3)$ dimensions. Molaee et al. [12] studied the DKP equation in $(1+2)$ dimensions for Hulthén potential using asymptotic iteration method. The exact solution of DKP with pseudoharmonic potential in the presence of magnetic field in $(1+2)$ dimensions

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has also been reported [13]. The main purpose of the present paper is to study the DKP equation for the Tietz–Hua potential in $(1+3)$-dimensional space-time for spin-one particles using the Nikiforov–Uvarov (NU) method [14, 15]. The Tietz–Hua potential is one of the best molecular potentials to describe the vibration energy spectra of diatomic molecules [16, 17]. The Tietz–Hua potential takes the form [16–18]

$$V(r) = D\left[\frac{1-\mathrm{e}^{-b_h(r-r_e)}}{1-c_h\,\mathrm{e}^{-b_h(r-r_e)}}\right]^2, \quad b_h = a(1-c_h), \tag{1}$$

where the parameters r, r_e, b_h, D, and c_h are the internuclear distance, the molecular bond length, the Morse constant, the dissociation energy, and the potential constant, respectively [18, 19]. The Morse and Coulomb potentials are the special cases of the Tietz–Hua potential. The organization of the paper is as follows. In Sec. 1, we present the theory of DKP. Section 2 is devoted to the theory of DKP in $(1+3)$ dimensions. We review the NU method in Sec. 3. In Sec. 4, we derive the solutions of DKP equation. Section 5 is devoted to results and discussion. We give a brief conclusion in the final section.

1. THEORY OF DKP FORMULATION

The relativistic DKP equation for a free spin-0 or spin-1 particle of mass *m* is

$$(i\beta^\mu\partial_\mu - m)\,\psi_{\mathrm{DKP}} = 0, \tag{2}$$

where $\beta^\mu(\mu = 0,1,2,3)$ matrices satisfy the commutation relation

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\nu\lambda}\beta^\mu, \tag{3}$$

which defines the DKP algebra. The algebra generated by the β^N's has three irreducible representations: a ten-dimensional one that is related to $S = 1$, a five-dimensional one relevant for $S = 0$ (spinless particles), and one-dimensional representation which is trivial. In the spin-0 representation, β^μ are 5×5 matrices defined as $(i = 1,2,3)$

$$\beta^0 = \begin{pmatrix} \theta & \tilde{0} \\ \bar{0}_T & 0 \end{pmatrix}, \quad \beta^i = \begin{pmatrix} \tilde{0} & \rho^i \\ -\rho^i_T & 0 \end{pmatrix}, \tag{4}$$

where $\bar{O}, \tilde{o}, o$ are 2×2, 2×3, 3×3 zero matrices, respectively, and

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\rho^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

However, for spin-one particle, the β^μ are 10×10 matrices defined as

$$\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^T & 0 & I & I \\ \bar{0}^T & I & 0 & 0 \\ \bar{0}^T & 0 & 0 & 0 \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ \bar{0}^T & 0 & 0 & -is_i \\ e^T & 0 & 0 & 0 \\ \bar{0}^T & -is_i & 0 & 0 \end{pmatrix}, \tag{6}$$

where s_i are usually well-known 3×3 spin-one matrices defined as

$$\bar{0} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

2. THE DKP EQUATION IN $(1+3)$-DIMENSIONAL SPACE-TIME

The DKP equation in the presence of an interaction term in $(1+3)$ dimensions is written as

$$(i\beta^0\partial_0 + i\beta^1\partial_1 + i\beta^2\partial_2 - i\beta^3\partial_3 - m - U)\,\psi(x,y,z,t) = 0. \tag{8}$$

