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THE GENERALIZED COULOMB INTERACTIONS FOR RELATIVISTIC SCALAR BOSONS

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Approximate analytical solutions of the Duffin–Kemmer–Petiau (DKP) equation are obtained for the truncated Coulomb and the generalized Cornell, Richardson and Song–Lin potentials via the quasi-exact analytical ansatz approach.

Получены приближенные аналитические решения уравнения Даффина–Кеммера–Петье для случаев усеченных кулоновских потенциалов, а также обобщенных потенциалов Корнелла, Ричардсона и Сонг–Лина в квазиточном аналитическом предположении.

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INTRODUCTION

We recognize the Duffin–Kemmer–Petiau (DKP) equation as the counterpart of the Klein–Gordon and Proca equations. In other words, this relativistic equation provides us with a basis to consider both spin-zero and spin-one bosons in a unified basis [1–4]. The equation, under the vector potential, resembles the Klein–Gordon (KG) equation, and consequently many people strictly believed the equations are completely equivalent. Now, however, there are serious doubts about the equivalence [5–10]. Fabulous discussions on the equivalence challenge can be also found in the interesting papers of Pimentel and Fainberg [11–14]. Some physicists state that the DKP equation provides us with a more elaborate basis to analyze physical interaction in comparison with its counterparts [15–19]. From another point of view, the DKP equation becomes a quite appealing research topic for the theoretical community as the Proca equation is not sufficiently analyzed when compared with other wave equations of quantum mechanics. This point becomes more important when we recall the high number of spin-one systems. As another merit of the equation we can mention its successful predictions in various fields of physics from subatomic to large-scale physics [20–26]. On the other hand, to our best knowledge, despite the large number of interactions studied within the framework of relativistic wave equations, the so-called truncated Coulomb potential has not been investigated in the case of relativistic bosonic systems. The truncated Coulomb potential has yielded successful outcomes in atomic, molecular and particle physics [27–37]. This

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is also true for the interesting phenomenological Richardson [38–40], Cornell-like [41, 42] and Song–Lin [43, 44] interactions which yield acceptable physical predictions due to their structure which include both confining and nonconfining terms.

As the final point in the introduction, it should be mentioned that there are other parallel approaches to consider two relativistic spin-1/2 fermions including the Bethe–Salpeter equation [45] and the two-body Dirac equation [46, 47]. Each of these approaches does have its own merits and disadvantages. In particular, the Bethe–Salpeter equation is a quite powerful technique to investigate the bound states when the interaction and the masses are known [47]. The equation, however, possesses such a complicated mathematical structure that it is not often considered in its full four-dimensional form and we have to consider some approximate schemes. The approach, even in the case of simple interactions such as the Coulomb term, leads to cumbersome problems. For example, the so-called ladder approximation does not lead to the correct one-body limit and does not respect gauge invariance [47].

Here, we are going to combine these two concepts of potential model, i.e., the spin-zero DKP equation and the introduced interaction. Our work, which considers the spin-zero version of the DKP equation, is organized as follows. We first review the DKP equation as compact as possible. Next, introducing the modified Coulomb potential which includes the ordinary Coulomb potential plus the so-called truncated Coulomb potential, as well as the generalized Cornell, Richardson and Song–Lin potentials, we work on the equation for vanishing scalar term. To solve our equations, as common techniques of quantum mechanics do not work, we propose an ansatz solution [48–50] to the problem and thereby report the eigenfunctions and the energy spectrum.

