Письма в ЭЧАЯ. 2006. Т. 3, № 6(135). С. 84–89

УДК 539.12.01

A NOTE ABOUT THE T'HOOFT ANSATZ FOR $SU(N)$ REAL TIME GAUGE THEORIES

J. Manjavidze [a, b, 1], *V. Voronyuk* [b, 2]

[a] Institute of Physics, Tbilisi, Georgia
[b] Joint Institute for Nuclear Research, Dubna

The t'Hooft ansatz which reduces the classical Yang–Mills theory to the $\lambda\phi^4$ one is under consideration. It is shown that in the framework of this ansatz the real-time classical solutions for the arbitrary $SU(N)$ gauge group are obtained by embedding $SU(2)\times SU(2)$ into $SU(N)$. It is argued that this group structure is the only possibility in the framework of the considered ansatz. New explicit solutions for $SU(3)$ and $SU(5)$ gauge groups are shown.

Рассматривается анзац т'Хофта, переводящий классическую теорию Янга–Милса в теорию $\lambda\phi^4$. Показано для произвольной $SU(N)$-калибровочной группы, что в рамках этого анзаца классические решения в реальном времени получаются вложением $SU(2)\times SU(2)$ в $SU(N)$. Показано, что это единственная возможность в рамках данного анзаца. Приведен явный вид построенного решения для $SU(3)$- и $SU(5)$-калибровочных групп.

INTRODUCTION

In order to simplify the problem of solving a Yang–Mills equation for the vector field, t'Hooft et al. offered the ansatz for the Euclidean space [1]. It reduces the Yang–Mills equation to the equation for a single scalar field ϕ. The $SU(2)$ classical solutions discovered by means of this ansatz are well known [2] and were used to generate $SU(N)$ solutions by simply embedding $SU(2)$ into $SU(N)$ [3].

One of them allows the coordinate transformation to the Minkowski space so that it becomes nonsingular, real and possesses a finite action and energy [2, 4].

The $SU(2)$ gauge group was assumed for both the Euclidean and the Minkowski space (see also [5]), while the experimental analysis shows that QCD is the $SU(3)$ gauge theory [6]. So, the knowledge of the real-time classical solution for QCD is important since it allows one to analyze the nonperturbative corrections [7] to the observables.

In this article we will try to find a $SU(N)$ solution by means of the t'Hooft ansatz. The only condition we assume for the ansatz is the following: it must reduce the Yang–Mills equation to the real scalar $\lambda\phi^4$ theory. We will solve this condition and will show that the only solution of the classical Yang–Mills equation in the framework of the t'Hooft ansatz is embedding $SU(2)\times SU(2)$ into $SU(N)$.

[1] E-mail: joseph@nu.jinr.ru
[2] E-mail: vadimv@nu.jinr.ru

1. DEFINITION OF ANSATZ

Let us start from the Yang–Mills equation in the matrix form

$$\partial^\mu F_{\mu\nu} + ig[A^\mu, F_{\mu\nu}] = 0, \tag{1}$$

where

$$A_\mu = t_a A_{a\mu},$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu],$$

t_a are generators of the gauge group.

Let us consider the t'Hooft ansatz without any assumptions about gauge group

$$A_\mu(x) = \frac{1}{g}\eta_{\mu\nu}\,\partial^\nu \ln\phi(x),$$

where $\eta_{\mu\nu}$ are some matrices. We will consider that $A_\mu(x)$ satisfies the Lorentz gauge condition: $\partial^\mu A_\mu = 0$ and so $\eta_{\mu\nu}$ are antisymmetric over μ and ν matrices. It is assumed that $\eta_{\mu\nu}$ are constant in this gauge.

It is necessary to take the equality

$$-i[\eta_{\mu\sigma}, \eta_{\nu\rho}] = \eta_{\mu\nu}g_{\rho\sigma} - \eta_{\mu\rho}g_{\sigma\nu} + \eta_{\sigma\rho}g_{\mu\nu} - \eta_{\sigma\nu}g_{\mu\rho} \tag{2}$$

in order to reduce the Yang–Mills equation to the equation for the single scalar field. As the result of substitution of ansatz with the property (2) into the Yang–Mills equation (1), we have

$$\Box\phi + \lambda\phi^3 = 0, \tag{3}$$

where λ is an arbitrary integration constant. Emphasize that Eq. (3) is the result of (2), this reduction is valid for any gauge group.

Therefore, the problem (1) was divided into two parts: the searching of $\eta_{\mu\nu}$ from the algebraic equality (2) and the solving of equation (3) for $\phi(x)$.

Particular solutions of equation (3) are known (see [2, 4, 8, 9]) and we will not consider this question.

The matrices $\eta_{\mu\nu}$ can be written in a convenient form

$$\eta_{\mu\nu} = -\varepsilon_{0\mu\nu\kappa}X_\kappa + ig_{0\mu}Y_\nu - ig_{0\nu}Y_\mu, \qquad \kappa = 1, 2, 3, \tag{4}$$

since they are antisymmetric, where $\varepsilon_{0123} = 1$; the unknown X_i and Y_i are matrices in the group space, $X_0 = 0$, $Y_0 = 0$, $X_i = -X^i$, $Y_i = -Y^i$.

