Письма в ЭЧАЯ. 2009. Т. 6, № 5(154). С. 602–606

RADIATIVE CORRECTIONS TO MUON DECAY IN LEADING AND NEXT-TO-LEADING APPROXIMATIONS FOR ELECTRON SPECTRUM

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We have noted that the electron spectrum of muon decay in the leading logarithmic approximation calculated in two lowest orders of the perturbation theory in the paper of Berman (1958), can be reproduced by the parton language. This fact permits one to generalize the result to all orders of the perturbation theory using the structure function method.

Показано, что спектр электрона в распаде мюона в лидирующем приближении может быть воспроизведен в терминах уравнений эволюции партонных операторов твиста два. На основе результатов вычисления спектра с учетом радиационных поправок низшего порядка предложено выражение для спектра, учитывающее поправки во всех порядках, в лидирующем и следующем за ним приближениях в терминах структурных функций лептона.

PACS: 13.35.Bv, 13.40.Ks

ELECTRON SPECTRUM IN LEADING AND NEXT-TO-LEADING APPROXIMATIONS

The lowest order radiative corrections (RC) to the muon weak decay width were calculated about fifty years ago [**?**]. The result for the electron energy spectrum in muon decay including RC was obtained in the form

$$\frac{dW^{(1)}(x)}{dx} = \frac{dW_B(x)}{dx}\Big[1 + \frac{\alpha}{2\pi}h(x)\Big],\ x = \frac{E_e}{E_{\max}},\ h(x) = A(x) + LB(x),\ L = \ln\frac{M^2}{m^2} \quad (1)$$

with the spectrum in the Born approximation

$$\frac{dW_B(x)}{dx} = \frac{G^2M^5}{96\pi^3}x^2(3-2x), \quad (2)$$

here M is the muon mass; m is the electron mass; L is the so-called «large logarithm» ($L \approx 11$). The result of the lowest order RC including is presented in expression $h(x)$, or in functions $A(x)$ and $B(x)$, respectively [**?**]

$$A(x) = 4\mathrm{Li}_2(x) - \frac{2\pi^2}{3} - 4 + 2[3\ln(1-x) - 2\ln x + 1]\ln x - 2\frac{1+x}{x}\ln(1-x) + \frac{(1-x)(5+17x-16x^2)}{3x^2(3-2x)}\ln x + \frac{(1-x)(-22x+34x^2)}{3x^2(3-2x)}, \quad (3)$$

$$B(x) = 3 + 4\ln\frac{1-x}{x} + \frac{(1-x)(5+17x-34x^2)}{3x^2(3-2x)}.$$

One must remark that the result of the calculations does not suffer from the ultraviolet and the infrared divergences. Besides it satisfies Kinoshita–Lee–Nauenberg (KLN) theorem [**?**] about the cancellation of mass singularities, namely the total width is finite in the limit of zero electron mass

$$\int_0^1 dx\, \frac{dW_B(x)}{dx} B(x) = 0. \tag{4}$$

The mechanism of the realization of KLN theorem can be understood from the positions of parton interpretation of Quantum Electrodynamics (QED). Really, one can be convinced in the validity of the relation

$$\begin{gathered}\frac{1}{2}x^2(3-2x)h(x) = (L-1)\int_x^1 \frac{dy}{y} y^2(3-2y) P\left(\frac{x}{y}\right) + K(x),\\ K(x) = \frac{1}{2}(A(x)+B(x))x^2(3-2x),\end{gathered} \tag{5}$$

where

$$P(z) = \left(\frac{1+z^2}{1-z}\right)_+ = \lim_{\Delta\to 0}\left[\frac{1+z^2}{1-z}\theta(1-z-\Delta) + \left(2\ln\Delta + \frac{3}{2}\right)\delta(1-z)\right]$$

is the kernel of the evolution equation of twist two operators. Using the property $\int_0^1 dx P(x) = 0$, one can validate Eq. (**??**)

$$\begin{aligned}\frac{G^2M^5}{96\pi^3}\frac{\alpha}{2\pi}\int_0^1 dx\, x^2(3-2x)B(x) &= \frac{\alpha}{2\pi}\int_0^1 dx \int_x^1 \frac{dy}{y}\frac{dW_B(y)}{dy} P\left(\frac{x}{y}\right) = \\ &= \frac{\alpha}{2\pi}\int_0^1 dy \frac{dW_B(y)}{dy}\int_0^y \frac{dx}{y} P\left(\frac{x}{y}\right) = 0.\end{aligned}$$

