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КОМПЬЮТЕРНЫЕ ТЕХНОЛОГИИ В ФИЗИКЕ

PROPERTIES OF GENERALIZED MATRIX SEQUENCE

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In case of a block-tridiagonal matrix, the problem of calculation of generalized double-point matrix sequence is examined. The general form of an inverse matrix of the bordered matrix is obtained when the initial matrix is singular. The criterion of existence of the generalized matrix sequence is found, and the algorithm of calculation of the sequence and the structure elements of the block-tridiagonal matrices is given.

Исследуется проблема вычислений обобщенной двухточечной матричной последовательности в случае блочно-трехдиагональной матрицы. В общей форме получена обратная матрица для окаймленной матрицы для случая, когда исходная матрица сингулярна. Найден критерий существования обобщенной матричной последовательности, и представлен алгоритм вычисления последовательности и структурных элементов блочно-трехдиагональных матриц.

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INTRODUCTION

Consider the matrix sequence

$$\Lambda_{i+1} = q_i - p_i \Lambda_i^{-1} r_i, \quad \Lambda_2 = q_1, \quad i = 2, \ldots, m, \tag{1}$$

where $\{q_i\}_{i=1}^m$ are quadratic-diagonal and $\{p_i, r_i\}_{i=2}^m$ are sub (off)-diagonal elements of the block-tridiagonal matrix in the form:

$$C = \begin{bmatrix} q_1 & r_2 & & \\ p_2 & q_2 & \ddots & \\ & \ddots & \ddots & r_m \\ & & p_m & q_m \end{bmatrix}. \tag{2}$$

Here, the orders of sub (off)-diagonal elements are defined by respective diagonal blocks q_{i-1} and q_i.

Let n_i be the orders of matrices q_i. We are interested in the problem of existence of sequence (1) when some of Λ_i are singularities, i.e., $\det(\Lambda_i) = 0$. It is known that the properties of elements Λ_i are connected with the properties of principal upper angular minors of matrix C (2). Consequently, when the singularity of Λ_i appears, the existence of sequence (1) will depend on the invertibility of the next bordered matrix:

$$\Lambda_i^{i+1} = \begin{bmatrix} \Lambda_i & r_i \\ p_i & q_i \end{bmatrix}. \tag{3}$$

1. METHOD OF CALCULATION OF THE GENERALIZED MATRIX SEQUENCE

It is known that in the case of nonsingularity of matrix Λ_i, the invertibility of matrix (3) is due to nonsingularity of matrix $q_i - p_i\Lambda_i^{-1}r_i$ [1, 2]. We consider certain cases with nonsingular Λ_i.

If the orders of submatrices q_{i-1} and q_i are equal, i.e., $n_{i-1} = n_i$, then the invertibility of one of the next submatrices $\Lambda_i - r_i q_i^{-1} p_i$, $r_i - \Lambda_i p_i^{-1} q_i$, and $p_i - q_i r_i^{-1}\Lambda_i$ is the invertibility criterion of matrix (3). All these cases reduce to the Frobenius Theorem [1].

Let now matrices p_i and r_i have the general form, i.e., submatrices q_{i-1} and q_i have different orders and $\det(q_i) = 0$. Then the next theorem takes place.

Theorem 1. The necessary and sufficient condition for invertibility of the general form of matrix (3) is the invertibility of the matrix $F_i = Q_i - P_i Q_{i-1}^{-1} P_i^T$, where $Q_i = q_i^T q_i + r_i^T r_i$, $P_i = r_i^T \Lambda_i + q_i^T p_i$, $Q_{i-1} = \Lambda_i^T \Lambda_i + p_i^T p_i$, T is the symbol of transposition. Here

$$\begin{bmatrix} \Lambda_i & r_i \\ p_i & q_i \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} E & -Q_{i-1}^{-1}P_i^T \\ & E \end{bmatrix} \begin{bmatrix} Q_{i-1}^{-1} & \\ & F_i^{-1} \end{bmatrix} \begin{bmatrix} E & \\ -P_i Q_{i-1}^{-1} & E \end{bmatrix} \begin{bmatrix} \Lambda_i^T & r_i^T \\ p_i^T & q_i^T \end{bmatrix}. \qquad (4)$$