We have

$$\psi(x,y,z,t) = \mathrm{e}^{-iE_{n,l}t}\,\psi_{n,l}(x,y,z), \tag{9}$$

where

$$\psi^T_{n,l}(r) = \{\varphi^{(1)}_{n,l}(r), \varphi^{(2)}_{n,l}(r), \varphi^{(3)}_{n,l}(r), \varphi^{(4)}_{n,l}(r), \varphi^{(5)}_{n,l}(r), \varphi^{(6)}_{n,l}(r), \varphi^{(7)}_{n,l}(r), \varphi^{(8)}_{n,l}(r), \varphi^{(9)}_{n,l}(r), \varphi^{(10)}_{n,l}(r))^T\}, \tag{10}$$

with the following definitions:

$$\varphi^{(1)}_{n,l}(r) = i\varphi, \tag{11a}$$

$$\mathbf{F}(r) = (\varphi^{(2)}_{n,l}(r), \varphi^{(3)}_{n,l}(r), \varphi^{(4)}_{n,l}(r)), \tag{11b}$$

$$\mathbf{G}(r) = (\varphi^{(5)}_{n,l}(r), \varphi^{(6)}_{n,l}(r), \varphi^{(7)}_{n,l}(r)), \tag{11c}$$

$$\mathbf{H}(r) = (\varphi^{(8)}_{n,l}(r), \varphi^{(9)}_{n,l}(r), \varphi^{(10)}_{n,l}(r)). \tag{11d}$$

However, for elastic scattering, the interaction is defined as [1–3, 11]

$$U = S(r) + PS_\mu(r) + \beta^0 V(r) + \beta^0 PV_p(r), \tag{12}$$

where each term has a specific Lorentz character. Two Lorentz vectors may be written as β^μ and $P\beta^\mu$ by assuming rotational invariance and parity conservation [1–3, 11]. The term $P = (\beta^\mu\beta_\mu = 2) = \mathrm{diag}\,(1,1,1,1,0,0,0,0,0,0)$ is a projection operator for the spin-one sector of the DKP equation. Thus, from Eq. (8), we obtain

$$i\boldsymbol{\nabla}\times\mathbf{F} = m\mathbf{H}, \tag{13}$$

$$\boldsymbol{\nabla}\cdot\mathbf{G} = m\varphi, \tag{14}$$

$$E_{n,l}\mathbf{G} + i\boldsymbol{\nabla}\times\mathbf{H} = m\mathbf{F}, \tag{15}$$

$$(E_{n,l} - V(r))\mathbf{F} + \boldsymbol{\nabla}\varphi = m\mathbf{G}. \tag{16}$$

The spinor of the DKP equation is given as $\psi = (\varphi, A^1, A^2, A^3, E_1, E_2, E_3, -B_1, -B_2, -B_3)^T$. The DKP equation in $(1+3)$ dimensions resembles Maxwell's equation of the form

$$-i\nabla\times A = mB, \tag{17}$$

$$i\frac{\partial A}{\partial t} - i\nabla\varphi = mE, \tag{18}$$

$$i\frac{\partial E}{\partial t} - i\nabla\times B = mA, \tag{19}$$

$$i\nabla\cdot A = m\varphi. \tag{20}$$

Equations (17)–(20) are Maxwell's form of the DKP equation in $(1+3)$ dimensions.

Consequently, combining Eqs. (13)–(16) and (17)–(20), we obtain

$$\{E_{n,l}(E_{n,l}-V(r))-m_0^2\}\mathbf{F}(r)+\nabla^2\mathbf{F}(r)=0, \tag{21}$$

or more explicitly, we write

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+E_{n,l}^2-E_{n,l}V(r)-m^2-\frac{l(l+1)}{r^2}\right\}\mathbf{F}(r)=0. \tag{22}$$

By introducing $\mathbf{F}(r)=r^{-1}\mathbf{R}(r)$ into Eq. (22),

$$\begin{pmatrix}\varphi_{n,l}^2(r)\\ \varphi_{n,l}^3(r)\\ \varphi_{n,l}^4(r)\end{pmatrix}=r^{-1}\begin{pmatrix}R_{n,l}^2(r)\\ R_{n,l}^3(r)\\ R_{n,l}^4(r)\end{pmatrix} \tag{23}$$

leads to the second-order differential equation for the DKP in $(1+3)$ dimensions [20–25]:

$$\left\{\frac{d^2}{dr^2}+E_{n,l}^2-E_{n,l}V(r)-m^2-\frac{l(l+1)}{r^2}\right\}R_{n,l}(r)=0. \tag{24}$$

