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CURVATURE DECOMPOSITION AND THE EINSTEIN–YANG–MILLS EQUATIONS

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The Einstein equations are written in the form of equalities for $O(1,3)$ invariant components of the curvature tensor field. The «material part» of the curvature is expressed in terms of the energy-momentum tensor field. In this approach the self-gravitating Yang–Mills fields are considered, and a reduced set of equations are obtained for them in the general form. In the spherically symmetric space-time the equations are written explicitly for the $SU(2)$ gauge group. It is shown that the Bianchi identities allow one to except some of the gauge field equations.

Уравнения Эйнштейна записаны в виде соотношений для $O(1,3)$-инвариантных компонент поля тензора кривизны. При этом «материальная часть» кривизны выражена через компоненты тензора энергии-импульса. В этом подходе рассмотрены самогравитирующие поля Янга–Миллса и для них в общем виде получена приведенная система уравнений. Для случая сферически-симметричного пространства-времени уравнения выписаны явно для калибровочной группы $SU(2)$. Показано, что тождества Бианки позволяют исключить некоторые уравнения для калибровочного поля.

INTRODUCTION

The most of non-Abelian solutions in the Einstein–Yang–Mills (EYM) theory are obtained numerically [1]. Because of nonlinearity of the equations even the correct boundary problem formulation is sufficiently hard, although the existence of solutions has been proved rigorously in some cases. In this connection, analytical investigations of the inner structure of the EYM equations can be useful, firstly, in order to convert the system into a more convenient form for numerical calculation and, secondly, for a deeper understanding of the interaction between gravity and the Yang–Mills fields.

The aim of this paper is to rewrite the Einstein equations in terms of the components of $O(1,3)$ decomposition of the curvature tensor field and to apply the obtained system to the self-gravitating spherically symmetric $SU(2)$ Yang–Mills fields. The direct profit of this reformulation consists in the exception, by using the Bianchi identities, of some Yang–Mills equations from the EYM system; moreover, for the field of pure magnetic type they can be excepted at all. In Sec. 1 we consider briefly the curvature decomposition and the transformation of the Einstein equations in the general case. In Sec. 2 the obtained system is reduced to the spherically symmetric case, and the Bianchi identities are written explicitly. In Sec. 3, from this point of view, the nonstationary spherically symmetric $SU(2)$ EYM

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equations are obtained with remaining of the gauge freedom for three metric functions. A special gauge of the Kruskal type is also considered.

It is important to emphasize that the approach presented can be applied in the same manner to other self-gravitating systems of material fields, which are intensively investigated now [2–4].

1. CURVATURE DECOMPOSITION AND EINSTEIN EQUATIONS

Let M be a space-time, g a metric of the signature $(+\,-\,-\,-)$ on M, (e_i) and (e^i) $(i = 0,1,2,3)$ dual orthonormal tetrads of vector field and 1-forms in some region of M. We consider the curvature R as a (0, 4) tensor field with the well-known $O(1,3)$ decomposition, written briefly as [5]

$$R = C + H, \qquad H = \frac{1}{2} g \owedge r_0, \tag{1}$$

where r is the Ricci tensor field and r_0 its traceless part; $\owedge$ the Kulcarni–Nomizu product defined on tetrad elements by

$$(e^i \otimes e^j) \owedge (e^k \otimes e^l) = (e^i \wedge e^k) \otimes (e^j \wedge e^l).$$

Note that C is $O(1,3)$ reducible and can be decomposed into the sum $C = W + (s/24)\, g \owedge g$, where W is the Weyl tensor field and s the scalar curvature.

The Einstein equations $r - (1/2)sg = \kappa T$ are equivalent to

$$r_0 = \kappa T_0, \qquad s = -\kappa \operatorname{tr} T,$$

where T_0 is the traceless part of the energy-momentum tensor field and $\operatorname{tr} T$ its trace with respect to the metric. Therefore, in accordance with (1), they are equivalent to the equations

$$\operatorname{tr} C = -\operatorname{tr} T, \qquad H = \frac{\kappa}{2}\, g \owedge T_0. \tag{2}$$

The left sides of equations (2) should be expressed in terms of the metric functions. It is convenient to use the orthonormal basis $(\alpha^A, *\alpha^A)$ $(A = 1,2,3$ and $*$ is the Hodge star) defined by

$$\alpha^A = e^0 \wedge e^A, \qquad *\alpha^1 = e^3 \wedge e^2, \quad *\alpha^2 = e^1 \wedge e^3, \quad *\alpha^3 = e^2 \wedge e^1.$$

