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DISCRETIZATION AND ITS PROOF FOR NUMERICAL SOLUTION OF A CAUCHY PROBLEM FOR LAPLACE EQUATION WITH INACCURATELY GIVEN CAUCHY CONDITIONS ON AN INACCURATELY DEFINED ARBITRARY SURFACE

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A Cauchy problem for Laplace equation with inaccurately given Cauchy conditions on an inaccurately defined arbitrary surface is considered. Discretization was performed and proved to obtain numerical solution. An economic algorithm is proposed.

Рассмотрена проблема Коши для уравнения Лапласа с нестрого определенными условиями Коши на нестрого заданной произвольной поверхности. Проводится дискретизация, которая затем используется для получения численного решения. Представлен прикладной алгоритм.

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FORMULATION OF THE PROBLEM

A mixed version of Cauchy problem for Laplace equation in a rectangular cross section cylinder bounded by an arbitrary surface and a plane is considered. Boundary conditions of the first kind are given on lateral sides, Cauchy conditions are given on the arbitrary surface:

$$\begin{gathered} \Delta u(M) = 0, \quad M \in D(F,H), \\ u|_S = f, \quad \frac{\partial u}{\partial n}|_S = g, \\ u|_{x=0} = 0, \;\; u|_{x=l_x} = 0, \;\; u|_{y=0} = 0, \;\; u|_{y=l_y} = 0, \end{gathered} \tag{1}$$

where

$$\begin{gathered} D(F,H) = \{(x,y,z) : 0 < x < l_x,\, 0 < y < l_y,\, F(x,y) < z < H\}, \\ S = \{(x,y,z) : 0 < x < l_x,\, 0 < y < l_y,\, z = F(x,y)\}, \\ F \in C^2(\Pi),\, \Pi(z) = \{(x,y,z) : 0 < x < l_x,\, 0 < y < l_y,\, z = \text{const}\}. \end{gathered} \tag{2}$$

As we are given Cauchy conditions on the surface S, this mixed problem is close to Cauchy problem and thus it is ill-posed [1]. Note that the surface S, where Cauchy conditions are given, is described by the equation $z = F(x, y)$, where F is an arbitrary differentiable function, which does not permit the use of Fourier method to solve the problem (1).

A method applicable to a wide range of problems described by elliptic equations is proposed in [1]. The method is based on reducing the original problem to an integral equation of the first kind with the right-hand side of the integral equation being an integral of given functions over surface S. On the one hand, this makes it possible to get the exact solution in explicit form, and to use Tikhonov regularization to obtain stable solution on the other hand.

Keeping in mind applications of the problem, we consider the surface S as well as Cauchy conditions f and g on that surface being both measurement data, that is, being given approximately. Thus we have approximate functions $f^\delta, g^\delta, F^\mu$ instead of exact functions f, g and F such that

$$\|\, f^\delta - f \,\| \leqslant \delta, \quad \|\, g^\delta - g \,\| \leqslant \delta, \tag{3}$$

$$\|F^\mu - F\| \leqslant \mu. \tag{4}$$

In case when the surface S is given approximately, the right-hand side of the integral equation requires calculating the normal to this surface, or, actually, the gradient of the function F^μ, which is an ill-posed problem of differentiation of an inaccurately given function. A stable method of its solution is based upon Morozov approach to the problem of an unbounded operator reconstruction [2]. As an approximate value of the gradient of F^μ we will take the gradient of the extremal of Tikhonov functional, the extremal is obtained as a Fourier double series [3]:

$$W^\mu_\beta(x, y) = \sum_{n,m=1}^{\infty} \frac{\tilde{F}^\mu_{nm}}{1 + \beta\left[\left(\dfrac{\pi n}{l_x}\right)^2 + \left(\dfrac{\pi m}{l_y}\right)^2\right]} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \tag{5}$$

β is the regularization parameter. Accordingly,

$$\nabla_{xy} W^\mu_\beta(x, y) = \sum_{n,m=1}^{\infty} \frac{\tilde{F}^\mu_{nm}}{1 + \beta\left[\left(\dfrac{\pi n}{l_x}\right)^2 + \left(\dfrac{\pi m}{l_y}\right)^2\right]} \times \\ \times \left(\mathbf{i}\frac{\pi n}{l_x} \cos\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} + \mathbf{j}\frac{\pi m}{l_y} \cos\frac{\pi m y}{l_y} \sin\frac{\pi n x}{l_x}\right). \tag{6}$$