1. DKP EQUATION

The DKP Hamiltonian for scalar and vector interactions is [1–4]:

$$(\beta \cdot \mathbf{p}c + mc^2 + U_s + \beta^0 U_v^0)\psi(\mathbf{r}) = \beta^0 E\psi(\mathbf{r}), \tag{1}$$

where

$$\psi(\mathbf{r}) = \begin{pmatrix} \psi_{\text{upper}} \\ i\psi_{\text{lower}} \end{pmatrix}, \tag{2}$$

the upper and lower components, respectively, are

$$\psi_{\text{upper}} \equiv \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \quad \psi_{\text{lower}} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{3}$$

β^0 is the usual 5×5 matrix and U_s, U_v^0, respectively, represent the scalar and vector interactions. The equation, in $(3+0)$-dimensions, is written as [1–4]:

$$\begin{aligned} (mc^2 + U_s)\phi &= (E - U_v^0)\,\varphi + \hbar c \boldsymbol{\nabla} \cdot \mathbf{A}, \\ \boldsymbol{\nabla}\phi &= (mc^2 + U_s)\,\mathbf{A}, \\ (mc^2 + U_s)\,\varphi &= (E - U_v^0)\,\phi, \end{aligned} \tag{4}$$

where $\mathbf{A} = (A_1, A_2, A_3)$. In Eq. (3), ψ is a simultaneous eigenfunction of J^2 and J_3, i.e.,

$$
\begin{aligned}
J^2 \begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix} &= \begin{pmatrix} L^2\psi_{\text{upper}} \\ (L+S)^2\psi_{\text{lower}} \end{pmatrix} = J(J+1) \begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix}, \\
J_3 \begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix} &= \begin{pmatrix} L_3\psi_{\text{upper}} \\ (L_3+S_3)\psi_{\text{lower}} \end{pmatrix} = M \begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix},
\end{aligned} \tag{5}
$$

and the general solution is considered as

$$
\psi_{JM}(r) = \begin{pmatrix} f_{nJ}(r)Y_{JM}(\Omega) \\ g_{nJ}(r)Y_{JM}(\Omega) \\ i\sum\limits_L h_{nJL}(r)Y^M_{JL1}(\Omega) \end{pmatrix}, \tag{6}
$$

where spherical harmonics $Y_{JM}(\Omega)$ are of order J; $Y^M_{JL1}(\Omega)$ stands for the normalized vector spherical harmonics, and f_{nJ}, g_{nJ} and h_{nJL} represent the radial wave functions. The above equations yield the coupled differential equations [1–10]:

$$
\begin{aligned}
(E_{n,J} - U_v^0)F_{n,J}(r) &= (mc^2 + U_s)G_{n,J}(r), \\
\left(\frac{dF_{n,J}(r)}{dr} - \frac{J+1}{r}F_{n,J}(r)\right) &= -\frac{1}{\alpha_J}(mc^2 + U_s)H_{1,n,J}(r), \\
\left(\frac{dF_{n,J}(r)}{dr} + \frac{J}{r}F_{n,J}(r)\right) &= \frac{1}{\zeta_J}(mc^2 + U_s)H_{-1,n,J}(r),
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
-\alpha_J\left(\frac{dH_{1,n,J}(r)}{dr} + \frac{J+1}{r}H_{1,n,J}(r)\right) + \zeta\left(\frac{dH_{-1,n,J}(r)}{dr} - \frac{J}{r}H_{-1,n,J}(r)\right) = \\
= \frac{1}{\hbar c}\left((mc^2 + U_s)F_{n,J}(r) - (E_{n,J} - U_v^0)G_{n,J}(r)\right),
\end{aligned}
$$

which give [1–10]:

$$
\begin{aligned}
&\frac{d^2F_{n,J}(r)}{dr^2}\left[1 + \frac{\zeta_J^2}{\alpha_J^2}\right] - \frac{dF_{n,J}(r)}{dr}\left[\frac{U_s'}{(m+U_s)}\left(1 + \frac{\zeta_J^2}{\alpha_J^2}\right)\right] + \\
&\quad + F_{n,J}(r)\Bigg[-\frac{J(J+1)}{r^2}\left(1 + \frac{\zeta_J^2}{\alpha_J^2}\right) + \frac{U_s'}{(m+U_s)}\left(\frac{J+1}{r} - \frac{\zeta_J^2}{\alpha_J^2}\frac{J}{r}\right) - \\
&\qquad - \frac{1}{\alpha_J^2}\left((m+U_s)^2 - (E_{n,J} - U_v^0)^2\right)\Bigg] = 0,
\end{aligned} \tag{8}
$$