Denote the elements of the inverse matrix for (3) by $\omega_{i-1}, \beta_i, c_i$ and ω_i, i.e.,

$$\begin{bmatrix} \Lambda_i & r_i \\ p_i & q_i \end{bmatrix}^{-1} = \begin{bmatrix} \omega_{i-1} & c_i \\ \beta_i & \omega_i \end{bmatrix}.$$

For example, consider the estimation of the next perturbation analysis [3–5]. For matrix (3) such type of estimation has the form:

$$||(\tilde{\Lambda}_i^{i+1})^{-1} - (\Lambda_i^{i+1})^{-1}|| \leqslant \max(||\omega_{i-1}||, ||\beta_i||)\max(\mu, ||\theta_{i-1}c_i||)\frac{1}{1-\mu},$$

where

$$\mu = ||E - \tilde{\Lambda}_i^{i+1}(\Lambda_i^{i+1})^{-1}||, \quad \tilde{\Lambda}_i^{i+1} = \begin{bmatrix} \Lambda_i + \theta_{i-1} & r_i \\ p_i & q_i \end{bmatrix}$$

is the perturbation matrix obtained instead of (3), as a result of computation.

Indeed, the matrix consequence

$$B_j = B_{j-1}(E - \Theta_{j-1}), \quad \Theta_{j-1} = E - \tilde{\Lambda}_i^{i+1} B_{j-1}, \quad B_0 = (\Lambda_i^{i+1})^{-1}, \quad j = 1, 2, \ldots$$

tends to $(\tilde{\Lambda}_i^{i+1})^{-1}$ at $||\Theta_0|| \equiv \mu < 1$.

Having expressed Θ_{j-1} through Θ_0 we receive

$$B_j = B_0 - B_0(\Theta_0 + \Theta_0^2 + \ldots),$$

where

$$\Theta_0 = \begin{bmatrix} \theta_{i-1}\omega_{i-1} & \theta_{i-1}c_i \\ 0 & 0 \end{bmatrix}.$$

Hence,

$$\Theta_0 + \Theta_0^2 + \ldots = \begin{bmatrix} E + \theta_{i-1}\omega_{i-1} + (\theta_{i-1}\omega_{i-1})^2 + \ldots & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{i-1}\omega_{i-1} & \theta_{i-1}c_i \\ 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} (E - \theta_{i-1}\omega_{i-1})^{-1} & \theta_{i-1}c_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{i-1}\omega_{i-1} & \theta_{i-1}c_i \\ 0 & 0 \end{bmatrix}.$$

So then,

$$||B_j - (\Lambda_i^{i+1})^{-1}|| = \left|\left| \begin{bmatrix} \omega_{i-1} \\ \beta_i \end{bmatrix} (E - \theta_{i-1}\omega_{i-1})^{-1} [\theta_{i-1}\omega_{i-1},\ \theta_{i-1}c_i] \right|\right| \leqslant$$

$$\leqslant \max(||\omega_{i-1}||, ||\beta_i||) \max(\mu, ||\theta_{i-1}c_i||) \frac{1}{1-\mu}.$$

The invertibility of matrix Λ_i leads to the continuous existence of consequence (1), i.e., the subsequent elements of this consequence will be functions of Λ_i. When $\det(\Lambda_i) = 0$, the continuity will depend on nonsingularity of Λ_i^{i+1} (3).