3. THE NIKIFOROV–UVAROV METHOD

The parametric form of the Nikiforov–Uvarov method takes the form [14, 15]

$$\frac{d^2\psi}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d\psi}{ds}+\frac{1}{s^2(1-\alpha_3 s)^2}\left\{-\xi_1 s^2+\xi_2 s-\xi_3\right\}\psi(s)=0. \tag{25}$$

The energy eigenvalues and eigenfunctions, respectively, satisfy the following sets of equations:

$$\alpha_2 n+(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0, \tag{26}$$

$$\psi_n(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\alpha_{13}/\alpha_3}P_n^{(\alpha_{10}-1,\alpha_{11}/\alpha_3-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{27}$$

where

$$\begin{gathered}\alpha_4=\frac{1}{2}(1-\alpha_1),\quad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\quad \alpha_6=\alpha_5^2+\xi_1,\\ \alpha_7=2\alpha_4\alpha_5-\xi_2,\quad \alpha_8=\alpha_4^2+\xi_3,\quad \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\\ \alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\quad \alpha_{11}=\alpha_2-2\alpha_5+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right),\\ \alpha_{12}=\alpha_4+\sqrt{\alpha_8},\quad \alpha_{13}=\alpha_5-\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right),\end{gathered} \tag{28}$$

and P_n is the orthogonal Jacobi polynomial.

4. APPROXIMATE SOLUTIONS OF THE DKP EQUATION IN $(1+3)$ DIMENSIONS

In this section, we shall solve the DKP equation in $(1+3)$ dimensions for the Tietz–Hua potential using the parametric form of the NU method. Substituting Eq. (1) into Eq. (24) yields

$$\left\{\frac{d^2}{dr^2}+E_{n,l}^2-m^2-E_{n,l}D\left(\frac{1-\mathrm{e}^{-b_h(r-r_e)}}{1-c_h\,\mathrm{e}^{-b_h(r-r_e)}}\right)^2-\frac{l(l+1)}{r^2}\right\}R_{n,l}(r)=0. \qquad (29)$$

In order to solve the approximate solution of Eq. (30) for the Tietz–Hua potential, we invoke the Pekeris approximation for the centrifugal term $1/r^2$ as [26, 27] (see Appendix for the details of this approximation)

$$\frac{l(l+1)}{r^2}\approx\frac{l(l+1)}{r_e^2}\left[D_0+D_1\frac{\mathrm{e}^{-\alpha x}}{1-c_h\,\mathrm{e}^{-\alpha x}}+D_2\frac{\mathrm{e}^{-2\alpha x}}{(1-c_h\,\mathrm{e}^{-\alpha x})^2}\right]. \qquad (30)$$

Now by using the transformation $y=\mathrm{e}^{-\alpha x}$, we find the following second-order differential equation for the DKP equation with Tietz–Hua potential:

$$\frac{d^2R_{n,l}(y)}{dy^2}+\frac{(1-c_hy)}{y(1-c_hy)}\frac{dR_{n,l}(y)}{dy}+\frac{1}{y^2(1-c_hy)^2}\left\{Ay^2+By+C\right\}R_{n,l}(y)=0, \qquad (31)$$

where,

$$A=\frac{c_h^2(E_{n,l}^2-m^2)}{\alpha^2}-\frac{E_{n,l}D}{\alpha^2}-\frac{c_h^2l(l+1)D_0}{\alpha^2r_e^2}+\frac{c_hl(l+1)D_1}{\alpha^2r_e^2}-\frac{l(l+1)D_2}{\alpha^2r_e^2}, \qquad (32)$$

$$B=-\frac{2c_h(E_{n,l}^2-m^2)}{\alpha^2}+\frac{2E_{n,l}D}{\alpha^2}+\frac{2c_hl(l+1)D_0}{\alpha^2r_e^2}-\frac{l(l+1)D_1}{\alpha^2r_e^2}, \qquad (33)$$

$$C=\frac{(E_{n,l}^2-m^2)}{\alpha^2}-\frac{E_{n,l}D}{\alpha^2}-\frac{l(l+1)D_0}{\alpha^2r_e^2}. \qquad (34)$$

