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TIME-DEPENDENT EXACTLY SOLVABLE MODELS FOR QUANTUM COMPUTING

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A time-dependent periodic Hamiltonian admitting exact solutions is applied to construct a set of universal gates for quantum computer. The time evolution matrices are obtained in an explicit form and used to construct logic gates for computation. A way of obtaining entanglement operator is discussed, too. The method is based on transformation of soluble time-independent equations into time-dependent ones by employing a set of special time-dependent transformation operators.

Периодически зависящий от времени гамильтониан, допускающий точные решения, используется для построения универсального набора квантовых вентилей для квантовых компьютеров. Показано, как конструировать логические гейты на основе полученных в явном виде матриц эволюции. Обсуждается также способ получения операторов запутывания. Метод основан на преобразовании стационарной задачи в нестационарные с помощью специальных зависящих от времени операторов преобразования.

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INTRODUCTION

Recent studies of quantum computation have attracted considerable interest in both theoretical and experimental physics. The physical realization of the qubit register and a universal set of one-qubit and two-qubit logic gates is an important problem of quantum computation [1–3]. In this paper we shall construct one-qubit and two-qubit gates with desired properties controlled by time-dependent Hamiltonian.

A quantum computer is composed of a set of qubits which can be manipulated in a controlled way. Any quantum two-level systems can be taken to create qubits. *A computation process corresponds to the evolution of the set of the qubits according to a specific unitary operator, for example, evolution operator $U(t)$.* A general operation is decomposed into a discrete sequence in time of operations — quantum gates. The simplest unit of quantum information is a quantum bit, or qubit. The qubit is a vector in a two-dimensional Hilbert space, which can be presented as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. The basis vectors $|0\rangle$ and $|1\rangle$ are chosen as $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, $|\psi\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$. Here α and β are complex

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coefficients, which satisfy the condition $|\alpha|^2 + |\beta|^2 = 1$. Then $|\psi\rangle$ is the normalized vector, and α^2 and β^2 characterize the probabilities of the results $|0\rangle$ and $|1\rangle$, correspondingly. The 2nd order matrices $\mathcal{U}(2\times 2)$ transform one-qubit states and describe their evolution in time:

$$|\psi_f\rangle = \mathcal{U}(2\times 2)|\psi_0\rangle, \quad \mathcal{U}(t) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

Such transformations in quantum computation determine one-qubit quantum operations — *quantum gates*.

The formalism of quantum mechanics is usually applied not to individual systems but to ensembles of systems. In quantum computation, the state of the computer of n qubits can be expressed as a vector $|\Psi\rangle$ in a space of dimension 2^n. Vector $|\Psi\rangle$ of the quantum register from n qubits is expressed as a complex linear superposition of 2^n basis states:

$$|\Psi\rangle = \sum_{k=0}^{2^n-1} a_k |j_k\rangle. \tag{1}$$

Here a_k are projections of the vector $|\Psi\rangle$ on the directions of basis states $|j_0\rangle, |j_1\rangle, \dots, |j_{2^n-1}\rangle$, $\sum_k a_k^2 = 1$. Basis states $|j\rangle = |i_1, i_2, \dots, i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \dots |i_n\rangle$, $\; i_1, i_2, \dots, i_n = \{0, 1\}$ are presented as

$$\begin{array}{c} |j_0\rangle = |0\rangle \otimes |0\rangle \dots \otimes |0\rangle \\ |j_1\rangle = |0\rangle \otimes |0\rangle \dots \otimes |1\rangle \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ |j_{2^n-1\rangle} = |1\rangle \otimes |1\rangle \dots \otimes |1\rangle \end{array} . \tag{2}$$

The transformation of an initial state vector $|\Psi_0\rangle$ into the final vector $|\Psi_f\rangle$ models the process of calculation on quantum computer

$$|\Psi_f\rangle = \mathcal{U}(2^n \times 2^n)|\Psi_0\rangle.$$

Vectors $|\Psi_0\rangle$ and $|\Psi_f\rangle$ are vectors in the 2^n Hilbert space. The transformation matrices $\mathcal{U}(2^n \times 2^n; t)$ define the dynamic evolution of the quantum system from n qubits. At the same time, the matrices $\mathcal{U}(2^n \times 2^n; t)$ provide the process of quantum computing at each fixed moment.