where $\alpha_J = \sqrt{(J+1)/(2J+1)}$, $f_{nJ}(r) = F(r)/r$, $g_{nJ}(r) = G(r)/r$, $h_{nJJ\pm1} = H_{\pm1}/r$ and $\zeta_J = \sqrt{J/(2J+1)}$. When $U_s = 0$, we recover the well-known formula [1–10]:

$$
\left(\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2} + (E_{n,J} - U_v^0)^2 - m^2\right)F_{n,J}(r) = 0. \tag{9}
$$

2. THE GENERALIZED TRUNCATED COULOMB INTERACTION

2.1. The Ground-State Solution. Here, we study the generalization of the truncated Coulomb potential [27]:

$$U_v^0(r) = \frac{g}{r} - \frac{f}{r+\beta}. \tag{10}$$

From Eqs. (9) and (10), we obtain

$$\left[\frac{d^2}{dr^2} - \frac{2Eg + \dfrac{2fg}{\beta}}{r} + \frac{g^2 - J(J+1)}{r^2} + \right.$$
$$\left. + \frac{2Ef + \dfrac{2fg}{\beta}}{r+\beta} + \frac{f^2}{(r+\beta)^2} + (E^2 - m^2)\right] F_{n,J}(r) = 0. \tag{11}$$

For the wave function, we propose an ansatz of the form [48–50]:

$$F_{n,J} = h_n(r)\,\exp\left(s_J(r)\right), \tag{12}$$

where

$$h_n(r) = \begin{cases} 1, & \text{if} \quad n = 0, \\ \prod_{i=1}^{n}(r - \alpha_i^n), & \text{if} \quad n \geqslant 1, \end{cases} \tag{13}$$

and

$$s(r) = \delta \ln(r) + \eta \ln(r+\beta) + \xi r. \tag{14}$$

For the ground state, $h_0(r) = 1$ and we find

$$F''_{0,J}(r) = \left[\left(\frac{2\delta\eta}{\beta} + 2\delta\xi\right)\frac{1}{r} + (\delta^2 - \delta)\frac{1}{r^2} + \right.$$
$$\left. + \left(-\frac{2\delta\eta}{\beta} + 2\eta\xi\right)\frac{1}{r+\beta} + (\eta^2 - \eta)\frac{1}{(r+\beta)^2} + \xi^2\right] F_{0,J}(r). \tag{15}$$

By comparing Eq. (11) with Eq. (15), we have

$$\frac{2\delta\eta}{\beta} + 2\delta\xi = 2Eg + \frac{2fg}{\beta}, \tag{16a}$$

$$\delta^2 - \delta = J(J+1) - g^2, \tag{16b}$$

$$-\frac{2\delta\eta}{\beta} + 2\eta\xi = -\left(2Ef + \frac{2fg}{\beta}\right), \tag{16c}$$

$$\eta^2 - \eta = -f^2, \tag{16d}$$

$$\xi^2 = -(E^2 - m^2). \tag{16e}$$

For fixed values of μ, $\tilde{m}$, β, g, the above system of five equations determines the sets of variables $E_{0,J}$, δ, η, ξ, f.

2.2. The Case of $n = 1$. For $n = 1$, we consider $h_n(r)$ as $h_1(r) = r - a_1^1$ and from Eqs. (11) and (14) we arrive at

$$F''_{1,J}(r) = \left[\left(\frac{2\delta\eta}{\beta} + 2\delta\xi\right)\frac{1}{r} + (\delta^2 - \delta)\frac{1}{r^2} + \left(-\frac{2\delta\eta}{\beta} + 2\eta\xi\right)\frac{1}{r+\beta} + \right. \\ \left. +(\eta^2 - \eta)\frac{1}{(r+\beta)^2} + \xi^2 + \frac{2\delta}{r(r-a_1^1)} + \frac{2\eta}{(r+\beta)(r-a_1^1)} + \frac{2\xi}{r-a_1^1}\right] F_{1,J}(r), \quad (17)$$