Comparing Eqs. (31) and (25), we find the following parameter sets:

$$\alpha_1=1,\quad \alpha_2=c_h,\quad \alpha_3=c_h,\quad \xi_1=-A,\quad \xi_2=B,\quad \xi_3=-C \qquad (35)$$

and from Eq. (28) we get the following:

$$\alpha_4=0,\quad \alpha_5=-\frac{c_h}{2},$$

$$\alpha_6=\frac{c_h^2}{4}-\frac{c_h^2(E_{n,l}^2-m^2)}{\alpha^2}+\frac{E_{n,l}D}{\alpha^2}+\frac{c_h^2l(l+1)D_0}{\alpha^2r_e^2}-\frac{c_hl(l+1)D_1}{\alpha^2r_e^2}+\frac{l(l+1)D_2}{\alpha^2r_e^2}, \qquad (36)$$

$$\alpha_7=\frac{2c_h(E_{n,l}^2-m^2)}{\alpha^2}-\frac{2E_{n,l}D}{\alpha^2}-\frac{2c_hl(l+1)D_0}{\alpha^2r_e^2}+\frac{l(l+1)D_1}{\alpha^2r_e^2}, \qquad (37)$$

$$\alpha_8=-\frac{(E_{n,l}^2-m^2)}{\alpha^2}+\frac{E_{n,l}D}{\alpha^2}+\frac{l(l+1)D_0}{\alpha^2r_e^2}, \qquad (38)$$

$$\alpha_9 = \frac{E_{n,l}Dc_h^2}{\alpha^2} - \frac{2E_{n,l}Dc_h}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{c_h^2}{4} + \frac{l(l+1)D_2}{\alpha^2 r_e^2}, \tag{39}$$

$$\alpha_{10} = 1 + 2\sqrt{-\frac{(E_{n,l}^2 - m^2)}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{l(l+1)D_0}{\alpha^2 r_e^2}}, \tag{40}$$

$$\alpha_{11} = 2c_h + 2\Bigg\{\sqrt{\frac{E_{n,l}Dc_h^2}{\alpha^2} - \frac{2E_{n,l}Dc_h}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{c_h^2}{4} + \frac{l(l+1)D_2}{\alpha^2 r_e^2}} + \\ + c_h\sqrt{-\frac{(E_{n,l}^2 - m^2)}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{l(l+1)D_0}{\alpha^2 r_e^2}}\Bigg\}, \tag{41}$$

$$\alpha_{12} = \sqrt{-\frac{(E_{n,l}^2 - m^2)}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{l(l+1)D_0}{\alpha^2 r_e^2}}, \tag{42}$$

$$\alpha_{13} = -\frac{c_h}{2} - \Bigg\{\sqrt{\frac{E_{n,l}Dc_h^2}{\alpha^2} - \frac{2E_{n,l}Dc_h}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{c_h^2}{4} + \frac{l(l+1)D_2}{\alpha^2 r_e^2}} + \\ + c_h\sqrt{-\frac{(E_{n,l}^2 - m^2)}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{l(l+1)D_0}{\alpha^2 r_e^2}}\Bigg\}. \tag{43}$$