Considering the process in the Born approximation as a «hard» process and applying Collins factorization theorem about the contributions of the short and long distances, one can generalize the lowest order result to include all terms of the sort $(\alpha L/\pi)^n$ (leading logarithmical approximation (LLA)) as well as the terms of the sort $\alpha(\alpha L/\pi)^n$ (next-to-leading approximation (NLO)) in the form (it is the main result of our paper)

$$\frac{dW(x)}{dx} = \int_x^1 \frac{dy}{y}\frac{dW_B(y)}{dy} D\Big(L, \frac{x}{y}\Big)\Big(1 + \frac{\alpha}{\pi}K(y)\Big) \tag{6}$$

with

$$D(L,z) = \delta(1-z) + \frac{\alpha}{2\pi}(L-1)P(z) + \frac{1}{2!}\Big(\frac{\alpha}{2\pi}(L-1)\Big)^2 P^{(2)}(z) + \ldots, \tag{7}$$

and

$$K(y) = \Big[2\mathrm{Li}_2(y) - \frac{\pi^2}{3} - \frac{1}{2} + \Big[3\ln(1-y) - 2\ln y + 1\Big]\ln y - \frac{1+y}{y}\ln(1-y) + \\ + \frac{1-y}{6y^2(3-2y)}\Big[(5+17y-16y^2)\ln y + 5(1-y)\Big] + 2\ln\frac{1-y}{y}\Big]y^2(3-2y). \quad (8)$$

For the numerical calculations one can use for the structure function $D(L,z)$ from Eq. (**??**) the «smoothed» (but equivalent) form [**?**]

$$D(L,z) = \frac{\beta}{2}(1-z)^{\frac{\beta}{2}-1}\Big(1+\frac{3}{8}\beta\Big) - \frac{\beta}{4}(1+z) + O(\beta^2), \quad \beta = \frac{2\alpha}{\pi}(L-1). \quad (9)$$

One can find useful relation

$$\int_x^1 \frac{dy}{y^2} D\Big(L, \frac{x}{y}\Big)\Psi(y) = \int_x^1 \frac{dy}{y^2} D\Big(L, \frac{x}{y}\Big)\Big[\Psi(y) - \Psi(x)\Big] + \frac{1}{x}\Psi(x)\int_x^1 dz D(L,z) \quad (10)$$

with

$$\Psi(y) = y^3(3-2y)\Big(1 + \frac{\alpha}{\pi}K(y)\Big),$$

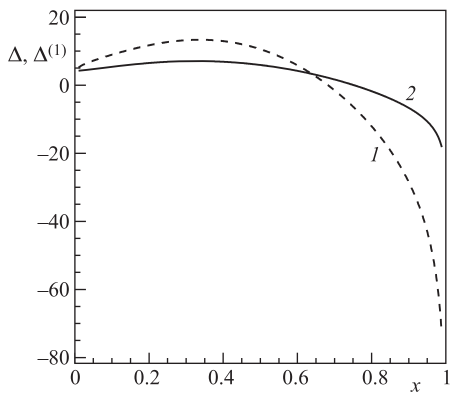
$$\int_x^1 dz D(L,z) = (1-x)^{\beta/2}\Big(1+\frac{3}{8}\beta\Big) - \frac{\beta}{8}(1-x)(3+x) + O(\beta^2).$$

CONCLUSION

For the comparison we have given in the Figure the numerical values of the quantity $\Delta^{(1)} = \frac{96\pi^3}{G^2M^5}\left(\frac{dW^{(1)}(x)}{dx} - \frac{dW_B(x)}{dx}\right)\frac{\pi}{\alpha} = \frac{1}{2}h(x)x^2(3-2x)$ — curve *1*, and the quantity $\Delta = \frac{96\pi^3}{G^2M^5}\left(\frac{dW(x)}{dx} - \frac{dW_B(x)}{dx}\right)$ — curve *2*, calculated in LLA and NLO approximations (see (**??**), (**??**)). One can see that the spectrum contrary to the result of the lowest order of perturbation theory is well defined in the whole region of x including $x \to 0$ and $x \to 1$. The total width with RC does not contain «large logarithm» due to the property $\int_0^1 dz D(L,z) = 1$. In our approach we have obtained

$$W = W_B\Big[1 - \frac{\alpha}{2\pi}\Big(\pi^2 - \frac{25}{4}\Big) + O(\alpha^2)\Big]. \quad (11)$$

Let us note that the terms of order α^2 were calculated in [**?**]. In papers [**?**, **?**, **?**] the one-loop and two-loop terms for the correction to the electron energy spectrum were calculated. These terms as well have a singular behavior at $x = 1$, and only the conversion with structure function provides the smooth behavior which is seen from the Figure.



The deviation of the electron spectrum in muon decay from the spectrum in the Born approximation: for the lowest order $\Delta^{(1)}$ (*1*), for all orders Δ (*2*) of perturbation theory

The higher $\beta^n, n = 2, 3, 4$ iterations of the structure functions can be found in [4, 9].

The contribution from emission of electron–positron pairs can be obtained from (6) replacing $D = D^\gamma \to D^\gamma + D^{e^+e^-}$ [4, 9].

The case of a polarized muon can be considered in the same way, not touched here.

First the structure function method was applied to describe the $\pi e2$ decay in [**?**].

Acknowledgements. One of us (E. A. K.) is grateful to the Institute of Physics, SAS for warm hospitality during the time of his stay. E. A. K. acknowledges the support of INTAS (grant No. 05-1000008-8328). The work was also supported in part by the Slovak Grant Agency for Sciences VEGA, Grant No. 2/7116.

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Received on January 26, 2009.