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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

# THE ENERGY EIGENVALUES OF THE DIRAC EQUATION WITH THE MODIFIED ECKART AND THE MODIFIED DEFORMED HYLLERAAS POTENTIALS BY SHAPE INVARIANCE APPROACH

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By using the supersymmetry quantum mechanics method, we approximately solve the Dirac equation for the modified Eckart and the modified deformed Hylleraas potentials including the Coulomb-like tensor potential under spin and pseudospin symmetry. We obtain approximate energy eigenvalues and the corresponding wave functions in terms of the Jacobi polynomial under the spin and pseudospin symmetries limit. In order to test the accuracy of our work, we compare our numerical results with those of the Nikiforov–Uvarov (NU) method. This shows that our results are consistent with those found in the literature.

В работе представлено приближенное решение уравнения Дирака с модифицированным потенциалом Эккарта и модифицированным деформированным потенциалом Хиллерааса, включающим в себя кулонподобный тензорный потенциал, в предположении спиновой и псевдоспиновой симметрии. Решение получено с помощью суперсимметричного квантово-механического метода. Приближенные значения собственной энергии и соответствующие им волновые функции в терминах полиномов Якоби вычислены в пределе спиновой и псевдоспиновой симметрий. Чтобы проверить точность полученного решения, проведено сравнение численных результатов с результатами метода Никифорова–Уварова. Сравнение показывает, что результаты работы согласуются с теми, что уже представлены в литературе.

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## INTRODUCTION

It is well known that the exact solutions of the Dirac equation for some physical potentials play an important role in relativistic and nonrelativistic quantum mechanics. In recent years, some authors have investigated the Dirac theory for spin and pseudospin symmetries. The concept of spin symmetry occurs in nuclei when the potential difference

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$\triangle(\mathbf{r}) = V(\mathbf{r}) - S(\mathbf{r}) = \mathrm{const}$. For the pseudospin symmetry counterpart it occurs when the sum potential $\Sigma(\mathbf{r}) = V(\mathbf{r}) + S(\mathbf{r}) = \mathrm{const}$. The Dirac equation under spin and pseudospin symmetries for some physical potentials, such as the harmonic oscillator, Pöschl–Teller potential, Woods–Saxon potential, Morse potential, ring-shaped nonspherical harmonic oscillator, Eckart potential, three-parameter potential function as a diatomic molecule model, and others, has been investigated [1–25]. One of the challenging problems in solving wave equations, such as the Dirac, Klein–Gordon or Schrödinger equations, is finding analytical solutions. Due to many interactions that cannot be exactly solved, one needs to resort to approximate techniques to obtain analytical solutions. The present aim of this paper is to try and attempt to study the Dirac equation for the modified deformed Hylleraas and the modified Eckart potentials plus the Coulomb-like tensor interaction [26–29].

The modified Eckart and the modified deformed Hylleraas potentials, $V_1$ and $V_2$, are given by [30]:

$$\begin{aligned} &V_1 = v_0 + v_1 \mathrm{csch}^2(\alpha r) + v_2 \coth(\alpha r) = v_0 + v_1 \left[\frac{4\,\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}\right] + v_2 \left[\frac{1+\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right], \\ &V_2 = \frac{v_3}{b}\left[\frac{a-\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right], \quad V(\mathbf{r}) = V_1 + V_2, \end{aligned} \tag{1}$$

where $v_i (i = 0, \ldots, 3)$, $a$, $b$ and $\alpha$ are constants. The potential $V(\mathbf{r})$ has been plotted in Fig. 1.

The tensor Coulomb potential is given by [31–33],

$$U = -\frac{T}{r}, \quad r > R_c, \tag{2}$$

where $T = \dfrac{Z_a Z_b e^2}{4\pi\varepsilon_0}$ ($Z_a$ and $Z_b$ denote the charges of the projectile $a$ and the target nuclei $b$) and the Coulomb radius is $R_c = 7.78$ fm. The energy solution of the Dirac equation for this potential is obtained by the NU method [30]. We shall attempt to calculate the approximate energy eigenvalues and energy eigenfunctions of the modified Eckart and the

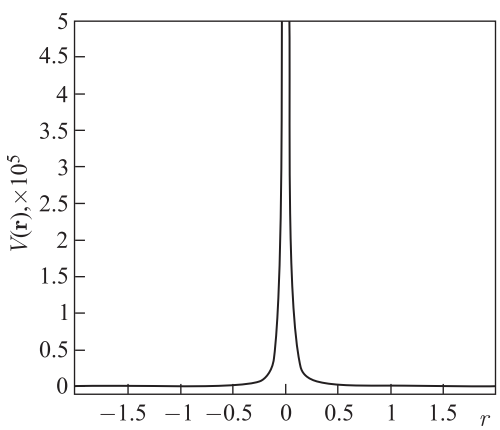


Fig. 1. The plot of potential (1) versus $r$ in the range of $[-2, 2]$ for the given parameters $v_0 = 0.1$, $v_1 = 0.5$, $v_2 = 0.4$, $v_3 = -0.8$, $a = 0.04$, $b = \alpha = 0.01$

modified deformed Hylleraas potentials including a tensor coupling by employing a proper approximation scheme for the spin-orbit coupling for both conditions of spin and pseudospin symmetries using supersymmetry quantum mechanics (SUSYQM) and compare our result of the SUSYQM method with the NU method already reported.

## 1. THE DIRAC EQUATION INCLUDING TENSOR INTERACTION

The Dirac equation for a spin 1/2 single particle in the field of an attractive scalar potential $S(\mathbf{r})$, a repulsive potential $V(\mathbf{r})$ and a tensor potential $U(\mathbf{r})$ is given by ($\hbar = c = 1$),

$$[\boldsymbol{\alpha}\cdot\mathbf{p} + \beta(M + S(\mathbf{r})) - i\beta\alpha\cdot\hat{r}U(\mathbf{r})]\Psi_{n\kappa}(\mathbf{r}) = [E_{n\kappa} - V(\mathbf{r})]\Psi_{n\kappa}(\mathbf{r}), \tag{3}$$

where $E_{n\kappa}$ is the relativistic energy of the system; $p = -i\nabla$ is the three-dimensional momentum operator and $M$ is the mass of the fermionic particle. $\alpha$ and $\beta$ are the $4\times 4$ Dirac matrices given as [34],

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4}$$

where $I$ is the $2\times 2$ unitary matrix and $\boldsymbol{\sigma}$ are the three-vector spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

For spherical nuclei, the total angular momentum $\mathbf{J}$ and spin matrix operator $\hat{K} = -\beta(\boldsymbol{\sigma}\times \mathbf{L}+1)$ commute with the Dirac Hamiltonian. The eigenvalues of spin-orbit coupling operator $\hat{K}$ are $\kappa = (j+1/2) > 0$ and $\kappa = -(j+1/2) < 0$ for the unaligned spin $j = l - 1/2$ and the aligned spin $j = l + 1/2$, respectively. In the relativistic quantum mechanics, the set $(H^2, K, J^2, J_z)$ can be taken as the complete set of the conservative quantities. The spinor wave functions can be classified according to their angular momentum $j$, spin-orbit quantum number $\kappa$, and the radial quantum number $n$, which can be written as follows:

$$\Psi_{n\kappa}(\mathbf{r}) = \begin{pmatrix} F_{n\kappa}(\mathbf{r})\, Y^{l}_{jm}(\theta,\varphi) \\ iG_{n\kappa}(\mathbf{r})\, Y^{\tilde{l}}_{jm}(\theta,\varphi) \end{pmatrix}, \tag{6}$$