Thus, by employing these parametric parameters in Eq. (26), we can easily obtain the analytic energy equation for the DKP equation for the Tietz–Hua potential as

$$c_h n + \frac{(2n+1)c_h}{2} + \\ + (2n+1)\Bigg\{\sqrt{\frac{E_{n,l}Dc_h^2}{\alpha^2} - \frac{2E_{n,l}Dc_h}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{c_h^2}{4} + \frac{l(l+1)D_2}{\alpha^2 r_e^2}} + \\ + c_h\sqrt{-\frac{(E_{n,l}^2 - m^2)}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{l(l+1)D_0}{\alpha^2 r_e^2}}\Bigg\} + n(n-1)c_h - \\ - \frac{2E_{n,l}D}{\alpha^2} - \frac{2E_{n,l}Dc_h}{\alpha^2} - \frac{4l(l+1)c_h D_0}{\alpha^2 r_e^2} + \frac{l(l+1)D_1}{\alpha^2 r_e^2} + \\ + 2\sqrt{\left\{\frac{E_{n,l}Dc_h^2}{\alpha^2} - \frac{2E_{n,l}Dc_h}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{c_h^2}{4} + \frac{l(l+1)D_2}{\alpha^2 r_e^2}\right\}} + \\ + 2\sqrt{\left\{-\frac{(E_{n,l}^2 - m^2)}{\alpha^2} + \frac{E_{n,l}D}{\alpha^2} + \frac{l(l+1)D_0}{\alpha^2 r_e^2}\right\}} = 0. \tag{44}$$

Consequently, in order to give a complete description of the DKP equation, we determine the corresponding wave function as

$$
\begin{aligned}
R_{n,l}(r) = N_{n,l}(\mathrm{e}^{-\alpha(r-r_e)})^{\sqrt{-\frac{(E_{n,l}^2-m^2)}{\alpha^2}+\frac{E_{n,l}D}{\alpha^2}+\frac{l(l+1)D_0}{\alpha^2 r_e^2}}} \times \\
\times (1-c_h\,\mathrm{e}^{-\alpha(r-r_c)})^{\frac{1}{2}+\frac{1}{c_h}\left\{\sqrt{\frac{E_{n,l}Dc_h^2}{\alpha^2}-\frac{2E_{n,l}Dc_h}{\alpha^2}+\frac{E_{n,l}D}{\alpha^2}+\frac{c_h^2}{4}+\frac{l(l+1)D_2}{\alpha^2 r_e^2}}\right\}} \times \\
\times P_n^{\left(2\sqrt{-\frac{(E_{n,l}^2-m^2)}{\alpha^2}+\frac{E_{n,l}D}{\alpha^2}+\frac{l(l+1)D_0}{\alpha^2 r_e^2}},\quad \frac{2}{c_h}\left\{\sqrt{\frac{E_{n,l}Dc_h^2}{\alpha^2}-\frac{2E_{n,l}Dc_h}{\alpha^2}+\frac{E_{n,l}D}{\alpha^2}+\frac{c_h^2}{4}+\frac{l(l+1)D_2}{\alpha^2 r_e^2}}\right\}\right)} \times \\
\times \left(1-2c_h\,\mathrm{e}^{-\alpha(r-r_e)}\right), \quad (45)
\end{aligned}
$$

where $N_{n,l}$ is the normalization constant.

5. RESULTS AND DISCUSSION

Furthermore, we compute the energy spectrum for different n with the potential variables D, D_0, D_1, D_2, r_e. Thus, in Table 1 it is observed that with decreasing r_e values of the potential parameter the energy level of the system increases. Similarly, in Table 2 with increasing D value of the potential parameter the energy eigenvalues decrease, and in Table 3 with rising values of D_1 the energy eigenvalues increase accordingly. Also, in Table 4 with increasing values of D_2 result in increased energies and, finally, in Table 5 with increasing

Table 1. $D = 1$, $D_0 = 0.5$, $D_1 = 0.25$, $D_2 = 0.735$, $L = 1$, $C = 0.3$, $\alpha = 0.1$, $m = 1$