According to [1] stable approximate solution of the problem (1) can be presented as

$$u^{\delta,\mu}_\alpha(M) = v^{\delta,\mu}_\alpha(M) - \Phi^{\delta,\mu}(M), \quad M \in D(F, H), \tag{7}$$

$D(F, H)$ is given by (2), $\Phi^{\delta,\mu}$ is given by

$$\Phi^{\delta,\mu}(M) = -\int\limits_{\Pi(0)} \Big[g^\delta(P)\varphi(M, P) n^\mu_1(P) - \\ - f^\delta(P)(\nabla_P \varphi(M, P), \mathbf{n}^\mu_1(P))\Big]_{P = P(x, y, W^\mu_\beta) \in S^\mu} dx_P dy_P, \tag{8}$$

$\mathbf{n}_1^\mu = \mathbf{n}_{1,\beta(\mu)}^\mu = \nabla_{xy} W_{\beta(\mu)}^\mu - \mathbf{k}$, $\beta(\mu) = \mathrm{const}\,\mu$, $n_1^\mu = |\mathbf{n}_1^\mu|$, W_β^μ is given by (5),

$$\varphi(M,P) = \frac{2}{\pi l_x l_y} \sum_{n,m=1}^{\infty} \frac{\mathrm{e}^{-\pi\sqrt{\frac{n^2}{l^2}+\frac{m^2}{l^2}}|z_M - z_P|}}{\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}} \times \\ \times \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y}, \quad (9)$$

function $v_\alpha^{\mu,\delta}$ is given by

$$v_\alpha^{\mu,\delta}(M) = \sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}^{\mu,\delta}(a)\,\mathrm{e}^{\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(z_M - a)}}{1 + \alpha\,\mathrm{e}^{2\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(H-a)}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}. \quad (10)$$

Here $\tilde{\Phi}_{nm}^{\mu,\delta}(a)$ are Fourier coefficients of function $\Phi^{\mu,\delta}(M)|_{M\in\Pi(a)}$:

$$\tilde{\Phi}_{nm}^{\mu,\delta}(a) = \frac{4}{l_x l_y} \int_0^{l_x} dx \int_0^{l_y} dy\ \ \Phi^{\delta,\mu}(x,y,a) \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \quad (11)$$

α is the regularization parameter. According to (2) notation, the a value is taken such that

$$a < \min_{(x,y)\in\Pi(0)} F(x,y).$$

The theorem of uniform convergence of the approximate stable solution of the problem (1) to the exact one is proved [1].

NUMERICAL SOLUTION

Here we will take a close look at the discretization of the problem (1) to obtain numerical solution in case when the surface S and the Cauchy conditions f and g are given approximately (3), (4).

Let the rectangle $\Pi(0)$ given by (2) be covered with uniform grid $(N_x+1)\times(N_y+1)$ such that

$$x_i = i\frac{l_x}{N_x},\ i = 0,\dots,N_x, \\ y_j = j\frac{l_y}{N_y},\ j = 0,\dots,N_y, \quad (12)$$

then the Fourier series

$$\Phi(x,y) = \sum_{n,m=1}^{\infty} \tilde{\Phi}_{nm} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}$$

substituted for the sum

$$\Phi(x_i, y_j) = \sum_{n=1}^{N_x-1} \sum_{m=1}^{N_y-1} \tilde{\Phi}_{nm}^N \sin \frac{\pi n x_i}{l_x} \sin \frac{\pi m y_j}{l_y}, \quad i = 0, \dots, N_x, \quad j = 0, \dots, N_y,$$

and the Fourier coefficients

$$\tilde{\Phi}_{nm} = \frac{4}{l_x l_y} \int_0^{l_x} dx \int_0^{l_y} dy \;\; \Phi(x, y) \sin \frac{\pi n x}{l_x} \sin \frac{\pi m y}{l_y}$$

are calculated as

$$\tilde{\Phi}_{nm}^N = \frac{4}{N_x N_y} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} \Phi(x_i, y_j) \sin \frac{\pi n x_i}{l_x} \sin \frac{\pi m y_j}{l_y}, \quad n = 1, \dots, N_x, \quad m = 1, \dots, N_y.$$

Norms (3) and (4) are regarded as finite sums, functions $f, g, F, f^\delta, g^\delta, F^\mu$ are regarded as traces of continuous functions on the grid (12).