where $F_{n\kappa}(\mathbf{r})$ and $G_{n\kappa}(\mathbf{r})$ are the radial wave functions of the upper and lower spinor components of the Dirac spinors, respectively. $Y^{\tilde{l}}_{jm}(\theta,\varphi)$ and $Y^{l}_{jm}(\theta,\varphi)$ are the spherical harmonic functions, and $m$ is the projection of the total angular momentum on the $z$ axis. The pseudo-orbital and the orbital angular momentum quantum numbers for pseudospin symmetry $\tilde{l}$ and spin symmetry $l$ are the labels of the upper and lower components, respectively. Substituting Eq. (6) into Eq. (3) and using the relations as $(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{i}\boldsymbol{\sigma}\cdot(\mathbf{A}\cdot\mathbf{B})$ and $(\boldsymbol{\sigma}\cdot\mathbf{P}) = \boldsymbol{\sigma}\cdot\hat{r}\left(\hat{r}\mathbf{P} + i\dfrac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r}\right)$ with the following properties [35]:

$$\boldsymbol{\sigma}\cdot\mathbf{L}\begin{cases} Y^{\tilde{l}}_{jm}(\theta,\varphi) \\ Y^{l}_{jm}(\theta,\varphi) \end{cases} = \begin{cases} (\kappa-1)Y^{\tilde{l}}_{jm}(\theta,\varphi), \\ -(\kappa-1)Y^{l}_{jm}(\theta,\varphi), \end{cases} \tag{7}$$

$$\boldsymbol{\sigma}\cdot\hat{r}\begin{cases} Y^{\tilde{l}}_{jm}(\theta,\varphi) \\ Y^{l}_{jm}(\theta,\varphi) \end{cases} = \begin{cases} -Y^{l}_{jm}(\theta,\varphi), \\ -Y^{\tilde{l}}_{jm}(\theta,\varphi), \end{cases} \tag{8}$$

one obtains two-coupled differential equation for the upper and lower radial wave functions $F_{n\kappa}(\mathbf{r})$ and $G_{n\kappa}(\mathbf{r})$ as follows:

$$\left[\frac{d}{dr}+\frac{\kappa}{r}-U(\mathbf{r})\right]F_{n\kappa}(\mathbf{r})=[M+E_{nk}-\triangle(\mathbf{r})]G_{n\kappa}(\mathbf{r}), \tag{9a}$$

$$\left[\frac{d}{dr}-\frac{\kappa}{r}+U(\mathbf{r})\right]G_{n\kappa}(\mathbf{r})=[M-E_{nk}+\Sigma(\mathbf{r})]F_{n\kappa}(\mathbf{r}), \tag{9b}$$

where $\triangle(\mathbf{r})=V(\mathbf{r})-S(\mathbf{r})$ and $\Sigma(\mathbf{r})=V(\mathbf{r})+S(\mathbf{r})$. Eliminating $F_{n\kappa}(\mathbf{r})$ and $G_{n\kappa}(\mathbf{r})$ from Eqs. (9a) and (9b), we obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components, respectively:

$$\begin{gathered}\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(\mathbf{r})-\frac{dU(\mathbf{r})}{dr}-U^2(\mathbf{r})+\right.\\ \left.+\frac{d\triangle(\mathbf{r})/dr}{E_{n\kappa}+M-\triangle(\mathbf{r})}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(\mathbf{r})\right)\right]F_{n\kappa}(\mathbf{r})=\\ =\left[M-E_{n\kappa}+\Sigma(\mathbf{r})\right]\left[M+E_{n\kappa}-\triangle(\mathbf{r})\right]F_{n\kappa}(\mathbf{r}),\end{gathered} \tag{10a}$$

$$\begin{gathered}\left[\frac{d^2}{dr^2}-\frac{\kappa\,(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(\mathbf{r})+\frac{dU(\mathbf{r})}{dr}-U^2(\mathbf{r})-\right.\\ \left.-\frac{d\Sigma(\mathbf{r})/dr}{M-E_{n\kappa}+\Sigma(\mathbf{r})}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(\mathbf{r})\right)\right]G_{n\kappa}(\mathbf{r})=\\ =\left[M+E_{n\kappa}-\triangle(\mathbf{r})\right]\left[M-E_{n\kappa}+\Sigma(\mathbf{r})\right]G_{n\kappa}(\mathbf{r}),\end{gathered} \tag{10b}$$

where $\kappa(\kappa+1)=l(l+1)$ and $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$. The quantum number $\kappa$ relates to the quantum numbers for spin symmetry $l$ and pseudospin symmetry $\tilde{l}$ as follows:

$$\kappa=\begin{cases}-(l+1)=-\left(j+\dfrac{1}{2}\right)\ (s_{1/2},p_{3/2},\ \text{etc.}), & j=l+\dfrac{1}{2},\quad \text{aligned spin } (\kappa<0),\\ +l=\left(j+\dfrac{1}{2}\right)\ (p_{1/2},d_{3/2},\ \text{etc.}), & j=l-\dfrac{1}{2},\quad \text{unaligned spin } (\kappa>0),\end{cases}$$

and the quasi-degenerate doublet structure can be expressed in terms of a pseudospin angular momentum $\tilde{s}=1/2$ and pseudo-orbital angular momentum $\tilde{l}$, which is defined as

$$\kappa=\begin{cases}-\tilde{l}=-\left(j+\dfrac{1}{2}\right)\ (s_{1/2},p_{3/2},\ \text{etc.}), & j=\tilde{l}-\dfrac{1}{2},\quad \text{aligned pseudospin } (\kappa<0),\\ \tilde{l}+1=\left(j+\dfrac{1}{2}\right)\ (d_{3/2},f_{5/2},\ \text{etc.}), & j=\tilde{l}+\dfrac{1}{2},\quad \text{unaligned pseudospin } (\kappa>0),\end{cases}$$

where $\kappa=\pm1,\pm2,\ldots$ For example, $(1p_{3/2},0f_{5/2})$ and $(1d_{5/2},0g_{7/2})$ can be considered as pseudospin doublets.

**1.1. Spin Symmetry Limit.** In the case of exact spin symmetry limit, $d\Delta/dr=0$ (or $\Delta(\mathbf{r})=C_s=\text{const}$) [36, 37]. Taking $\Sigma(\mathbf{r})$ as the modified Eckart and the modified deformed Hylleraas potentials and using the definition of the Coulomb-like tensor interaction, we can reduce Eq. (10a) into the following form:

$$\left[-\frac{d^2}{dr^2}+\frac{\eta_\kappa(\eta_\kappa+1)}{r^2}+\beta\Sigma(\mathbf{r})\right]F_{n\kappa}(\mathbf{r})=[\gamma^2]\,F_{n\kappa}(\mathbf{r}), \tag{11}$$

where $\kappa = l$ and $\kappa = -l-1$ for $\kappa > 0$ and $\kappa < 0$, respectively. Also, $\eta_\kappa = \kappa + T$, $\beta = (E_{n\kappa} + M - C_s)$ and $\gamma^2 = \beta(E_{n\kappa} - M)$.

**1.2. Pseudospin Symmetry Limit.** For the exact pseudospin symmetry limit, i.e., $d\Sigma/dr = 0$ (or $\Sigma = C_p = \mathrm{const}$), also with $\Delta(\mathbf{r})$ as the potential given in Eq. (1) and the Coulomb-like tensor interaction, Eq. (10b) becomes

$$\Bigg[-\frac{d^2}{dr^2}+\frac{\eta_\kappa(\eta_\kappa-1)}{r^2}+\beta'\Bigg(v_0+v_1\left[\frac{4\,\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}\right]+ \\ +v_2\left[\frac{1+\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right]+\frac{v_3}{b}\left[\frac{a-\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right]\Bigg)\Bigg]G_{n\kappa}(\mathbf{r})=[\gamma'^2]\,G_{n\kappa}(\mathbf{r}), \tag{12}$$

where $\kappa = -\tilde{l}$ and $\kappa = \tilde{l}+1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Also, $\beta' = (E_{n\kappa} - M - C_p)$ and $\gamma'^2 = \beta'(E_{n\kappa} + M)$.