Clearly, the realization of the transformation $\mathcal{U}(2^n \times 2^n)$ with $n > 3$ is a very difficult problem. As usual, one considers the presentation of $\mathcal{U}(2^n \times 2^n)$ as a production of second $U(2\times 2)$ order and forth $U(4\times 4)$ order matrices:

$$\mathcal{U}(2^n \times 2^n) = \prod_{i,j} \mathcal{U}_i(2\times 2) \otimes \mathcal{U}_j(2^2 \times 2^2). \tag{3}$$

As is known [1], a universal set of gates is given by 2×2 unitary operators and a unitary entangled operator 4×4 which acts on $C^2 \otimes C^2$. We shall show how it is possible to generate explicitly one-qubit logic gates from the time evolution matrices and give a way of obtaining entanglement operators.

1. A UNIVERSAL GATE SET

The universal one-qubit logic gates can be constructed from the time evolution matrices which we obtain in a closed analytic form. In our approach, the time-dependent periodic Hamiltonians admitting exact solutions are applied to control the time evolution of the one-qubit gates. The time-dependent Hamiltonians are obtained from time-independent soluble Hamiltonians and a set of unitary time-dependent transformations [4].

Suppose that the time evolution of the quantum system is governed by the Schrödinger equation

$$i\frac{\partial|\Psi(r,t)\rangle}{\partial t} = H(r,t)|\Psi(r,t)\rangle \tag{4}$$

with $\hbar = 1$ and T periodic time-dependent Hamiltonian, $H(t) = H(t+T)$.

Assume that the initial state of the qubit can be written in one of the states of the time-independent Hamiltonian $\widetilde{H}$:

$$\widetilde{H} = \boldsymbol{\sigma}\cdot\widetilde{\mathbf{B}} = \lambda\begin{pmatrix}\cos\widetilde{\theta} & \sin\widetilde{\theta}\\ \sin\widetilde{\theta} & -\cos\widetilde{\theta}\end{pmatrix}, \tag{5}$$

$\phi_1 = \cos\widetilde{\theta}/2|0\rangle + \sin\theta/2|1\rangle$ or $\phi_2 = -\sin\theta/2|0\rangle + \cos\widetilde{\theta}/2|1\rangle$. Taking the gauge transformation as

$$|\Psi(r,t)\rangle = \mathcal{S}(t)|\Phi(r,t)\rangle, \quad S(t) = \exp(-i\sigma_x\omega_1 t/2), \tag{6}$$

the time-independent Hamiltonian (5) is changed to the time-dependent one:

$$H(t) = \mathcal{S}(t)\widetilde{H}\mathcal{S}^\dagger(t) + i\dot{\mathcal{S}}(t)\mathcal{S}^\dagger(t). \tag{7}$$

The evolution operator $U(t) = \exp(-i\sigma_x\omega_1 t/2)\exp(-i\widetilde{H}t)$, corresponding to the time-dependent Hamiltonian

$$H(t) = \lambda\begin{pmatrix}\cos\widetilde{\theta}\cos(\omega_1 t) & \sin\widetilde{\theta} - \omega_1/2\lambda + i\cos\widetilde{\theta}\sin(\omega_1 t)\\ \sin\widetilde{\theta} - \omega_1/2\lambda - i\cos\widetilde{\theta}\sin(\omega_1 t) & -\cos\widetilde{\theta}\cos(\omega_1 t)\end{pmatrix},$$

is written as

$$U_1(t) = \begin{pmatrix}\cos(\omega_1 t/2) & -i\sin(\omega_1 t/2)\\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2)\end{pmatrix}\begin{pmatrix}\exp(-i\lambda t) & 0\\ 0 & \exp(i\lambda t)\end{pmatrix}. \tag{8}$$

The time evolution matrix $U(t)$ is the universal one-qubit gate, which is controlled by the time-dependent magnetic field parameters ω_1 and λ.