Let us insert (4) into (2). Then we obtain algebraic equations for X_i and Y_i. Because of antisymmetry of $\eta_{\mu\nu}$, it is convenient to examine only three cases:

1. $\mu = 0$, $\sigma = i$, $\nu = 0$, $\rho = j$, where $i, j = 1, 2, 3$. Then we have

$$[Y_i, Y_j] = i\varepsilon_{ijk}X_k; \tag{5}$$

2. $\mu = 0$, $\sigma = i$, $\nu = j$, $\rho = k$, where $i, j, k = 1, 2, 3$. It is easy to obtain

$$\varepsilon_{jks}[Y_i, X_s] = iY_j g_{ik} - iY_k g_{ij}.$$

So, we have

$$[Y_i, X_j] = i\varepsilon_{ijk} Y_k; \tag{6}$$

after changing the indices

3. $\mu = i$, $\sigma = j$, $\nu = k$, $\rho = s$, where $i, j, k, s = 1, 2, 3$. This case gives

$$-i[(-\varepsilon_{ijp} X_p), (-\varepsilon_{ksl} X_l)] = (-\varepsilon_{ikp} X_p) g_{sj} - (-\varepsilon_{isp} X_p) g_{jk} + (-\varepsilon_{jsp} X_p) g_{ik} - (-\varepsilon_{jkp} X_p) g_{is}.$$

After simplification and changing of the indices we have

$$[X_i, X_j] = i\varepsilon_{ijk} X_k. \tag{7}$$

The other cases can be easily reduced to these three ones.

It follows from (5)–(7) that

$$[\mathcal{J}_i, \mathcal{J}_j] = i\varepsilon_{ijk} \mathcal{J}_k, \qquad [\mathcal{K}_i, \mathcal{K}_j] = i\varepsilon_{ijk} \mathcal{K}_k, \tag{8}$$

$$[\mathcal{J}_i, \mathcal{K}_j] = 0,$$

where

$$\mathcal{J}_i = \frac{X_i + Y_i}{2}, \qquad \mathcal{K}_i = \frac{X_i - Y_i}{2}.$$

It follows from (8) that $N \times N$ matrices $\mathcal{J}_i$ and $\mathcal{K}_i$ are elements of the $SU(2) \times SU(2)$ group. Then the ansatz can be written as follows:

$$\eta_{\mu\nu} = (-\varepsilon_{0\mu\nu\kappa} \mathcal{J}_\kappa + i g_{0\mu} \mathcal{J}_\nu - i g_{0\nu} \mathcal{J}_\mu) + (-\varepsilon_{0\mu\nu\kappa} \mathcal{K}_\kappa - i g_{0\mu} \mathcal{K}_\nu + i g_{0\nu} \mathcal{K}_\mu), \quad \kappa = 1, 2, 3. \tag{9}$$

This is the general solution of (2) and, therefore, it is unique. There always exists a nonzero t'Hooft ansatz for any $N \geqslant 2$ since the representation of the $SU(2) \times SU(2)$ group by $N \times N$ matrices always exists. The meaning of such a representation is embedding $SU(2) \times SU(2)$ into $SU(N)$.

This ansatz gives complex potentials A_μ for real ϕ; however, one can check that it leads to a real Lagrangian density. Therefore, one can expect that there exists some complex gauge transformation which makes it real as it was done for $SU(2)$ [4].

Let us consider the solutions for $SU(2)$, $SU(3)$ and $SU(5)$ groups.

1.1. *SU*(2). For the $SU(2)$ gauge group the only solution is (either $\mathcal{J}_i$ or $\mathcal{K}_i$ is equal to zero)

$$X_i = \pm Y_i = \frac{\sigma_i}{2}.$$

Then we obtain the well-known solution [1,2] which can be written in a component form

$$\eta_{a\mu\nu} = -\varepsilon_{0a\mu\nu} \mp i g_{0\mu} g_{a\nu} \pm i g_{0\nu} g_{a\mu}.$$

1.2. *SU*(3). For the $SU(3)$ gauge group also either $\mathcal{J}_i$ or $\mathcal{K}_i$ is equal to zero, so we have

$$X_i = \pm Y_i.$$

There exist both reducible and irreducible representations of the $SU(2)$ group in terms of 3×3 matrices.

Reducible Representation. The $SU(3)$ group contains three independent $SU(2)$ subgroups which do not form direct product. So there exist three independent solutions:

(I): $X_1^{(\mathrm{I})} = t_1$, $X_2^{(\mathrm{I})} = t_2$, $X_3^{(\mathrm{I})} = t_3$.