Equations (11) and (12) cannot be solved analytically due to the pseudocentrifugal term $\dfrac{\eta_\kappa(\eta_\kappa \pm 1)}{r^2}$. Thus, we use the Pekeris approximation scheme for the centrifugal term

$$\frac{1}{r^2}\approx 4\frac{\alpha^2\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}, \tag{13}$$

which is plotted in Fig. 2. This shows that it is such a good approximation for small values of the parameter $\alpha$.

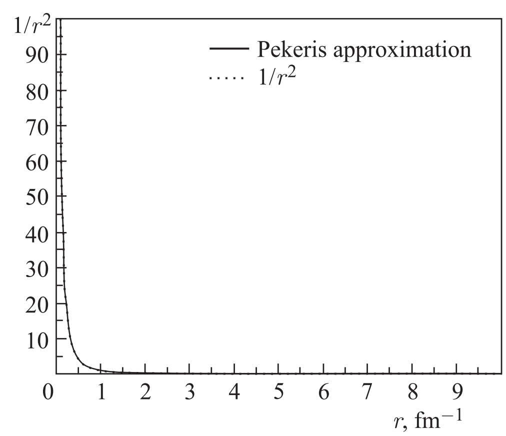


Fig. 2. The shape of expression $1/r^2$ and the Pekeris approximation for $\alpha = 0.01$

**1.3. Solution of the Dirac Equation with Spin Symmetry Limit.** If we define $\tilde{E}_{n\kappa}$, $\tilde{V}_1$ and $\tilde{V}_2$ as

$$\tilde{E}_{n\kappa} = \beta\left[E_{n\kappa} - M - \left(v_0 - v_2 + \frac{v_3}{b}\right)\right], \tag{14a}$$

$$\tilde{V}_1 = -4\alpha^2\eta_\kappa(\eta_\kappa + 1) + \beta\left[2v_2 + \frac{v_3}{b}(a-1) - 4v_1\right], \tag{14b}$$

$$\tilde{V}_2 = 4\alpha^2\eta_\kappa(\eta_\kappa + 1) + 4\beta v_1, \tag{14c}$$

and using the approximation given in Eq. (13) and substituting $V(\mathbf{r})$ into Eq. (11), the following Schrödinger-like equation for the upper wave function is obtained:

$$\left[-\frac{d^2}{dr^2} + \tilde{V}_1\frac{1}{1-\mathrm{e}^{-2\alpha r}} + \tilde{V}_2\frac{1}{(1-\mathrm{e}^{-2\alpha r})^2}\right]F_{n\kappa}(\mathbf{r}) = \left[\tilde{E}_{n\kappa}\right]F_{n\kappa}(\mathbf{r}). \tag{15}$$

We now use the SUSYQM method and shape invariance approach [38, 39] to solve Eq. (15). The ground state function $F_{0\kappa}(\mathbf{r})$ can be written in the form of

$$F_{0\kappa}(\mathbf{r}) = \exp\left(-\int W(\mathbf{r})dr\right), \tag{16}$$

where $W(\mathbf{r})$ is called superpotential in SUSYQM. Putting Eq. (16) into Eq. (15) yields an equation for $W(\mathbf{r})$ of

$$W^2(\mathbf{r}) - \frac{dW(\mathbf{r})}{dr} = \tilde{V}_1\frac{1}{1-\mathrm{e}^{-2\alpha r}} + \tilde{V}_2\frac{1}{(1-\mathrm{e}^{-2\alpha r})^2} - \tilde{E}_{0\kappa}, \tag{17}$$

where $\tilde{E}_{0\kappa}$ refers to the ground-state energy. As we can see, Eq. (17) is the nonlinear Riccati equation. Inserting the superpotential $W(\mathbf{r})$ of the form

$$W(\mathbf{r}) = -A + B\frac{1}{(1-\mathrm{e}^{-2\alpha r})}, \tag{18}$$

where $A$ and $B$ are constants, and substituting this expression into Eq. (17), we obtain the following relations:

$$-2AB - 2\alpha B = \tilde{V}_1, \tag{19a}$$

$$B^2 + 2\alpha B = \tilde{V}_2, \tag{19b}$$

$$A^2 = -\tilde{E}_{0\kappa}. \tag{19c}$$

Substituting Eq. (18) into Eq. (16) leads one to obtain the ground-state upper component

$$F_{0\kappa}(\mathbf{r}) = N_0\exp(Ar)\left(\frac{-\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right)^{B/2\alpha}. \tag{20}$$

In the present work, we will deal with bound state solutions, i.e., the radial part of the wave function $\Psi_{n\kappa}(\mathbf{r})$ must satisfy the boundary condition that $F_{n\kappa}(\mathbf{r})/r$ becomes zero when

$r \to \infty$ and $F_{n\kappa}(\mathbf{r})/r$ is finite if $r = 0$. By considering the function $F_{0\kappa}(\mathbf{r})$ satisfying the boundary condition and solving Eqs. (19a)–(19c), one can obtain

$$A = \frac{-(\tilde{V}_1 + \tilde{V}_2)}{2B} + \frac{B}{2}, \tag{21a}$$

$$B = -\alpha \left[ 1 + \sqrt{1 + \frac{\tilde{V}_2}{\alpha^2}} \right]. \tag{21b}$$

In terms of superpotential $W(\mathbf{r})$, the two supersymmetric partner potentials $V_{\pm}(\mathbf{r})$ are obtained as follows:

$$V_{+}(\mathbf{r}) = W^2 + \frac{dW}{dr} = \left[ \frac{\tilde{V}_1 + \tilde{V}_2}{2B} - \frac{B}{2} \right]^2 + \frac{B^2 - 2\alpha B}{(1 - \mathrm{e}^{-2\alpha r})^2} + \frac{\tilde{V}_1 + \tilde{V}_2 - B^2 + 2\alpha B}{1 - \mathrm{e}^{-2\alpha r}}, \tag{22}$$

$$V_{-}(\mathbf{r}) = W^2 - \frac{dW}{dr} = \left[ \frac{\tilde{V}_1 + \tilde{V}_2}{2B} - \frac{B}{2} \right]^2 + \frac{B^2 + 2\alpha B}{(1 - \mathrm{e}^{-2\alpha r})^2} + \frac{\tilde{V}_1 + \tilde{V}_2 - B^2 - 2\alpha B}{1 - \mathrm{e}^{-2\alpha r}}. \tag{23}$$

Considering $a_0 = B$ and $a_1 = B - 2\alpha$, we can easily show that the two partner potentials $V_{+}(\mathbf{r}, a_0)$ and $V_{-}(\mathbf{r}, a_1)$ satisfy the following relationship:

$$V_{+}(\mathbf{r}, a_0) = V_{-}(\mathbf{r}, a_1) + R(a_1), \tag{24}$$

and it shows that $V_{+}(\mathbf{r}, a_0)$ and $V_{-}(\mathbf{r}, a_1)$ can be determined by using the shape invariance approach

$$R(a_1) = \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2B} - \frac{B}{2} \right)^2 - \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2(B - 2\alpha)} - \frac{B - 2\alpha}{2} \right)^2. \tag{25}$$