$\vert n, L\rangle$	$E_{n,L}(r_e = 0.8)$	$E_{n,L}(r_e = 0.6)$	$E_{n,L}(r_e = 0.4)$	$E_{n,L}(r_e = 0.2)$
$\vert 0, 1\rangle$	1.481327	1.873292	2.659163	5.033787
$\vert 1, 1\rangle$	1.569213	1.952952	2.727026	5.085633
$\vert 2, 1\rangle$	1.650396	2.026419	2.789954	5.134708
$\vert 3, 1\rangle$	1.725140	2.093980	2.848160	5.181092
$\vert 4, 1\rangle$	1.793679	2.155881	2.901829	5.224855
$\vert 5, 1\rangle$	1.856213	2.212334	2.951125	5.266064

Table 2. $r_e = 0.8$, $D_0 = 0.5$, $D_1 = 0.25$, $D_2 = 0.735$, $L = 1$, $C = 0.3$, $\alpha = 0.1$, $m = 1$

$\vert n, L\rangle$	$E_{n,L}(D = 1)$	$E_{n,L}(D = 1.2)$	$E_{n,L}(D = 1.4)$	$E_{n,L}(D = 1.6)$
$\vert 0, 1\rangle$	1.481327	1.421915	1.360072	1.298128
$\vert 1, 1\rangle$	1.569213	1.516866	1.460061	1.401468
$\vert 2, 1\rangle$	1.650396	1.605917	1.555013	1.500652
$\vert 3, 1\rangle$	1.725140	1.689213	1.644959	1.595604
$\vert 4, 1\rangle$	1.793679	1.766894	1.729941	1.686277
$\vert 5, 1\rangle$	1.856213	1.839088	1.810011	1.772646

values of D_0 the energies come down. We show the plot of the wave functions for different n versus r in the figure.

Table 3. $r_e = 0.8,\ D_0 = 0.5,\ D_2 = 0.735,\ L = 1,\ C = 0.3,\ \alpha = 0.1,\ m = 1$

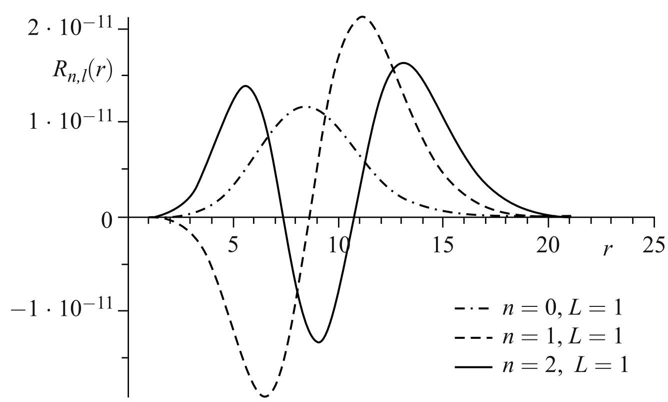
$\vert n, L\rangle$	$E_{n,L}(D_1 = 0.25)$	$E_{n,L}(D_1 = 0.45)$	$E_{n,L}(D_1 = 0.65)$	$E_{n,L}(D_1 = 0.85)$
$\vert 0, 1\rangle$	1.481327	1.605698	1.715942	1.813434
$\vert 1, 1\rangle$	1.569213	1.683088	1.783849	1.872583
$\vert 2, 1\rangle$	1.650396	1.754395	1.846123	1.926421
$\vert 3, 1\rangle$	1.725140	1.819797	1.902877	1.975013
$\vert 4, 1\rangle$	1.793679	1.879448	1.954208	2.018406
$\vert 5, 1\rangle$	1.856213	1.933482	2.000194	2.056630

Table 4. $r_e = 0.8,\ D_0 = 0.5,\ D_1 = 1,\ L = 1,\ C = 0.3,\ \alpha = 0.1,\ m = 1$