Theorem 2. Let in consequence (1) the Λ_i be singular. Then the criterion of continuous existence of (1) will be the invertibility of matrix (3) and

$$\Lambda_{i+2} = q_{i+1} - p_{i+1}\omega_i r_{i+1}. \tag{5}$$

In case of existence of consequence with discontinuation, i.e., singularity of matrix (3), necessary and sufficient condition of invertibility of matrix C (2) will be the invertibility of the induced matrix of the following form:

$$C_{\text{ind}} = \bar{\Lambda}_i^{i+1} = \begin{bmatrix} \Lambda_i & r_i \\ p_i & q_i - \theta_i \end{bmatrix}, \tag{6}$$

where $\theta_i = r_{i+1}(Q_{i+1}^m)^{-1}p_{i+1},\ Q_{i+1}^m$ is an submatrix of C (2).

In case of singularity of matrix Q_{i+1}^m and at $\det(\Lambda_j) = 0 (i < j \leqslant m+1)$ we have the next matrix-factorized decomposition [6]:

$$C = \begin{bmatrix} E & & & & \\ p_{i-1}(Q_1^i)^{-1} & E & r_{i+1}(Q_{i+2}^j)^{-1} & & \\ & & E & & \\ & & p_{j-1}(Q_{i+2}^j)^{-1} & E & \\ & & & & E \end{bmatrix} \times$$

$$\times \begin{bmatrix} Q_1^i & & & & \\ & \bar{\Lambda}_i^{i+1} & 0 & -r_{i+1}(Q_{i+2}^j)^{-1}r_{j-1} & \\ & 0 & Q_{i+2}^j & 0 & \\ & -p_{j-1}(Q_{i+2}^j)^{-1}p_{i+1} & 0 & \Lambda_j^{j+1} & r_{j+1} \\ & & & p_{j+1} & Q_{j+2}^m \end{bmatrix} \times$$

$$\times \begin{bmatrix} E & (Q_1^i)^{-1}r_{i-1} & & & \\ & E & & & \\ & (Q_{i+2}^j)^{-1}p_{i+1} & E & (Q_{i+2}^j)^{-1}r_{j-1} & \\ & & & E & \\ & & & & E \end{bmatrix},$$

where $p_{k-1}(Q_l^k)^{-1}$ and $(Q_l^k)^{-1}r_{k-1}$ are the matrix half-lines and half-columns, the dimensions of which be detemined by appropriate diagonal blocks Q_l^k and q_{k-1}; $r_{i+1}(Q_{i+2}^j)^{-1}r_{j-1}$ and $p_{j-1}(Q_{i+2}^j)^{-1}p_{i+1}$ are the matrix elements whose elements will depend on appropriate diagonal elements q_i and q_{j-1}.

Let Λ_j^{j+1} be singular. If the submatrix Q_1^j of matrix C is nonsingular, then by virtue of Theorem 2 the matrix $\bar{\Lambda}_i^{i+1}$ also is nonsingular and here

$$\Lambda_j = q_{j-1} - p_{j-1}(Q_1^j)^{-1}r_{j-1} = \\ = q_{j-1} - p_{j-1}(\Lambda_{j-1}^{-1} + (Q_{i+2}^j)^{-1}p_{i+1}(\bar{\Lambda}_i^{i+1})^{-1}r_{i+1}(Q_{i+2}^j)^{-1})r_{j-1}$$

may be nonsingular. In case, when Λ_j is singular, then $C_{\text{ind}} = \bar{\Lambda}_j^{j+1}$. This type of discontinuation of consequence (1) at the point $(i+1)$ is named the *II type of discontinuation*.