The remainder $R(a_1)$ is independent of $r$. The ground-state energy of $V_{-}(\mathbf{r})$ is zero $\tilde{E}_{0\kappa}^{(-)} = 0$. The energy eigenvalues of the shape invariance potential $V_{-}(\mathbf{r})$ are given by

$$\begin{aligned} \tilde{E}_{n\kappa}^{(-)} = \sum_{\kappa=1}^{n} R(a_\kappa) = R(a_1) + R(a_2) + \ldots + R\left(a_n\right) = \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2B} - \frac{B}{2} \right)^2 - \\ - \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2\left(B - 2\alpha\right)} - \frac{B - 2\alpha}{2} \right)^2 + \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2\left(B - 2\alpha\right)} - \frac{B - 2\alpha}{2} \right)^2 - \\ - \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2\left(B - 4\alpha\right)} - \frac{B - 4\alpha}{2} \right)^2 + \ldots + \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2\left(B - 2\left(n - 1\right)\alpha\right)} - \frac{B - 2\left(n - 1\right)\alpha}{2} \right)^2 - \\ - \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2(B - 2n\alpha)} - \frac{B - 2n\alpha}{2} \right)^2 = \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2B} - \frac{B}{2} \right)^2 - \left( \frac{\tilde{V}_1 + \tilde{V}_2}{2(B - 2n\alpha)} - \frac{B - 2n\alpha}{2} \right)^2, \end{aligned} \tag{26}$$

where quantum number $n = 1, 2, 3, \ldots$

Combining Eqs. (19c), (24) and (26) yields the solution for $\tilde{E}_{n\kappa}$ in Eq. (15) as

$$\tilde{E}_{n\kappa} = \tilde{E}_{n\kappa}^{(-)} + \tilde{E}_{0\kappa} = -\left(\frac{\tilde{V}_1 + \tilde{V}_2}{2(B - 2n\alpha)} - \frac{B - 2n\alpha}{2}\right)^2, \tag{27}$$

where we have employed $\tilde{E}_{0\kappa} = -\left(\frac{\tilde{V}_1 + \tilde{V}_2}{2B} - \frac{B}{2}\right)^2$. The energy equation for the modified Eckart and the modified deformed Hylleraas potentials $V(\mathbf{r})$ is obtained under the condition of spin symmetry, so by incorporating Eq. (14a) into (27) and using Eq. (21b), we get

$$E_{n\kappa}^2 - M^2 - C_s\left(E_{n\kappa} - M - v_0 + v_2 - \frac{v_3}{b}\right) + (E_{n\kappa} + M)\left(-v_0 + v_2 - \frac{v_3}{b}\right) =$$
$$= -\frac{1}{4}\left[\frac{(E_{n\kappa} + M - C_s)(2v_2 + (v_3/b)(a-1))}{\alpha + 2n\alpha + \alpha\sqrt{1 + 4\eta_\kappa(\eta_\kappa + 1) + \frac{4(E_{n\kappa} + M - C_s)v_1}{\alpha^2}}} - \right.$$
$$\left. - \left(\alpha + 2n\alpha + \alpha\sqrt{1 + 4\eta_\kappa(\eta_\kappa + 1) + \frac{4(E_{n\kappa} + M - C_s)v_1}{\alpha^2}}\right)\right]^2, \tag{28}$$

where the quantum number $n = 0, 1, 2, \ldots$

Finally, to obtain the energy eigenfunctions, we define a new variable of the form $x = \mathrm{e}^{-2\alpha r}$, then the upper spinor radial wave functions $F_{n\kappa}(\mathbf{x})$ may be written as [29]:

$$F_{n\kappa}(\mathbf{x}) = D_n x^{\sqrt{-\frac{\beta\delta}{4\alpha^2}}}(1-x)(1-x)^{\frac{1}{2}+\sqrt{\frac{1}{4}\left[1+4\eta_\kappa(\eta_\kappa+1)-\frac{\beta(\lambda+\delta-\mu)}{\alpha^2}\right]}} \times$$
$$\times P_n^{\left(2\sqrt{-\frac{\beta\delta}{4\alpha^2}},\, 2\sqrt{\frac{1}{4}\left[1+4\eta_\kappa(\eta_\kappa+1)-\frac{\beta(\lambda+\delta-\mu)}{\alpha^2}\right]}\right)}, \tag{29}$$

where $D_n$ is the normalized constant and $\delta = \left(E_{n\kappa} - M - v_0 - v_2 - \frac{av_3}{b}\right)$, $\mu = (2E_{n\kappa} - 2M - 2v_0 + 4v_1 - \frac{v_3}{b}(1+a))$ and $\lambda = \left(E_{n\kappa} - M - v_0 + v_2 - \frac{v_3}{b}\right)$.

The lower component of the Dirac equation can be calculated by applying Eq. (9a) as

$$G_{n\kappa}(\mathbf{r}) = \frac{1}{M + E_{nk} - C_s}\left[\frac{d}{dr} + \frac{\kappa}{r} - U(\mathbf{r})\right]F_{n\kappa}(\mathbf{r}), \tag{30}$$

where $M + E_{nk} \neq C_s - M$.

**1.4. Solution of the Dirac Equation with Pseudospin Symmetry Limit.** Following the same procedures as explained in Subsec. 1.3, we obtain the energy eigenvalues equation corresponding to the pseudospin symmetry condition:

$$E_{n\kappa}^2 - M^2 - C_p\left(E_{n\kappa} + M - v_0 + v_2 - \frac{v_3}{b}\right) + (E_{n\kappa} - M)\left(-v_0 + v_2 - \frac{v_3}{b}\right) =$$

$$= -\frac{1}{4}\left[\frac{(M - E_{n\kappa} + C_p)(-2v_2 - (v_3/b)(a-1))}{\alpha + 2n\alpha + \alpha\sqrt{1 + 4\eta_\kappa(\eta_\kappa - 1) - \frac{4(M - E_{n\kappa} + C_p)v_1}{\alpha^2}}} - \left(\alpha + 2n\alpha + \alpha\sqrt{1 + 4\eta_\kappa(\eta_\kappa - 1) - \frac{4(M - E_{n\kappa} + C_p)v_1}{\alpha^2}}\right)\right]^2. \quad (31)$$

Further, we can easily obtain the lower spinor radial wave functions $F_{n\kappa}(\mathbf{r})$ directly via the spin symmetry solution through the transformations

$$F_{n\kappa} \leftrightarrow G_{n\kappa}, \quad E_{n\kappa} \leftrightarrow -E_{n\kappa}, \quad C_{ps} \leftrightarrow -C_s, \quad V(\mathbf{r}) \leftrightarrow -V(\mathbf{r}), \quad \kappa \leftrightarrow \kappa + 1, \quad (32)$$

and the corresponding wave functions in spin symmetry limit are obtained as

$$G_{n\kappa}(\mathbf{x}) = D_n' x^{\sqrt{-\frac{\beta'\delta'}{4\alpha^2}}}(1-x)(1-x)^{\frac{1}{2}+\sqrt{\frac{1}{4}\left[1+4\eta_\kappa(\eta_\kappa-1)-\frac{\beta'(\lambda'+\delta'-\mu')}{\alpha^2}\right]}} \times \times P_n^{\left(2\sqrt{-\frac{\beta'\delta'}{4\alpha^2}},2\sqrt{\frac{1}{4}\left[1+4\eta_\kappa(\eta_\kappa-1)-\frac{\beta'(\lambda'+\delta'-\mu')}{\alpha^2}\right]}\right)}, \quad (33)$$

where $D_n'$ is the normalized constant and $\delta' = \left(E_{n\kappa} + M - v_0 - v_2 - \frac{av_3}{b}\right)$, $\mu' = (2E_{n\kappa} + 2M - 2v_0 + 4v_1 - \frac{v_3}{b}(1+a))$ and $\lambda' = (E_{n\kappa} + M - v_0 + v_2 - \frac{v_3}{b})$.