$\vert n, L\rangle$	$E_{n,L}(D_2 = 0.735)$	$E_{n,L}(D_2 = 0.935)$	$E_{n,L}(D_2 = 1.3)$	$E_{n,L}(D_2 = 2)$
$\vert 0, 1\rangle$	1.481327	1.573431	1.689564	1.820336
$\vert 1, 1\rangle$	1.569213	1.654737	1.762068	1.881822
$\vert 2, 1\rangle$	1.650396	1.729453	1.828131	1.937046
$\vert 3, 1\rangle$	1.725140	1.797853	1.888027	1.986253
$\vert 4, 1\rangle$	1.793679	1.860171	1.941975	2.029634
$\vert 5, 1\rangle$	1.856213	1.916601	1.990154	2.067332

Table 5. $r_e = 0.8,\ D_2 = 0.735,\ D_1 = 1,\ L = 1,\ C = 0.3,\ \alpha = 0.1,\ m = 1$

$\vert n, L\rangle$	$E_{n,L}(D_0 = 0.5)$	$E_{n,L}(D_0 = 1)$	$E_{n,L}(D_0 = 1.5)$	$E_{n,L}(D_0 = 2)$
$\vert 0, 1\rangle$	1.481327	1.438368	1.268730	0.966948
$\vert 1, 1\rangle$	1.569213	1.555186	1.420232	1.166871
$\vert 2, 1\rangle$	1.650396	1.663779	1.560455	1.349039
$\vert 3, 1\rangle$	1.725140	1.764669	1.690452	1.515990
$\vert 4, 1\rangle$	1.793679	1.858311	1.811103	1.669649
$\vert 5, 1\rangle$	1.856213	1.945105	1.923146	1.811524



The wave functions vs. r

CONCLUSIONS

In this paper, we used the NU method to obtain the complete set of the energy spectrum and the corresponding wave function for the relativistic first-order DKP equation in the presence of Tietz–Hua potential in $(1+3)$ dimensions for spin-one particles. Also, we used the Pekeris approximation for deriving the results. Our results can be useful in related fields such as particle and nuclear physics [26].

Appendix
Pekeris approximation to the centrifugal term

The centrifugal term $l(l+1)/r^2$ can be expanded around $r = r_e$ by Taylor power expansion of $x = (r - r_e)/r_e$ as

$$V(r) = \frac{l(l+1)}{r^2} = \frac{l(l+1)}{r_e^2}\left(1 - 2x + 3x^2 - 4x^2 + \ldots\right). \tag{46}$$

Also, we can conversely approximate the expression in the form

$$\tilde{V}(r) = \frac{l(l+1)}{r_e^2}\left(D_0 + D_1\frac{\mathrm{e}^{-\alpha x}}{1 - c_h\,\mathrm{e}^{-\alpha x}} + D_2\frac{\mathrm{e}^{-2\alpha x}}{(1 - c_h\,\mathrm{e}^{-\alpha x})^2}\right), \tag{47}$$

where $\alpha = b_h r_e$ and D_i is constant $(i = 0, 1, 2)$ [27, 28]. By Taylor expanding of (47) up to the second-order term x^2 and comparing equal power of x with Eq. (46), we can obtain the expansion coefficient of D_0, D_1 and D_2 as follows:

$$D_0 = 1 - \frac{1}{\alpha}(3 + c_h)(1 - c_h) + \frac{3}{\alpha^2}(1 - c_h)^2, \tag{48}$$

$$D_1 = \frac{2}{\alpha}(1 - c_h)^2(2 + c_h) - \frac{6}{\alpha^2}(1 - c_h)^2, \tag{49}$$

$$D_2 = -\frac{1}{\alpha}(1 - c_h)^3(1 + c_h) + \frac{3}{\alpha^2}(1 - c_h)^4. \tag{50}$$

In the limiting case of $c_h = 0$, we recover the Pekeris approximation [28]:

$$\lim_{c_h \to 0} D_0 = 1 - \frac{3}{\alpha} + \frac{3}{\alpha^2}, \tag{51}$$

$$\lim_{c_h \to 0} D_1 = \frac{4}{\alpha} - \frac{6}{\alpha^2}, \tag{52}$$

$$\lim_{c_h \to 0} D_2 = -\frac{1}{\alpha} + \frac{3}{\alpha^2}. \tag{53}$$

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