or

$$(r - a_1^1)F''_{1,J}(r) = \left[\left(\frac{-2a_1^1\delta\eta}{\beta} - 2a_1^1\delta\xi + \delta^2 + \delta\right)\frac{1}{r} - a_1^1(\delta^2 - \delta)\frac{1}{r^2} + \right. \\ + \left(\eta^2 + \eta + (-a_1^1 - \beta)\left(\frac{-2\delta\eta}{\beta} + 2\eta\xi\right)\right)\frac{1}{r+\beta} - (a_1^1 + \beta)(\eta^2 - \eta)\frac{1}{(r+\beta)^2} + \\ \left. + (-a_1^1\xi^2 + 2\xi + 2\eta\xi + 2\xi\delta) + \xi^2 r\right] F_{1,J}(r). \quad (18)$$

On the other hand, Eq. (11) gives

$$(r - a_1^1)F''_{1,J}(r) = \left[-\left(a_1^1\left(2Eg + \frac{2fg}{\beta}\right) - J(J+1) + g^2\right)\frac{1}{r} + a_1^1(g^2 - J(J+1))\frac{1}{r^2} - \right. \\ - \left(-(\beta + a_1^1)\left(\frac{2fg}{\beta} + 2Eg\right) + f^2\right)\frac{1}{r+\beta} + (a_1^1 + \beta)f^2\frac{1}{(r+\beta)^2} - \\ \left. - ((-2Eg + 2Ef) + a_1^1(E^2 - m^2)) - (E^2 - m^2)r\right] F_{1,J}(r). \quad (19)$$

By comparing Eqs. (18) and (19), we have

$$\frac{-2a_1^1\delta\eta}{\beta} - 2a_1^1\delta\xi + \delta^2 + \delta = -\left(a_1^1\left(2Eg + \frac{2fg}{\beta}\right) - J(J+1) + g^2\right), \quad (20a)$$

$$-(\delta^2 - \delta) = g^2 - J(J+1), \quad (20b)$$

$$\left(\eta^2 + \eta + (-a_1^1 - \beta)\left(\frac{-2\delta\eta}{\beta} + 2\eta\xi\right)\right) = -\left(-(\beta + a_1^1)\left(\frac{2fg}{\beta} + 2Eg\right) + f^2\right), \quad (20c)$$

$$-(\eta^2 - \eta) = f^2, \quad (20d)$$

$$-a_1^1\xi^2 + 2\xi + 2\eta\xi + 2\xi\delta = -((-2Eg + 2Ef) + a_1^1(E^2 - m^2)), \quad (20e)$$

$$\xi^2 = -(E^2 - m^2). \quad (20f)$$

By solving the above equations for the fixed values of β, g, we can find $E_{1,J}$, δ, η, ξ, f, a_1^1.

2.3. The Special Case of the Coulomb Potential. Considering $f = 0$ or $V(r) = g/r$, the equation changes into

$$\left[\frac{d^2}{dr^2} - \frac{J'(J'+1)}{r^2} + \frac{g'}{r} + E'_{n,J'}\right] F_{n,J'}(r) = 0, \tag{21}$$

where

$$\begin{gathered} J' = \frac{1}{2}\left[-1 \pm \sqrt{1 + 4(-g^2 + J(J+1))}\right], \\ g' = -2E_{n,J}g, \quad E'_{n,J'} = E^2_{n,J} - m^2. \end{gathered} \tag{22}$$