The upper component of the Dirac equation can be calculated by applying Eq. (9b) as

$$F_{n\kappa}(\mathbf{r}) = \frac{1}{[M - E_{nk} + C_p]}\left[\frac{d}{dr} - \frac{\kappa}{r} + U(\mathbf{r})\right]G_{n\kappa}(\mathbf{r}), \quad (34)$$

where $E_{nk} \neq M + C_p$.

## 2. DISCUSSIONS AND NUMERICAL RESULTS

The relativistic energy eigenvalues under the spin and pseudospin symmetry conditions are plotted in Figs. 1–8 for several states in the presence of tensor interaction $T = 1$. In Figs. 3, 4 and 5, the energy eigenvalues are illustrated in terms of parameters $\alpha$, $a$ and $b$ for both spin and pseudospin symmetry conditions. It can be seen that by increasing in amounts of $\alpha$, $a$ and $b$ in spin symmetry limit, the energy eigenvalues are increased and they are decreased by increasing in amounts of $\alpha$, $a$ and $b$ in pseudospin symmetry limit. In Figs. 6 and 7, the energy eigenvalues are plotted for parameters $v_1$ and $v_2$. It is clear that with the increase in $v_1$ and $v_2$, the energy eigenvalues are increased for either spin or pseudospin symmetry conditions, respectively. In addition, in Fig. 8, the energy eigenvalues are decreased by increasing in $v_3$ for both spin and pseudospin symmetry limits.

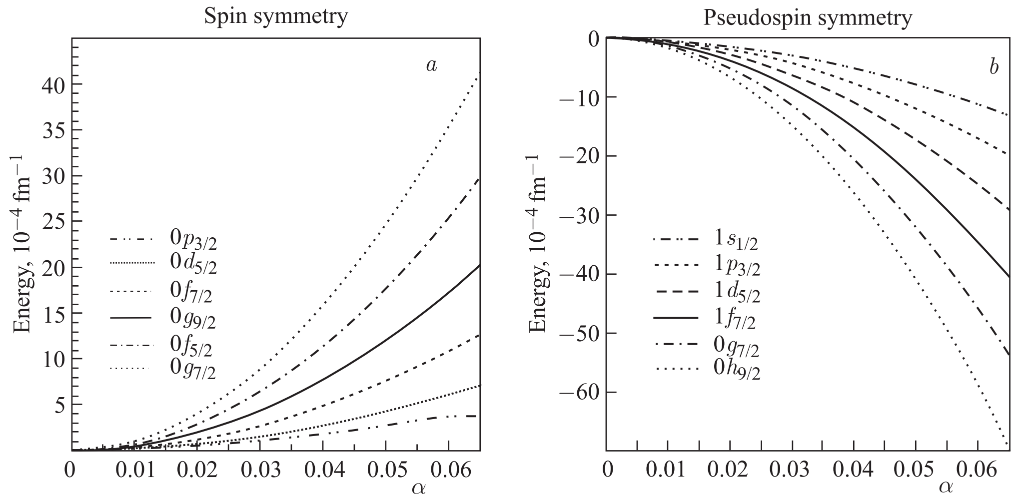


Fig. 3. Energy eigenvalue in terms of different values of $\alpha$ for spin (*a*) and pseudospin (*b*) symmetry limits with: *a*) $a = 0.04$, $b = 0.01$, $M = 5$ fm$^{-1}$, $C_s = 5$, $v_0 = -0.1$, $v_1 = -0.5$, $v_2 = -0.4$, $v_3 = 0.8$; *b*) $a = 0.04$, $b = 0.01$, $M = 5$ fm$^{-1}$, $C_p = -5$, $v_0 = 0.1$, $v_1 = 0.5$, $v_2 = 0.4$, $v_3 = -0.8$

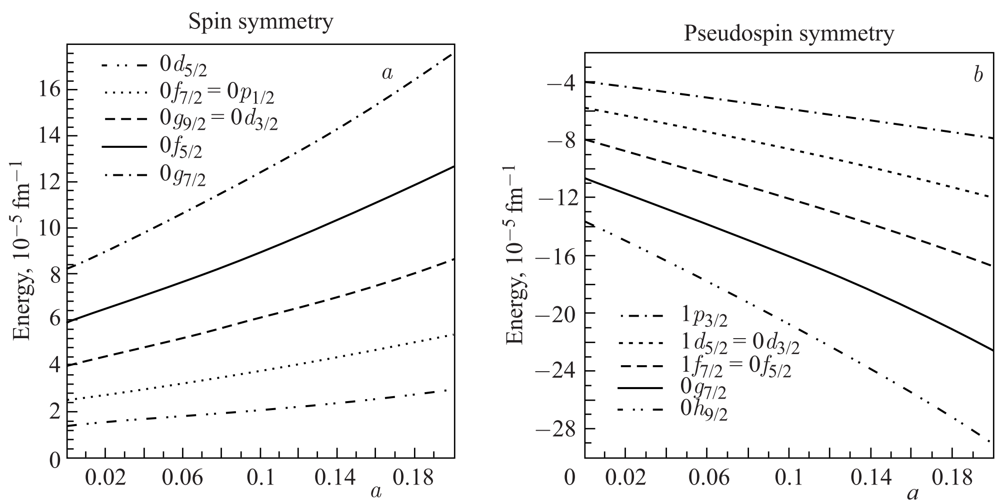


Fig. 4. Energy eigenvalue in terms of different values of $a$ for spin (*a*) and pseudospin (*b*) symmetry limits with: *a*) $\alpha = 0.01$, $b = 0.01$, $M = 5$ fm$^{-1}$, $C_s = 5$, $v_0 = -0.1$, $v_1 = -0.5$, $v_2 = -0.4$, $v_3 = 0.8$; *b*) $\alpha = 0.01$, $b = 0.01$, $M = 5$ fm$^{-1}$, $C_p = -5$, $v_0 = 0.1$, $v_1 = 0.5$, $v_2 = 0.4$, $v_3 = -0.8$

Moreover, in Tables 1 and 2, the numerical results are calculated for different states of $n$ and $l$. We consider the same set of spin doublets as $(ns_{1/2}, (n-1)d_{3/2})$, $(np_{3/2}, (n-1)f_{5/2})$, $(nd_{5/2}, (n-1)g_{7/2}), \ldots$ (where each pair is considered as a spin doublet) and the same set of pseudospin doublets as $(np_{1/2}np_{3/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, $(ng_{7/2}, ng_{9/2}), \ldots$ (where each pair is considered as a pseudospin doublet). Also, from Tables 1 and 2 it is

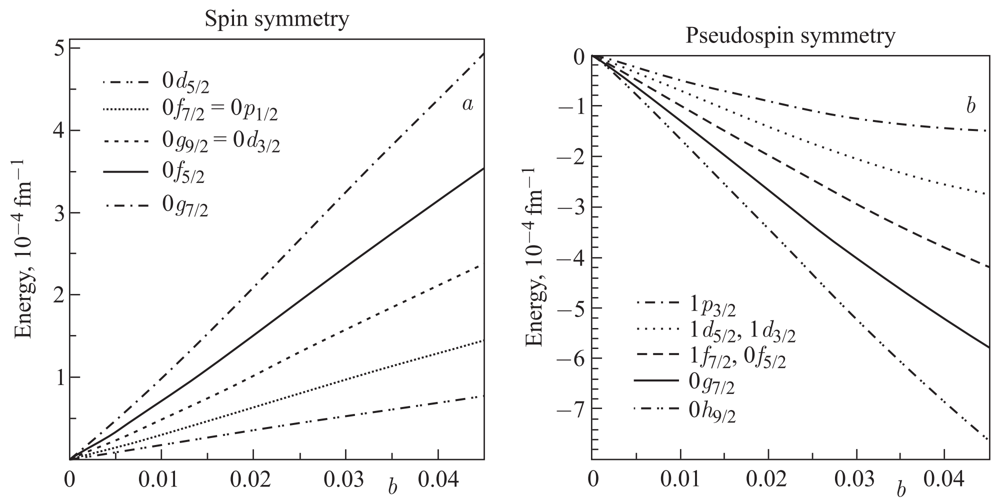


Fig. 5. Energy eigenvalue in terms of different values of $b$ for spin ($a$) and pseudospin ($b$) symmetry limits with: $a$) $a = 0.04$, $\alpha = 0.01$, $M = 5$ fm$^{-1}$, $C_s = 5$, $v_0 = -0.1$, $v_1 = -0.5$, $v_2 = -0.4$, $v_3 = 0.8$; $b$) $a = 0.04$, $\alpha = 0.01$, $M = 5$ fm$^{-1}$, $C_p = -5$, $v_0 = 0.1$, $v_1 = 0.5$, $v_2 = 0.4$, $v_3 = -0.8$