An important one-bit transformation is the operation of negation or inversion operation $\mathsf{NOT} = \sigma_x$. The gate NOT can be obtained from (8) at $\omega_1 t = \pi$ and $\lambda t = 2n\pi$ and then after multiplication of the result by i:

$$\mathsf{NOT} = iU_1(\omega_1 t = \pi, \lambda t = 2n\pi) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{9}$$

The transformation NOT exchanges $|0\rangle$ and $|1\rangle$, e.g., $\mathsf{NOT}(a|0\rangle + b|1\rangle) = a|1\rangle + b|0\rangle$. Another special one-qubit gate can be obtained from (8) at $\omega_1 t = \pi$ and $\lambda t = \pi/2$ and after multiplication of the result by i:

$$\mathsf{Y} = iU_1(\omega_1 t = \pi, \lambda t = \pi/2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y. \tag{10}$$

The special gate Z is obtained from (8) at $\omega_1 t = 4\pi$ and $\lambda t = \pi/2$ and after multiplication by i:

$$\mathsf{Y} = iU_1(\omega_1 t = 4\pi, \lambda t = \pi/2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z. \tag{11}$$

Now let us obtain another important single-bit transformation. It is the Hadamard transformation defined by

$$\mathsf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z). \tag{12}$$

When applied to $|0\rangle$ and to $|1\rangle$, H creates the superposition of states with the equal probability

$$\mathsf{H}|0\rangle = H \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = H \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

If the initial state of the qubit is $|0\rangle$, then the evolution matrix $U(t)$ corresponding to the time-dependent Hamiltonian (8) is written as

$$U(t) = \exp(-i\sigma_x \omega_1 t/2) \exp(-i\sigma_z \lambda t) \exp(-i\sigma_y \widetilde{\theta}/2) = \\ = \begin{pmatrix} \cos(\omega_1 t/2) & -i\sin(\omega_1 t/2) \\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2) \end{pmatrix} \begin{pmatrix} \exp(-i\lambda t) & 0 \\ 0 & \exp(i\lambda t) \end{pmatrix} \times \\ \times \begin{pmatrix} \cos(\widetilde{\theta}/2) & -\sin(\widetilde{\theta}/2) \\ \sin(\widetilde{\theta}/2) & \cos(\widetilde{\theta}/2) \end{pmatrix}. \tag{13}$$

At $t = 0$, $\widetilde{\theta} = \pi/2$ and any ω_1, λ, from (13) we obtain the gate

$$U(\omega_1, \lambda; t = 0, \widetilde{\theta} = \pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{14}$$

To obtain the Hadamard gate, we multiply NOT by the gate $U(\omega_1, \lambda; t = 0, \widetilde{\theta} = \pi/2)$. Therefore, the Hadamard gate H is a result of the sequence of two transformations:

$$\mathsf{H} = iU_1(\pi, 2\pi n, \widetilde{\theta} = 0) U(\omega_1, \lambda, \widetilde{\theta} = \pi/2; t = 0). \tag{15}$$

Here $U_1(t) = U(t; \widetilde{\theta} = 0)$ was used. Applied to n bits, H generates superposition of all 2^n possible states, which can be considered as a binary representation of the numbers from 0 to $2^n - 1$:

$$(H \otimes H \otimes \ldots \otimes H)|00\ldots0\rangle = \\ = \tfrac{1}{\sqrt{2^n}} \left((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes \ldots \otimes (|0\rangle + |1\rangle) \right) = \tfrac{1}{\sqrt{2^n}} \textstyle\sum_{k=0}^{2^n-1} |j_k\rangle. \tag{16}$$