Substitute the function φ (9) in (8) for its finite sum — function $\varphi^N(M, P)$, substitute the integral (8) itself for composite trapezium formula on grid (12) over $\Pi(0)$:

$$\Phi^{\delta,\mu,NT}(M) = -\frac{l_x l_y}{N_x N_y} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} \Big[g^\delta(x_i, y_j) \varphi^N(M, P_{ij}^\mu) n_1^\mu(x_i, y_j) - \\ - f^\delta(x_i, y_j) (\nabla_P \varphi^N(M, P_{ij}^\mu), \mathbf{n}_1^\mu(x_i, y_j)) \Big], \;\; P_{ij}^\mu = (x_i, y_j, W_\beta^\mu(x_i, y_j)), \quad (13)$$

$$W_\beta^\mu(x_i, y_j) = \sum_{n=1}^{N_x-1} \sum_{m=1}^{N_y-1} \frac{\tilde{F}_{nm}^\mu}{1 + \beta \left[\left(\frac{\pi n}{l_x} \right)^2 + \left(\frac{\pi m}{l_y} \right)^2 \right]} \sin \frac{\pi n x_i}{l_x} \sin \frac{\pi m y_j}{l_y}. \quad (14)$$

Omitting the proof we get the estimate $|\Phi^{\delta,\mu,NT}(M) - \Phi(M)|$:

$$|\Phi^{\delta,\mu,NT}(M) - \Phi(M)|_{M \in \Pi(a)} \leqslant C_1 \sqrt{\mu} + C_2 \delta + C_3 N_x^{-2} + C_4 N_y^{-2} + \\ + C_5 \, \mathrm{e}^{-\pi d \, \min[N_x/l_x, N_y/l_y]} \, \min[N_x/l_x, N_y/l_y] = \\ = \Delta(\mu, \delta, N_x, N_y), \quad d = \min_{(x,y)} F(x, y) - a. \quad (15)$$

The heart of the algorithm for solving the problem (1) is calculation of «discrete» Fourier coefficients of continuous function $\Phi^{\delta,\mu}$ (8):

$$\tilde{\Phi}_{nm}^{\delta,\mu,NT}(a) = \frac{4}{N_x N_y} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} \Phi^{\delta,\mu,NT}(x_i, y_j, a) \sin \frac{\pi n x_i}{l_x} \sin \frac{\pi m y_j}{l_y}, \\ n = 1, \dots, N_x - 1, \;\; m = 1, \dots, N_y - 1. \quad (16)$$

To calculate $N_x N_y$ values of function $\Phi^{\delta,\mu,NT}$, each of them being an integral calculated as a sum of $N_x N_y$ terms, we need the order of $(N_x N_y)^3$ operations, since there is a double Fourier series under the integral sign, producing $N_x N_y$ operations for every fixed pair of M and P. This is the decisive factor since we need the order of $(N_x N_y)^2$ operations to calculate Fourier coefficients having calculated function values. We can reduce the scope of work if we calculate Fourier coefficients as integrals

$$\tilde{\Phi}_{nm}^{\delta,\mu,NT}(a) = \frac{4}{l_x l_y} \int_0^{l_x} dx \int_0^{l_y} dy \;\; \Phi^{\delta,\mu,NT}(x,y,a) \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \tag{17}$$

and perform the integration in (17) under the sum sign in (13), and after that integrate the function φ^N and its derivatives series term by term. With regard to orthogonality of sines we have

$$\begin{aligned}
\tilde{\Phi}_{nm}^{\delta,\mu,NT}(a) = {} & \frac{n}{l_x\sqrt{\dfrac{n^2}{l_x^2}+\dfrac{m^2}{l_y^2}}} \frac{2}{N_x N_y} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} f^\delta(x_i,y_j) n_{1,x}^\mu(x_i,y_j) \times \\
& \times \mathrm{e}^{-\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(W_\beta^\mu(x_i,y_j)-a)} \cos\frac{\pi n x_i}{l_x} \sin\frac{\pi m y_j}{l_y} + \\
& + \frac{m}{l_y\sqrt{\dfrac{n^2}{l_x^2}+\dfrac{m^2}{l_y^2}}} \frac{2}{N_x N_y} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} f^\delta(x_i,y_j) n_{1,y}^\mu(x_i,y_j) \times \\
& \times \mathrm{e}^{-\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(W_\beta^\mu(x_i,y_j)-a)} \sin\frac{\pi n x_i}{l_x} \cos\frac{\pi m y_j}{l_y} + \\
& + \frac{2}{N_x N_y} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} f^\delta(x_i,y_j)\, \mathrm{e}^{-\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(W_\beta^\mu(x_i,y_j)-a)} \sin\frac{\pi n x_i}{l_x} \sin\frac{\pi m y_j}{l_y} - \\
& - \frac{1}{\pi\sqrt{\dfrac{n^2}{l_x^2}+\dfrac{m^2}{l_y^2}}} \frac{2}{N_x N_y} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} g^\delta(x_i,y_j) n_1^\mu(x_i,y_j) \times \\
& \times \mathrm{e}^{-\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(W_\beta^\mu(x_i,y_j)-a)} \sin\frac{\pi n x_i}{l_x} \sin\frac{\pi m y_j}{l_y},
\end{aligned} \tag{18}$$