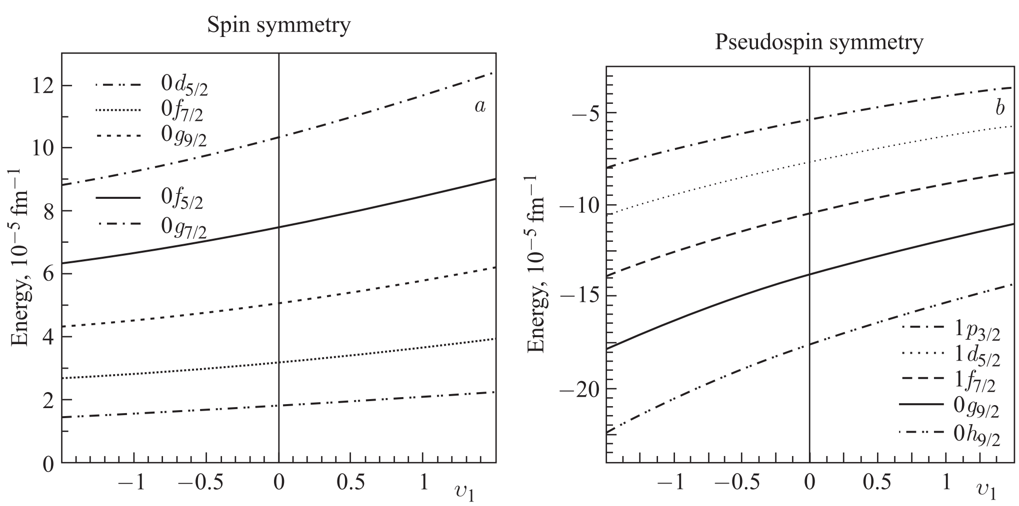


Fig. 6. Energy eigenvalue in terms of different values of $v_1$ for spin ($a$) and pseudospin ($b$) symmetry limits with: $a$) $a = 0.04$, $b = \alpha = 0.01$, $M = 5$ fm$^{-1}$, $C_s = 5$, $v_0 = -0.1$, $v_2 = -0.4$, $v_3 = 0.8$; $b$) $a = 0.04$, $b = \alpha = 0.01$, $M = 5$ fm$^{-1}$, $C_p = -5$, $v_0 = 0.1$, $v_2 = 0.4$, $v_3 = -0.8$

obvious that when the tensor interaction is denied $T = 0$, the states are degenerate, while when we consider the effect of tensor interaction with the amount $T = 1$, those degeneracies are vanished. For instance, in spin doublet $(0p_{3/2}, 0p_{1/2})$ in the absence of tensor interaction $T = 0$, the energy eigenvalues for states with $\kappa = -2$ and $\kappa = 1$ were equal to $E_{0,-2} = E_{0,1} = 0.000036016$, while in the presence of tensor interaction $T = 1$, the

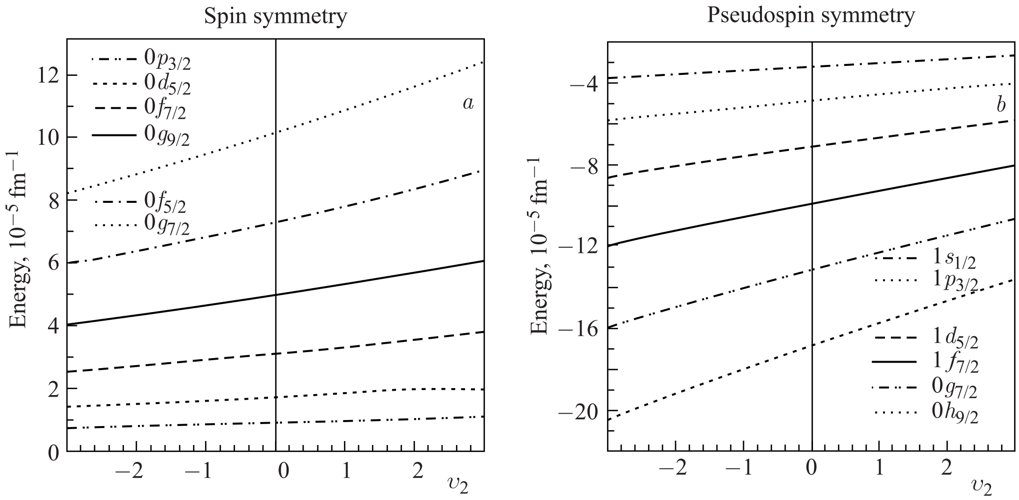


Fig. 7. Energy eigenvalue in terms of different values of $v_2$ for spin (*a*) and pseudospin (*b*) symmetry limits with: *a*) $a = 0.04$, $b = \alpha = 0.01$, $M = 5$ fm$^{-1}$, $C_s = 5$, $v_0 = -0.1$, $v_1 = -0.5$, $v_3 = 0.8$; *b*) $a = 0.04$, $b = \alpha = 0.01$, $M = 5$ fm$^{-1}$, $C_p = -5$, $v_0 = 0.1$, $v_1 = 0.5$, $v_3 = -0.8$

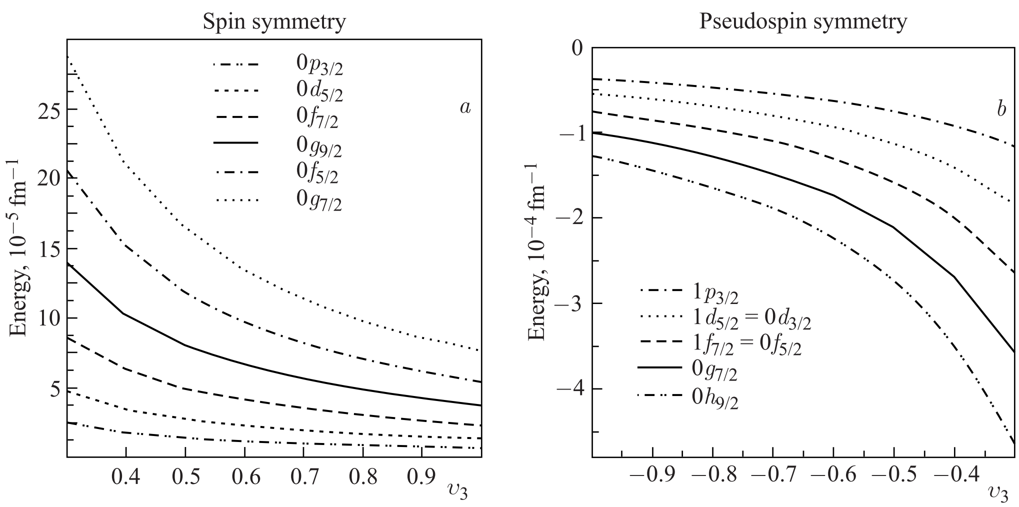


Fig. 8. Energy eigenvalue in terms of different values of $v_3$ for spin (*a*) and pseudospin (*b*) symmetry limits with: *a*) $a = 0.04$, $b = \alpha = 0.01$, $M = 5$ fm$^{-1}$, $C_s = 5$, $v_0 = -0.1$, $v_1 = -0.5$, $v_2 = -0.4$; *b*) $a = 0.04$, $b = \alpha = 0.01$, $M = 5$ fm$^{-1}$, $C_p = -5$, $v_0 = 0.1$, $v_1 = 0.5$, $v_2 = 0.4$

energy eigenvalues turn to $E_{0,-2} = 0.000008726$ and $E_{0,1} = 0.000081539$. As a further matter, for both symmetry conditions, the energy eigenvalues of every state in the absence of tensor interaction $T = 0$, are equal to the next state when the tensor interaction is present. For example, the energy state of $0p_{3/2}$ when $T = 0$ is equal to $0d_{5/2}$