To make this formalism more visual, one can write the curvature decomposition in the matrix form as follows:

$$R = (\alpha^T \; *\alpha^T) \begin{pmatrix} P & S^T \\ S & Q \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ *\alpha \end{pmatrix} = (\alpha^T \; *\alpha^T) \left(\begin{pmatrix} U & V \\ V & -U \end{pmatrix} + \begin{pmatrix} K & -L \\ L & K \end{pmatrix} \right) \otimes \begin{pmatrix} \alpha \\ *\alpha \end{pmatrix}. \tag{3}$$

Here

$$U = \frac{1}{2}(P - Q), \quad V = \frac{1}{2}(S + S^T), \quad K = \frac{1}{2}(P + Q), \quad L = \frac{1}{2}(S - S^T)$$

are 3×3 matrices, and two summands in (3) give us C and H. Now equations (2) take the form

$$\operatorname{tr} U = \frac{\kappa}{4} \operatorname{tr} T, \qquad K_{11} = \frac{\kappa}{4}(T_{11} - T_{00} - T_{22} - T_{33}),$$
$$K_{22} = \frac{\kappa}{4}(T_{22} - T_{00} - T_{11} - T_{33}), \qquad K_{33} = \frac{\kappa}{4}(T_{33} - T_{00} - T_{11} - T_{22}), \tag{4}$$
$$K_{AB} = \frac{\kappa}{2} T_{AB} \ (A \neq B), \quad L_{32} = \frac{\kappa}{2} T_{01}, \quad L_{13} = \frac{\kappa}{2} T_{02}, \quad L_{21} = \frac{\kappa}{2} T_{03}.$$

These are the Einstein equations written in terms of the components of the $O(1,3)$ curvature decomposition.

2. SPHERICALLY SYMMETRIC SPACE-TIME

We assume the metric in the standard form

$$ds^2 = A^2 dt^2 - B^2 dr^2 - C^2 (d\theta^2 + \sin^2\theta \, d\phi^2),$$

where A, B, C are functions of t and r.

Let

$$e_0 = \frac{1}{A}\partial_t, \qquad e_1 = \frac{1}{B}\partial_r, \qquad e_2 = \frac{1}{C}\partial_\theta, \qquad e_3 = \frac{1}{C\sin\theta}\partial_\phi,$$

and lower indices 0, 1 denote the derivatives in the directions of e_0 and e_1; i. e., $f_0 = e_0 f$, $f_1 = e_1 f$ for any function f. The space-time is of type I by Petrov, and the nonvanishing coefficients of matrices (4) are

$$U = \operatorname{diag}(v, -u, -u), \quad K = \operatorname{diag}(f,\, h,\, h), \quad L_{32} = q = \frac{C_{01}}{C} - \frac{B_0 C_1}{BC},$$
$$v = -\frac{1}{2}\left(\frac{A_{11}}{A} + \frac{C_1^2 - C_0^2 - 1}{C^2} - \frac{B_{00}}{B}\right), \qquad u = \frac{1}{2}\left(\frac{C_{11}}{C} + \frac{A_1 C_1}{AC} - \frac{C_{00}}{C} - \frac{B_0 C_0}{BC}\right), \tag{5}$$
$$f = -\frac{1}{2}\left(\frac{A_{11}}{A} - \frac{C_1^2 - C_0^2 - 1}{C^2} - \frac{B_{00}}{B}\right), \qquad h = \frac{1}{2}\left(-\frac{C_{11}}{C} + \frac{A_1 C_1}{AC} - \frac{C_{00}}{C} + \frac{B_0 C_0}{BC}\right).$$

The Bianchi identities have the form

$$(u-h)_1 + (f - h + v + u)\frac{C_1}{C} - 2h\frac{A_1}{A} + q_0 + q\frac{C_0}{C} + 2q\frac{B_0}{B} = 0,$$
$$(f-v)_1 + 2(f - h - v - u)\frac{C_1}{C} + 2q\frac{C_0}{C} = 0,$$
$$(u+h)_0 + (f + h + v + u)\frac{C_0}{C} + 2h\frac{B_0}{B} - q_1 - q\frac{C_1}{C} - 2q\frac{A_1}{A} = 0,$$
$$(f-v)_0 + 2(f + h - v - u)\frac{C_0}{C} - 2q\frac{C_1}{C} = 0.$$

From these equalities one can easily obtain

$$\frac{(fC^4)_1}{C^4} - 2\frac{(A^2C^2h)_1}{A^2C^2} + 2\frac{(B^2C^2q)_0}{B^2C^2} = (v - 2u)_1. \tag{6}$$

$$\frac{(fC^4)_0}{C^4} + 2\frac{(B^2C^2h)_0}{B^2C^2} - 2\frac{(A^2C^2q)_1}{A^2C^2} = (v-2u)_0. \tag{7}$$

If the Einstein equations hold, we can exchange f, h, q and $v - 2u = \operatorname{tr} U$ for the right sides of equations (5). Then equations (6) and (7) should be considered as ones for fields and matter. It turns out that dynamics of the perfect fluid and electrodynamics are contained in these equations. This is true in the general case, not only for the spherical symmetry, except the pure wave fields. However, for non-Abelian gauge fields the situation is more complicated.