If the submatrix Q_1^j is singular, then $\bar{\Lambda}_i^{i+1}$ will be singular. Then the induced matrix will have the following form:

$$C_{\text{ind}} = \begin{bmatrix} \bar{\Lambda}_i^{i+1} & r_{i+1}(Q_{i+2}^j)^{-1}r_{j-1} \\ p_{j-1}(Q_{i+2}^j)^{-1}p_{i+1} & \bar{\Lambda}_j^{j+1} \end{bmatrix},$$

where $\bar{\Lambda}_i^{i+1}$ and $\bar{\Lambda}_j^{j+1}$ are nonsingular matrices. This type of discontinuation of consequence (1) at the point $(i+1)$ is named the *I type of discontinuation.*

2. ALGORITHM AND RESULTS

For the nonsingular block-tridiagonal matrix C (2), if its principal upper angular n_{i-1}-order minors vanish, for any i from $(2 \leqslant i \leqslant m)$, then the corresponding element Λ_i of the consequence will be singular [2, 3, 7, 8]. As a corollary of Theorem 2, the following matrix-factorization decompositions for C take place:

$$C = \begin{bmatrix} E & & & & & & & & \\ -\beta_2 & E & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & -\beta_{i-1} & E & & & & & \\ & & & 0 & E & & & & \\ & & & -\beta_{i+1i} & -\tilde{\beta}_{i+1} & E & & & \\ & & & & & -\beta_{i+2} & E & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & -\beta_m & E \end{bmatrix} \times$$

$$\times \begin{bmatrix} \Lambda_2 & & & & & & & \\ & \ddots & & & & & & \\ & & \Lambda_{i-1} & & & & & \\ & & & \Lambda_i & r_i & & & \\ & & & p_i & q_i & & & \\ & & & & & \Lambda_{i+2} & & \\ & & & & & & \ddots & \\ & & & & & & & \Lambda_m \end{bmatrix} \times$$

$$\times \begin{bmatrix} E & -c_2 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & E & -c_{i-1} & & & & & \\ & & & E & 0 & -c_{ii+1} & & & \\ & & & & E & -\tilde{c}_{i+1} & & & \\ & & & & & E & -c_{i+2} & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & E & -c_m \\ & & & & & & & & E \end{bmatrix}, \quad (7)$$

where

$$\beta_j = -p_j \Lambda_j^{-1}, \quad c_j = -\Lambda_j^{-1} r_j, \quad j = 2, \ldots, \quad i-1, i+2, \ldots m,$$
$$\tilde{\beta}_{i+1} = -p_{i+1}\omega_i, \quad \tilde{c}_{i+1} = -\omega_i r_{i+1},$$

and when $\det(\Lambda_i) = 0$, we attain $\beta_{i+1} = -p_{i+1}$, $c_{i+1} = -r_{i+1}$, $\beta_{i+1i} = \beta_{i+1}\beta_i$, $c_{ii+1} = c_i c_{i+1}$.

Below we will give the *algorithm* of calculation of consequence (1) and structure elements (β, c) of matrix (7):

Start of computations:
$i = 1, \Lambda_{i+1} = q_i$;
$(1)^0$ $i = i + 1$;
If $\det(\Lambda_i) = 0$, then $(3)^0$, otherwise $(2)^0$.

$(2)^0$ $\beta_i = -p_i\Lambda_i^{-1}, c_i = -\Lambda_i^{-1}r_i, \Lambda_{i+1} = q_i - p_i\Lambda_i^{-1}r_i;$
If $i = m$, then computations are over, otherwise $(1)^0$.

$(3)^0$ If $\det(q_i) = 0$, then $(6)^0$, otherwise $(4)^0$.

$(4)^0$ $f_i = \Lambda_i - r_i q_i^{-1} p_i;$
If $\det(f_i) = 0$, then computations are over, otherwise $(5)^0$.

$(5)^0$ $\omega_i = q_i^{-1} + q_i^{-1}p_i f_i^{-1} r_i q_i^{-1}, \beta_i = -q_i^{-1}p_i f_i^{-1}, c_i = -f_i^{-1} r_i q_i^{-1};$
If $i = m$, computations are over, otherwise $(14)^0$.

$(6)^0$ If $\det(p_i) = 0$, then $(9)^0$, otherwise $(7)^0$.

$(7)^0$ $f_i = \Lambda_i p_i^{-1} q_i - r_i;$
If $\det(f_i) = 0$, then computations are over, otherwise $(8)^0$.