*Table* 1. **Bound state energy eigenvalues of the Dirac equation in the spin symmetry limit with the modified Eckart plus the modified deformed Hylleraas potentials without and with a tensor interaction for** $v_0 = -0.1$, $v_1 = -0.5$, $v_2 = -0.4$, $v_3 = 0.8$, $a = 0.04$, $b = 0.01$, $\alpha = 0.01$, $C_s = 5$, $M = 5$ **fm**$^{-1}$

| $l$ | $n$, $\kappa < 0$ | $(l, j = l + 1/2)$ | $E_{n\kappa<0}$ | | | $n$, $\kappa > 0$ | $(l, j = l - 1/2)$ | $E_{n\kappa>0}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $T = 0$ | $T = 1$ | $T = 0$ [30] | | | $T = 0$ | $T = 1$ | $T = 0$ [30] |
| 1 | 0, –2 | $0p_{3/2}$ | 0.000036016 | 0.000008726 | 0.000016732 | 0, 1 | $0p_{1/2}$ | 0.000036016 | 0.000081539 | –0.000016732 |
| 2 | 0, –3 | $0d_{5/2}$ | 0.000081539 | 0.000036016 | 0.000030075 | 0, 2 | $0d_{3/2}$ | 0.000081539 | 0.000145343 | –0.000030075 |
| 3 | 0, –4 | $0f_{7/2}$ | 0.000145343 | 0.000081539 | 0.000048222 | 0, 3 | $0f_{5/2}$ | 0.000145343 | 0.000227431 | –0.000048222 |
| 4 | 0, –5 | $0g_{9/2}$ | 0.000227431 | 0.000145343 | 0.000071047 | 0, 4 | $0g_{7/2}$ | 0.000227431 | 0.000327807 | –0.000071047 |
| 1 | 1, –2 | $1p_{3/2}$ | 0.000078535 | 0.000031258 | 0.000047428 | 1, 1 | $1p_{1/2}$ | 0.000078535 | 0.000142285 | –0.000047428 |
| 2 | 1, –3 | $1d_{5/2}$ | 0.000142285 | 0.000078535 | 0.000069573 | 1, 2 | $1d_{3/2}$ | 0.000142285 | 0.000227431 | –0.000069573 |
| 3 | 1, –4 | $1f_{7/2}$ | 0.000224107 | 0.000142285 | 0.000096636 | 1, 3 | $1f_{5/2}$ | 0.000224107 | 0.000324145 | –0.000096636 |
| 4 | 1, –5 | $1g_{9/2}$ | 0.000324145 | 0.000224107 | 0.000128414 | 1, 4 | $1g_{7/2}$ | 0.000324145 | 0.000442438 | –0.000128414 |
| 1 | 2, –2 | $2p_{3/2}$ | 0.000135096 | 0.000022818 | 0.000089763 | 2, 1 | $2p_{1/2}$ | 0.000135096 | 0.000218152 | –0.000089763 |
| 2 | 2, –3 | $2d_{5/2}$ | 0.000218152 | 0.000135096 | 0.000122790 | 2, 2 | $2d_{3/2}$ | 0.000218152 | 0.000318427 | –0.000122790 |
| 3 | 2, –4 | $2f_{7/2}$ | 0.000318427 | 0.000218152 | 0.000159637 | 2, 3 | $2f_{5/2}$ | 0.000318427 | 0.000436653 | –0.000159637 |
| 4 | 2, –5 | $2g_{9/2}$ | 0.000436653 | 0.000318427 | 0.000200847 | 2, 4 | $2g_{7/2}$ | 0.000436653 | 0.000573014 | –0.000200847 |

*Table* 2. **Bound state energy eigenvalues of the Dirac equation in the pseudospin symmetry limit with the modified Eckart plus the modified deformed Hylleraas potentials without and with a tensor interaction for** $v_0 = 0.1$, $v_1 = 0.5$, $v_2 = 0.4$, $v_3 = -0.8$, $a = 0.04$, $b = 0.01$, $\alpha = 0.01$, $C_p = -5$, $M = 5$ **fm**$^{-1}$

| $\tilde{l}$ | $n$, $\kappa < 0$ | $(l, j)$ | $E_{n\kappa<0}$ | | | $n-1$, $\kappa > 0$ | $(l + 2, j + 1)$ | $E_{n\kappa>0}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $T = 0$ | $T = 1$ | $T = 0$ [30] | | | $T = 0$ | $T = 1$ | $T = 0$ [30] |
| 1 | 1, –1 | $1s_{1/2}$ | –0.000024278 | –0.000010484 | –0.000047109 | 0, 2 | $0d_{3/2}$ | –0.000024278 | –0.000043400 | –0.000047109 |
| 2 | 1, –2 | $1p_{3/2}$ | –0.000043400 | –0.000024278 | –0.000069088 | 0, 3 | $0f_{5/2}$ | –0.000043400 | –0.000067972 | –0.000069088 |
| 3 | 1, –3 | $1d_{5/2}$ | –0.000067972 | –0.000043400 | –0.000095952 | 0, 4 | $0g_{7/2}$ | –0.000067972 | –0.000098007 | –0.000095952 |
| 4 | 1, –4 | $1f_{7/2}$ | –0.000098007 | –0.000067972 | –0.000127498 | 0, 5 | $0h_{9/2}$ | –0.000098007 | –0.000133511 | –0.000127498 |
| 1 | 2, –1 | $2s_{1/2}$ | –0.000042760 | –0.000022818 | –0.000089215 | 1, 2 | $1d_{3/2}$ | –0.000042760 | –0.000067438 | –0.000089215 |
| 2 | 2, –2 | $2p_{3/2}$ | –0.000067438 | –0.000042760 | –0.000121973 | 1, 3 | $1f_{5/2}$ | –0.000067438 | –0.000097495 | –0.000121973 |
| 3 | 2, –3 | $2d_{5/2}$ | –0.000097495 | –0.000067438 | –0.000158541 | 1, 4 | $1g_{7/2}$ | –0.000097495 | –0.000436653 | –0.000158541 |
| 4 | 2, –4 | $2f_{7/2}$ | –0.000436653 | –0.000097495 | –0.000199445 | 1, 5 | $1h_{9/2}$ | –0.000436653 | –0.000840507 | –0.000199445 |
| 1 | 3, –1 | $3s_{1/2}$ | –0.000066177 | –0.000038731 | –0.000135881 | 2, 2 | $2d_{3/2}$ | –0.000066177 | –0.000096571 | –0.000135881 |
| 2 | 3, –2 | $3p_{3/2}$ | –0.000096571 | –0.000066177 | –0.000185984 | 2, 3 | $1f_{5/2}$ | –0.000096571 | –0.000132179 | –0.000185984 |
| 3 | 3, –3 | $3d_{5/2}$ | –0.000132179 | –0.000096571 | –0.000234045 | 2, 4 | $1g_{7/2}$ | –0.000132179 | –0.000173174 | –0.000234045 |
| 4 | 3, –4 | $3f_{7/2}$ | –0.000173174 | –0.000132179 | –0.000285193 | 2, 5 | $1h_{9/2}$ | –0.000173174 | –0.000219601 | –0.000285193 |

when $T = 1$, $E_{0,-2,(T=0)} = E_{0,-3,(T=1)} = 0.000036016$. Likewise, in Tables 3, 4 and 5, 6, the energy eigenvalues are obtained in terms of different values of $M$, $C_s$ and $C_p$ for the four arbitrary states $0p_{3/2}$, $0g_{7/2}$, $2s_{1/2}$, $1h_{9/2}$.