We assume the standard Lagrangian [2] of the classical EYM system for a semi-simple gauge group $\mathfrak{G}$, with Yang–Mills charge $\gamma = \sqrt{4\pi G}$ $(c = 1)$ and suitable normalization of the Killing product in Lie algebra $\mathfrak{g}$, so that

$$\frac{\kappa}{2}T_{km} = \frac{1}{4}g_{km}F_{ij}\cdot F^{ij} - g^{ij}F_{ik}\cdot F_{jm}.$$

Generally, it is more convenient to use invariant notations. For a Yang–Mills field

$$F = F_A \otimes \alpha^A + \tilde{F}_A \otimes *\alpha^A,$$

one can easily find in the right side of (2)

$$\frac{\kappa}{2}\, g \owedge T_0 = -F\dot{\otimes}F - *F\dot{\otimes} * F,$$

where the dot denotes the Killing product.

Therefore, equations (4) take the form

$$\operatorname{tr} U = 0, \quad K_{AB} = -F_A\cdot F_B - \tilde{F}_A\cdot\tilde{F}_B, \quad L_{AB} = F_A\cdot\tilde{F}_B - \tilde{F}_A\cdot F_B. \tag{8}$$

These are the Einstein part of the EYM equations for arbitrary Yang–Mills field in the general case.

3. SPHERICALLY SYMMETRIC $SU(2)$ EYM EQUATIONS

Note that the spherical symmetry of space-time does not demand the same symmetry of the Yang–Mills field for arbitrary gauge group, as it follows from (8). It is necessary only that the expressions in the right side of (8) be in accordance with the conditions $K_{AB} = 0$ $(A \neq B)$, $K_{22} = K_{33}$, $L_{12} = L_{13} = 0$. However, for $SU(2)$ gauge group the space-time spherical symmetry means also the same symmetry for the Yang–Mills field [4, 6].

The spherically symmetric gauge potential

$$\Omega = \frac{a}{A}\tau_1 \otimes e^0 - \frac{w}{C}\tau_2 \otimes e^3 + \left(\frac{w}{C}\tau_3 - \frac{\cot\theta}{C}\tau_1\right)\otimes e^3,$$

where $\tau_A \cdot \tau_B = \delta_{AB}$, gives us the Yang–Mills field

$$F = -\frac{a_1}{A}\tau_1 \otimes \alpha^1 + \left(-\frac{aw}{AC}\tau_2 + \frac{w_0}{C}\tau_3\right)\otimes\alpha^2 + \left(\frac{w_0}{C}\tau_2 + \frac{aw}{AC}\tau_3\right)\otimes\alpha^3 + \\ + \frac{w^2-1}{C^2}\tau_1\otimes *\alpha^1 + \frac{w_1}{C}\tau_2 \otimes *\alpha^2 - \frac{w_1}{C}\tau_3\otimes *\alpha^3,$$

and the Yang–Mills equation $d_\Omega * F = 0$ can be reduced to the set

$$\frac{(Aw_1)_1}{A} + \frac{w(1-w^2)}{C^2} + \frac{a^2 w}{A^2} - \frac{(Bw_0)_0}{B} = 0,$$
$$\left(\frac{C^2 a_1}{A}\right)_1 + \left(\frac{C^2 a_1}{A}\right)_0 = 2\frac{aw^2}{A}, \qquad \left(\frac{Baw^2}{A}\right)_0 = 0. \tag{9}$$