where $\mathbf{n}_1^\mu = (n_{1,x}^\mu, n_{1,y}^\mu, -1)$ is given by $\nabla_{xy} W_\beta^\mu - \mathbf{k}$; W_β^μ is given by (14) and $n_1^\mu = |\mathbf{n}_1^\mu| = \sqrt{1 + (n_{1,x}^\mu)^2 + (n_{1,y}^\mu)^2}$.

Note that $\Phi^{\delta,\mu,NT}(a) = 0$ for $n > N_x$, $m > N_y$, since the series $\varphi(M,P)$ (9) is substituted for the finite sum φ^N.

Also note that to calculate Fourier coefficients by (18) we need the order of $(N_x N_y)^2$ operations.

Discrete approximate solution $u_\alpha^{\delta,\mu,NT}$ of the problem (1) is given by

$$u_\alpha^{\delta,\mu,NT}(x_i,y_j,z)=v_\alpha^{\delta,\mu,NT}(x_i,y_j,z)-\Phi^{\delta,\mu,NT}(x_i,y_j,z),\quad (x_i,y_j,z)\in D(W_\beta^\mu,H),\tag{19}$$

$$v_\alpha^{\mu,\delta,NT}(x_i,y_j,z)=\sum_{n,m=1}^{\infty}\frac{\tilde{\Phi}_{nm}^{\mu,\delta,NT}(a)\,\mathrm{e}^{\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(z-a)}}{1+\alpha\,\mathrm{e}^{2\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(H-a)}}\sin\frac{\pi n x_i}{l_x}\sin\frac{\pi m y_j}{l_y}=$$
$$=\sum_{n=1}^{N_x-1}\sum_{m=1}^{N_y-1}\frac{\tilde{\Phi}_{nm}^{\mu,\delta,NT}(a)\,\mathrm{e}^{\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(z-a)}}{1+\alpha\,\mathrm{e}^{2\pi\sqrt{\frac{n^2}{l_x^2}+\frac{m^2}{l_y^2}}(H-a)}}\sin\frac{\pi n x_i}{l_x}\sin\frac{\pi m y_j}{l_y},\tag{20}$$

function $\Phi^{\delta,\mu,NT}$ is calculated by (13).

The theorem of uniform convergence of the discrete approximate stable solution of the problem (1) to the exact one is proved:

Theorem. *Let the solution of the problem (1) exist in $D(H,F)$, $\alpha=\alpha(\Delta)$, $\alpha(\Delta)\to 0$, $\Delta/\sqrt{\alpha(\Delta)}\to 0$ as $\Delta\to 0$. Then the function $u_{\alpha(\Delta)}$ given by (19), where according to (15), $\Delta=\Delta(\mu,\delta,N_x,N_y)=C_1\sqrt{\mu}+C_2\delta+C_3N_x^{-2}+C_3N_y^{-2}+C_5\,\mathrm{e}^{-\pi d\min[N_x/l_x,N_y/l_y]}$ $\min[N_x/l_x,N_y/l_y]$, converges uniformly to the exact solution of the problem (1) as $\delta\to 0$, $\mu\to 0$, $N_x\to\infty$, $N_y\to\infty$ in $D(F+\varepsilon,H-\varepsilon)$, where $\varepsilon>0$ is some fixed number as small as is wished.*

CONCLUSIONS

Research results may be applied to obtain numerical solutions of problems described by harmonic functions, for example, to a problem of a stationary temperature field analytical continuation toward its sources with the purpose of the sources identification [4].

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