*Table* 3. **The energy eigenvalues for different values of $M$ in spin symmetry limit: $C_s = 5$, $v_0 = -0.1$, $v_1 = -0.5$, $v_2 = -0.4$, $v_3 = 0.8$, $a = 0.04$, $b = 0.01$, $\alpha = 0.01$**

| $M$, fm$^{-1}$ | $E_{n\kappa<0}$ | | $E_{n\kappa>0}$ | |
|---|---|---|---|---|
| | $T=0$ ($0p_{3/2}$) | $T=1$ ($0p_{3/2}$) | $T=0$ ($0g_{7/2}$) | $T=1$ ($0g_{7/2}$) |
| 2 | 3.000012245 | 3.000003695 | 3.000029219 | 3.000040471 |
| 3 | 2.000006076 | 2.000003233 | 2.000059075 | 2.000035343 |
| 4 | 1.000005499 | 1.000002928 | 1.000023082 | 1.000031966 |
| 5 | 0.000016612 | 0.000008726 | 0.000021254 | 0.000097794 |
| 6 | –0.999995287 | –0.999990677 | –0.999980230 | –0.999895215 |
| 7 | –1.999981126 | –1.999990114 | –1.999919674 | –1.999888593 |
| 8 | –2.999980076 | –2.999989579 | –2.999982508 | –2.999975779 |
| 9 | –3.999979064 | –3.999989064 | –3.999910703 | –3.999876123 |
| 10 | –4.999978085 | –4.999988568 | –4.999906425 | –4.999870174 |

*Table* 4. **The energy eigenvalues for different values of $M$ in pseudospin symmetry limit: $C_p = -5$, $v_0 = 0.1$, $v_1 = 0.5$, $v_2 = 0.4$, $v_3 = -0.8$, $a = 0.04$, $b = 0.01$, $\alpha = 0.01$**

| $M$, fm$^{-1}$ | $E_{n\kappa<0}$ | | $E_{n\kappa>0}$ | |
|---|---|---|---|---|
| | $T=0$ ($2s_{1/2}$) | $T=1$ ($2s_{1/2}$) | $T=0$ ($1h_{9/2}$) | $T=1$ ($1h_{9/2}$) |
| 2 | –3.000058357 | –3.000030375 | –3.000182591 | –3.000238945 |
| 3 | –2.000051150 | –2.000026946 | –2.000159598 | –2.000208803 |
| 4 | –1.000046375 | –1.000024614 | –1.000144436 | –1.000188936 |
| 5 | 0.000042760 | –0.000022818 | –0.000436653 | –0.000173947 |
| 6 | 0.999960152 | 0.999978646 | 0.999876199 | 0.999386594 |
| 7 | 1.999847726 | 1.999979880 | 1.999883863 | 1.999348413 |
| 8 | 2.999964667 | 2.999980945 | 2.999476167 | 2.999311895 |
| 9 | 3.999966482 | 3.999981879 | 3.999449538 | 3.999864162 |
| 10 | 4.999825136 | 4.999982709 | 4.999901139 | 4.999242569 |

*Table* 5. **The energy eigenvalues for different values of $C_s$. Spin symmetry: $M = 5$ fm$^{-1}$, $v_0 = -0.1$, $v_1 = -0.5$, $v_2 = -0.4$, $v_3 = 0.8$, $a = 0.04$, $b = 0.01$, $\alpha = 0.01$**

| $C_s$ | $E_{n\kappa<0}$ | | $E_{n\kappa>0}$ | |
|---|---|---|---|---|
| | $T=0$ ($0p_{3/2}$) | $T=1$ ($0p_{3/2}$) | $T=0$ ($0g_{7/2}$) | $T=1$ ($0g_{7/2}$) |
| –4 | –8.999978571 | –8.999988814 | –8.999908549 | –8.999873128 |
| –2 | –6.999979566 | –6.999989319 | –6.999912888 | –6.999879161 |
| 0 | –4.999995708 | –4.999997711 | –4.999917370 | –4.999885391 |
| 2 | –2.999981668 | –2.999997568 | –2.999922028 | –2.999891865 |
| 4 | –0.999982797 | –0.999990970 | –0.999979529 | –0.999898656 |
| 6 | 1.000005268 | 1.000002806 | 1.000067880 | 1.000094113 |
| 8 | 3.000014679 | 3.000003068 | 3.000024201 | 3.000086220 |
| 10 | 5.000006456 | 5.000003434 | 5.000055658 | 5.000037571 |
| 12 | 7.000007697 | 7.000004088 | 7.000032381 | 7.000044854 |

*Table* 6. **The energy eigenvalues for different values of $C_p$. Pseudospin symmetry:** $M = 5$ **fm**$^{-1}$**,** $v_0 = 0.1$**,** $v_1 = 0.5$**,** $v_2 = 0.4$**,** $v_3 = -0.8$**,** $a = 0.04$**,** $b = 0.01$**,** $\alpha = 0.01$

| $C_p$ | $E_{n\kappa<0}$ | | $E_{n\kappa>0}$ | |
|---|---|---|---|---|
| | $T = 0$ ($2s_{1/2}$) | $T = 1$ ($2s_{1/2}$) | $T = 0$ ($1h_{9/2}$) | $T = 1$ ($1h_{9/2}$) |
| –12 | –7.000064472 | –7.000033190 | –7.000202209 | –7.000264674 |
| –10 | –5.000045288 | –5.000028453 | –5.000169593 | –5.000221905 |
| –8 | –3.000048572 | –3.000025693 | –3.000385790 | –3.000198065 |
| –6 | –1.000044460 | –1.000023665 | –1.000138368 | –1.000551720 |
| –4 | 0.999860385 | 0.999977947 | 0.999547791 | 0.999406468 |
| –2 | 2.999851817 | 2.999979287 | 2.999880191 | 2.999367268 |
| 0 | 4.999843744 | 4.999980431 | 4.999893355 | 4.999329971 |
| 2 | 6.999836064 | 6.999981427 | 6.999600162 | 6.999294146 |
| 4 | 8.999828708 | 8.999982306 | 8.999436545 | 8.999259508 |

To show accuracy of the present model, some numerical values of energy eigenstates are compared in Tables 1–6 for both spin and pseudospin symmetries with the NU method in the presence and absence of the Coulomb-like tensor interaction, respectively. The cause of a little difference in both conditions of spin and pseudospin symmetry limits with the reference [30] is in the present work, the Pekeris approximation scheme has been used for the centrifugal term in Eq. (15).

## CONCLUSION

In this paper, approximate analytical solutions of the Dirac equation with the modified Hylleraas plus Eckart potentials including a tensor interaction are studied by taking the SUSY method in the framework of spin and pseudospin symmetries. Clearly, degeneracy between the members of doublet states in spin and pseudospin symmetries is vanished by a tensor interaction. The obtained results for the modified Eckart and the modified deformed Hylleraas potentials are compared with those obtained through the NU method.

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