On the other hand, substituting

$$f \equiv -F_1 \cdot F_1 - \tilde{F}_1 \cdot \tilde{F}_1 = -\frac{a_1^2}{A^2} - \frac{(1-w^2)^2}{C^4},$$
$$h \equiv -F_2 \cdot F_2 - \tilde{F}_2 \cdot \tilde{F}_2 = -\left(\frac{aw}{AC}\right)^2 - \left(\frac{w_0}{C}\right)^2 - \left(\frac{w_1}{C}\right)^2,$$
$$q \equiv F_2 \cdot \tilde{F}_3 - \tilde{F}_2 \cdot F_3 = -2\frac{w_0 w_1}{C^2}$$

in (6) and (7), we get

$$w_1\left(\frac{(Aw_1)_1}{A} + \frac{w(1-w^2)}{C^2} + \frac{a^2 w}{A^2} - \frac{(Bw_0)_0}{B}\right) - \frac{a_1}{2A}\left(\left(\frac{C^2 a_1}{A}\right)_1 - 2\frac{aw^2}{A}\right) = 0,$$

$$w_0\left(\frac{(Aw_1)_1}{A} + \frac{w(1-w^2)}{C^2} + \frac{a^2 w}{A^2} - \frac{(Bw_0)_0}{B}\right) - \frac{a_1}{2A}\left(\frac{C^2 a_1}{A}\right)_0 - \frac{a}{AB}\left(\frac{Baw^2}{A}\right)_0 = 0.$$

These equations should be compared with (9) in order to see that only two (one, in the static case) of the Yang–Mills equations are independent. So, from (5), (8) and (9) one can obtain the complete EYM system in the form

$$\frac{A_{11}}{A} + 2\frac{A_1 C_1}{AC} + 2\frac{C_{11}}{C} + \frac{C_1^2 - C_0^2 - 1}{C^2} - \frac{B_{00}}{B} - 2\frac{C_{00}}{C} - 2\frac{B_0 C_0}{BC} = 0,$$
$$-\frac{1}{2}\left(\frac{A_{11}}{A} - \frac{C_1^2 - C_0^2 - 1}{C^2} - \frac{B_{00}}{B}\right) = \frac{a_1^2}{A^2} + \frac{(1-w^2)^2}{C^4},$$
$$\frac{1}{2}\left(-\frac{C_{11}}{C} + \frac{A_1 C_1}{AC} - \frac{C_{00}}{C} + \frac{B_0 C_0}{BC}\right) = \left(\frac{aw}{AC}\right)^2 + \left(\frac{w_0}{C}\right)^2 + \left(\frac{w_1}{C}\right)^2, \tag{10}$$
$$\frac{C_{01}}{C} - \frac{B_0 C_0}{BC} = -2\frac{w_0 w_1}{C^2}, \qquad \left(\frac{C^2 a_1}{A}\right)_0 = 0, \qquad \left(\frac{C^2 a_1}{A}\right)_1 = 2\frac{aw^2}{A};$$

if w depends on t only as in some cosmological models, the latter equation should be exchanged for $(Baw^2/A)_0 = 0$.

It is interesting to consider the special gauge and new variables:

$$A = B, \qquad \xi = (t+r)/2, \qquad \eta = (t-r)/2.$$

For simplicity we suppose $a = 0$, then system (10) reduces to

$$\frac{1}{A^2}\left(\frac{A_\eta}{A}\right)_\xi + 2\frac{C_{\eta\xi}}{A^2 C} + \frac{C_\eta C_\xi}{A^2 C^2} + \frac{1}{C^2} = 0,$$

$$-\frac{1}{A^2}\left(\frac{A_\eta}{A}\right)_\xi + \frac{C_\eta C_\xi}{A^2 C^2} + \frac{1}{C^2} = 2\frac{(1-w^2)^2}{C^4}, \tag{11}$$

$$-\frac{C_{\xi\xi}}{C} + 2\frac{A_\xi C_\xi}{AC} = 2\frac{{w_\xi}^2}{C^2}, \qquad -\frac{C_{\eta\eta}}{C} + 2\frac{A_\eta C_\eta}{AC} = 2\frac{{w_\eta}^2}{C^2}.$$

In spite of the special gauge this form of EYM system is universal in some respects. For example, assuming that the metric functions A, C depend on $r = \xi - \eta$ and $w = 0$, $A^2 = C'$, we obtain the Reissner–Nordstrem solution with unit magnetic charge and $A^2 = 1 - 2m/C + 1/C^2$. On the other hand, for the metric function depending on $\xi\eta$ and $w = 1$, the Kruskal ansatz

$$\left(\frac{C}{2m} - 1\right)\exp\left(\frac{C}{2m}\right) = -4\xi\eta, \qquad A^2 = \frac{32m^3}{C}\exp\left(-\frac{C}{2m}\right)$$

gives us the extended Schwarzschild solution.

CONCLUSION

The presented method of reduction of the Einstein equations allows one to efficiently take into account specific features of sources of gravitational field. Also the curvature decomposition separates the «material part» from the curvature tensor field, so that the Bianchi identities contain the dynamics equations for sources of gravity. We demonstrate this for the $SU(2)$ Yang–Mills field; however, the method can be applied to an arbitrary self-gravitating system without essential modification.

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