$(8)^0$ $\omega_i = f_i^{-1}\Lambda_i p_i^{-1}, \beta_i = -f_i^{-1}, c_i = p_i^{-1} + p_i^{-1} q_i f_i^{-1} \Lambda_i p_i^{-1};$
If $i = m$, computations are over, otherwise $(14)^0$.

$(9)^0$ If $\det(r_i) = 0$, then $(12)^0$, otherwise $(10)^0$.

$(10)^0$ $f_i = q_i r_i^{-1} \Lambda_i - p_i;$
If $\det(f_i) = 0$, then computations are over, otherwise $(11)^0$.

$(11)^0$ $\omega_i = r_i^{-1}\Lambda_i f_i^{-1}, \beta_i = r_i^{-1} + r_i^{-1}\Lambda_i f_i^{-1} q_i r_i^{-1}, c_i = -f_i^{-1};$
If $i = m$, computations are over, otherwise $(14)^0$.

$(12)^0$ $Q_{i-1} = \Lambda_i^T\Lambda_i + p_i^T p_i, \tilde{Q}_{i-1} = \Lambda_i\Lambda_i^T + r_i r_i^T;$
If $\det(Q_{i-1}) = 0$ or $\det(\tilde{Q}_{i-1}) = 0$, then computations are over, otherwise $(13)^0$.

$(13)^0$ $P_i = r_i^T\Lambda_i + q_i^T p_i, \tilde{P}_i = p_i\Lambda_i^T + q_i r_i^T;$
$Q_i = q_i^T q_i + r_i^T r_i, \tilde{Q}_i = q_i q_i^T + p_i p_i^T;$
$f_i = Q_i - P_i Q_{i-1}^{-1} P_i^T, \tilde{f}_i = \tilde{Q}_i - \tilde{P}_i \tilde{Q}_{i-1}^{-1} \tilde{P}_i^T;$
$\omega_i = f_i^{-1}(q_i^T - P_i Q_{i-1}^{-1} p_i^T), \beta_i = f_i^{-1}(r_i^T - P_i Q_{i-1}^{-1}\Lambda_i^T), c_i = (p_i^T - \Lambda_i^T \tilde{Q}_{i-1}^{-1}\tilde{P}_i^T)\tilde{f}_i^{-1};$
If $i = m$, then computations are over, otherwise $(14)^0$.

$(14)^0$ $i = i + 1, \beta_i = -p_i, c_i = -r_i, \tilde{\beta}_i = -p_i\omega_{i-1}, \tilde{c}_i = -\omega_{i-1}r_i, \Lambda_{i+1} = q_i - p_i\omega_{i-1}r_i;$
If $i = m$, then computations are over, otherwise $(1)^0$.
End of computations.

Example. Let the block-tridiagonal matrix be given

$$C = \begin{bmatrix} 1 & -1 & 1 & & & & \\ -1 & 1 & -1 & 1 & & & \\ 0 & -1 & 1 & -1 & 1 & & \\ & 0 & -1 & 1 & -1 & 1 & \\ & & 0 & -1 & 0 & 0 & 1 \\ & & & 1 & 1 & 1 & 0 \\ & & & & 0 & 1 & 0 \end{bmatrix}. \tag{8}$$

Computed values of consequence (1) and the structure elements of matrix (8) by algorithm $1^0 - 14^0$

i	Λ_{i+1}	ω_i	β_{i+1}	c_{i+1}	$\tilde{\beta}_{i+1}$	$\tilde{c}_{i+1}$
1	$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	—	$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$	—	—
2	Indef.	$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
3	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	—	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$[1 \quad 0]$	—	—
4	Indef.	0	—	—	—	—

The diagonal blocks have the orders $[2 \times 2]$ except for the last diagonal block. In this case Λ_2 and Λ_4 will be singular. Shown in the Table are the elements of consequence (1) and the structure elements of the matrix, which were computed on the base of algorithm 1^0–14^0.

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