1.1. Construction of Two-Qubit Gates. The 2nd order matrices $\mathcal{U}_i(2 \times 2)$ transform one-qubit states. The 4th order matrices $\mathcal{U}_j(2^2 \times 2^2)$ transform couples of one-qubit states. There are four basis states in 4th dimension Hilbert space for two-qubit systems building on one-qubit states $|0\rangle, |1\rangle$:

$$\{|00\rangle = |0\rangle \otimes |0\rangle, |01\rangle = |0\rangle \otimes |1\rangle, |10\rangle = |1\rangle \otimes |0\rangle, |11\rangle = |1\rangle \otimes |1\rangle\},$$

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any two-qubit state can be expressed as a superposition of these basis states:

$$|\Psi\rangle = c_{00}|00\rangle + c_{10}|10\rangle + c_{01}|01\rangle + c_{11}|11\rangle, \tag{17}$$

where $|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$.

Entanglement. A gate G is said to be entangling, if $|\Psi\rangle = G|\psi_1\rangle \otimes |\psi_2\rangle$ is not decomposable as a tensor product of two one-qubit states. If in (17) $c_{00}c_{11} - c_{01}c_{10} \neq 0$, then $|\Psi\rangle$ is an entangled state. The property $|\Psi_{12}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$ is called entanglement. In our case the entanglement operator is obtained from two independent systems with the use of unitary gauge time-dependent transformations, which lead to time-dependent periodic operators and entanglement of states.

One of the important two-qubit gates is the Controlled NOT=CNOT gate, which can be defined by

$$\mathsf{CNOT} = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes \sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{18}$$

1.2. Construction of the Hamiltonian with the Desired Entangled Operator. Let

$$H = h \otimes 1 + 1 \otimes h + \epsilon A, \tag{19}$$

where $\epsilon \in \{0, 1\}$ and h is a two-dimensional diagonal time-independent Hamiltonian in the form $h = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. The evolution operator of the matrix Schrödinger equation (4) with the Hamiltonian (19) is expressed as follows:

$$U(t) = (\mathrm{e}^{-iht} \otimes \mathrm{e}^{-iht})\mathrm{e}^{-iAt},$$

if the operator A commutes with the Hamiltonian $h \otimes 1 + 1 \otimes h$. We would like to get the entanglement operator $U(t)$ and to construct a corresponding Hamiltonian in the form (19). To this end, let us select the operator $R(t) = \mathrm{e}^{-iAt}$ in the form

$$R(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & -i\sin(t) & 0 \\ 0 & -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

Find $A(t)$ from

$$A = i\frac{dR(t)}{dt}R^{-1}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (21)$$

The matrix $h = \sigma_3/2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies the condition of commutation $[A, (h \otimes 1 + 1 \otimes h)]$. At last, substitution of e^{-iAt} and h into the evolution matrix $U(t)$ gives the entanglement operator

$$U(t) = \begin{pmatrix} \mathrm{e}^{it} & 0 & 0 & 0 \\ 0 & \cos(t) & -i\sin(t) & 0 \\ 0 & -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-it} \end{pmatrix}.$$

So, the entanglement operator has been obtained with the use of the unitary time-dependent transformation (20), which leads to the time-dependent periodic operator $U(t)$ and entanglement of states. We obtain the corresponding Hamiltonian (19) with A as given in (21).

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REFERENCES

1. *Bryglinski J. L., Bryglinski R.* Universal Quantum Gates in Mathematics of Quantum Computation. Boca Ratton, Florida: Chapman and Hall/ CRC Press, 2002.
2. *Kauffman L. H.* Braiding Operators are Universal Quantum Gates. quant-ph/0401090. 2004.
3. *Radtke T., Fritzshe S.* // Comp. Phys. Commun. 2005. V. 173. P. 91.
4. *Suzko A. A.* // Phys. Lett. A. 2003. V. 308. P. 267.