Equation (21) has the form of hydrogen atom whose solutions are well known:

$$F_{n,J'}(r) = r^{\frac{1}{2}+\sqrt{\frac{1}{4}+J'(J'+1)}} \exp\left(-\sqrt{-E'_{n,J'}}r\right) L_n^{1+2\sqrt{\frac{1}{4}+J'(J'+1)}}\left(2\sqrt{-E'_{n,J'}}\, r\right) \tag{23}$$

and

$$E_{n,J'} = \frac{m\left[(2n+1) + 2\sqrt{\frac{1}{4} + J'(J'+1)}\right]}{\sqrt{4g^2 + \left[(2n+1) + 2\sqrt{\frac{1}{4} + J'(J'+1)}\right]^2}}. \tag{24}$$

For $n = 0$, Eqs. (23) and (24) reduce to

$$F_{0,J'}(r) = r^{\frac{1}{2}+\sqrt{\frac{1}{4}+J'(J'+1)}} \exp\left(-\sqrt{-E'_{n,J'}}\, r\right) \tag{25}$$

and

$$E_{0,J'} = \frac{m\left[1 + 2\sqrt{\frac{1}{4} + J'(J'+1)}\right]}{\sqrt{4g^2 + \left[1 + 2\sqrt{\frac{1}{4} + J'(J'+1)}\right]^2}}. \tag{26}$$

Let us now compare the latter with our result. If we put $f = 0$ in Eq. (16), we have

$$2\delta\xi = -2E^2_{0,J'}g, \tag{27a}$$

$$\delta^2 - \delta = J'(J'+1), \tag{27b}$$

$$\xi^2 = -(E^2_{0,J'} - m^2), \tag{27c}$$

which determine δ and ξ as

$$\xi = -\sqrt{-(E^2_{0,J'} - m^2)}, \tag{28a}$$

$$\delta = \frac{1}{2}\left(1 \pm \sqrt{1 + 4J'(J'+1)}\right), \tag{28b}$$

and the eigenvalues for the case of $n = 0$ can be obtained as

$$2\frac{1}{2}\left[1 \pm \sqrt{1+4J'(J'+1)}\right]\left(-\sqrt{-(E_{0,J'}^2 - m^2)}\right) = -2E_{0,J'}g, \tag{29a}$$

or

$$E_{0,J'} = \frac{m\left[1+2\sqrt{\frac{1}{4}+J'(J'+1)}\right]}{\sqrt{4g^2 + \left[1+2\sqrt{\frac{1}{4}+J'(J'+1)}\right]^2}}, \tag{29b}$$

which is exactly the same as Eq. (26). For the wave function, we have

$$F_{0,J'}(r) = r^{\delta+\eta}\exp(\xi r). \tag{30}$$

From (14) and (16), by setting $\eta = 0$, we have

$$F_{0,J'}(r) = r^{\frac{1}{2}+\sqrt{\frac{1}{4}+J'(J'+1)}}\exp\left(-\sqrt{-(E_{0,J'}^2 - m^2)}\right)r, \tag{31}$$

which is the result of Eq. (25).

3. THE RICHARDSON POTENTIAL

The Richardson potential includes cubic and inverse terms and, by keeping the notation of [38–40], possesses the form

$$U_v^0(r) = \Lambda\left((\Lambda r)^3 - \frac{12}{\Lambda r}\right), \tag{32}$$

in which Λ is a constant parameter. Substituting the potential into Eq. (9) yields the linear second-order differential equation

$$\left[\frac{d^2}{dr^2} - 24\Lambda^4 r^2 - 2E\Lambda^4 r^3 + \Lambda^8 r^6 + \frac{24E}{r} + (-J(J+1)+144)\frac{1}{r^2} + E^2 - m^2\right]F_{n,J} = 0, \tag{33}$$

which is not a well-known differential equation. Let us now consider the ground-state case. In this case, we propose

$$s(r) = ar^4 + br + c\ln(r) \tag{34}$$

and substitute the solution in Eq. (13). Now, equating the corresponding powers on both sides gives the set of equations

$$\begin{gathered} 3a + 2ac = 6\Lambda^4, \quad 4ab = E\Lambda^4, \quad 16a^2 = -\Lambda^8, \quad bc = -12E, \\ c^2 - c = J(J+1) - 144, \quad b^2 = -(E^2 - m^2), \end{gathered} \tag{35}$$

which determines the spectrum.

4. THE GENERALIZED CORNELL POTENTIAL

In this section, we consider a generalization of the Cornell potential [41, 42]:

$$U_v^0(r) = v_0 + kr - \frac{e}{r} + \frac{f}{r^2}, \tag{36}$$

in which v_0, k, e and f are constant potential parameters. The corresponding equation then reads

$$\left[\frac{d^2}{dr^2} + (-2Ek + 2kv_0)r + k^2r^2 + (2Ee - 2ev_0 + 2kf)\frac{1}{r} + (-J(J+1) - 2Ef + 2fv_0 + e^2)\frac{1}{r^2} - \right.$$
$$\left. - \frac{2ef}{r^3} + \frac{f^2}{r^4} + (E^2 - 2Ev_0 + v_0^2 - 2ke - m^2)\right] F_{n,J} = 0. \tag{37}$$

Solution of the corresponding Riccati equation in this case determines the term in the exponent as

$$s(r) = ar^2 + br + \frac{c}{r} + d\ln(r), \tag{38}$$

and therefore the corresponding set of equations is obtained as

$$\begin{gathered} 4ab = 2Ek - 2kv_0, \quad 4a^2 = -k^2, \quad -4ac + 2bd = -2Ee + 2ev_0 - 2kf, \\ -d - 2cb + d^2 = J(J+1) + 2Ef - 2fv_0 - e^2, \quad c - cd = ef, \\ c^2 = -f^2, \quad 2a + 4ad + b^2 = -E^2 + 2Ev_0 - v_0^2 + 2ke + m^2. \end{gathered} \tag{39}$$

5. THE SONG–LIN POTENTIAL

In this section, we consider the term proposed by Song and Lin to analyze the quark systems [43, 44]. Although this is not a generalization of the Coulomb interaction, we include that to show the power of apparently simple ansatz approach. The potential considers the square root and its inverse

$$U_v^0(r) = Ar^{1/2} + Br^{-1/2}, \tag{40}$$

with A and B being constant parameters. In this case, we have to deal with the differential equation

$$\left[\frac{d^2}{dr^2} + A^2r - 2EAr^{1/2} - 2EBr^{-1/2} + \frac{B^2}{r} - \frac{J(J+1)}{r^2} + (E^2 + 2AB - m^2)\right] F_{n,J} = 0, \tag{41}$$

which can be solved via the term

$$s(r) = ar^{3/2} + br + c\ln(r), \tag{42}$$

and the corresponding energy relation is

$$\begin{gathered} \frac{9}{4}a^2 = -A^2, \quad 3ab = 2EA, \quad \frac{3}{4}a + 3ac = 2EB, \\ 2bc = -B^2, \quad c^2 - c = J(J+1), \quad b^2 = -E^2 - 2AB + m^2, \end{gathered} \tag{43}$$

which can be solved numerically to report the energy for given potential parameters.

CONCLUSIONS

The truncated Coulomb potential can be considered as the generalization of the Coulomb potential and has provided motivating results in various fields of physics. On the other hand, the generalized Cornell, Richardson and Song–Lin potentials have had successful predictions in particle physics. Bearing in mind the physical significance of these interactions, we considered this potential within the framework of the DKP equation. To our best knowledge, neither of our common tools of mathematical physics such as supersymmetry quantum mechanics, factorization and Nikiforov–Uvarov techniques can solve these problems in the analytical manner. Here, we applied the simple but powerful ansatz technique which is based on solving a related Riccati equation and thereby reported the solutions. With the aid of this approach, we avoided using cumbersome numerical methodologies and provided a more touchable solution to the problem. Although we only derived the first two states, the higher states can be similarly obtained by choosing $h_2(r) = (r - \alpha_1^2)(r - \alpha_2^2)$ for the second node, $h_3(r) = (r - \alpha_1^3)(r - \alpha_2^3)(r - \alpha_3^3)$ for the third node, etc.

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