In the component form we obtain

$$\eta_{1\,\mu\nu} = \begin{pmatrix} 0 & \pm i & 0 & 0 \\ \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{2\,\mu\nu} = \begin{pmatrix} 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 1 \\ \mp i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{3\,\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \pm i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \mp i & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{a\,\mu\nu} = 0, \quad a = 4, \ldots, 8;$$

(II): $X_1^{(\mathrm{II})} = t_4$, $X_2^{(\mathrm{II})} = t_5$, $X_3^{(\mathrm{II})} = \frac{1}{2}(\sqrt{3}\, t_8 + t_3)$;

(III): $X_1^{(\mathrm{III})} = t_6$, $X_2^{(\mathrm{III})} = t_7$, $X_3^{(\mathrm{III})} = \frac{1}{2}(\sqrt{3}\, t_8 - t_3)$.

The cases (II) and (III) are similar to the (I) with the difference in gauge indices.

Irreducible Representation. There also exists an irreducible representation of the $SU(2)$ group by 3×3 matrices,

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then in the component form we obtain

$$\eta_{1\,\mu\nu} = \sqrt{2} \begin{pmatrix} 0 & \pm i & 0 & 0 \\ \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{2\,\mu\nu} = \sqrt{2} \begin{pmatrix} 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 1 \\ \mp i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{3\,\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \pm i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \mp i & 0 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{4\,\mu\nu} = \eta_{5\,\mu\nu} = 0, \quad \eta_{6\,\mu\nu} = \eta_{1\,\mu\nu}, \quad \eta_{7\,\mu\nu} = \eta_{2\,\mu\nu}, \quad \eta_{8\,\mu\nu} = \sqrt{3}\, \eta_{3\,\mu\nu}.$$

1.3. *SU*(5). Considering the $SU(5)$ group it is interesting to examine the solution with both nonzero $SU(2)$ groups. If $\mathcal{J}_i$ or $\mathcal{K}_i$ is equal to zero, then the solution will be given by reducible or irreducible representation of the group in a way like $SU(3)$.

For the $\mathcal{J}_i$ one can take irreducible group presentation for the 3×3 matrices, for example, in the upper left corner and for the $\mathcal{K}_i$ one can take 2×2 group presentation for the lower right corner, and vice versa. It can be written in the obvious form

$$\begin{pmatrix} \mathcal{J} & & & 0 & 0 \\ SU(2) & & & 0 & 0 \\ 3\times 3 & & & 0 & 0 \\ 0 & 0 & 0 & \mathcal{K} & \\ 0 & 0 & 0 & SU(2) & 2\times 2 \end{pmatrix}.$$

Then the ansatz in component form is as follows:

$$\eta_{1\,\mu\nu} = \sqrt{2}\begin{pmatrix} 0 & \pm i & 0 & 0 \\ \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{2\,\mu\nu} = \sqrt{2}\begin{pmatrix} 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 1 \\ \mp i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{3\,\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \pm i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \mp i & 0 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{4\,\mu\nu} = \eta_{5\,\mu\nu} = 0, \quad \eta_{6\,\mu\nu} = \eta_{1\,\mu\nu}, \quad \eta_{7\,\mu\nu} = \eta_{2\,\mu\nu}, \quad \eta_{8\,\mu\nu} = \sqrt{3}\,\eta_{3\,\mu\nu}, \quad \eta_{9,\dots,20\,\mu\nu} = 0,$$

$$\eta_{21\,\mu\nu} = \begin{pmatrix} 0 & \mp i & 0 & 0 \\ \pm i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{22\,\mu\nu} = \begin{pmatrix} 0 & 0 & \mp i & 0 \\ 0 & 0 & 0 & 1 \\ \pm i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu},$$

$$\eta_{23\,\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \mp i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \pm i & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad \eta_{24\,\mu\nu} = 0.$$

If one believes that the $SU(5)$ theory is unification of electroweak and strong interactions, then indices $a = 1, \dots, 8$ correspond to the strong and $a = 21, \dots, 23$ to the electroweak interactions. But one can see that this solution cannot be used for this purpose.

CONCLUSIONS

In the framework of the ansatz the $SU(N)$ classical solutions always exist and each one is given by embedding $SU(2)\times SU(2)$ into $SU(N)$.

Let us assume that ϕ is invariant under $O(4)\times O(2)$ coordinate transformations [4,9]. In the framework of this prescription, we obtain the real solution of the Yang–Mills equation

$$A_0 = \pm\frac{x_0 x_a}{g y^2}\mathcal{J}_a \mp \frac{x_0 x_a}{g y^2}\mathcal{K}_a,$$

$$A_i = \frac{1}{gy^2}\left[-\varepsilon_{ain}x_n \pm \delta_{ai}\frac{1}{2}(1+x^2) \pm x_a x_i\right]\mathcal{J}_a + \\ + \frac{1}{gy^2}\left[-\varepsilon_{ain}x_n \mp \delta_{ai}\frac{1}{2}(1+x^2) \mp x_a x_i\right]\mathcal{K}_a,$$

where

$$y^2 = \frac{1}{4}(1-x^2)^2 + x_0^2, \qquad \varepsilon_{123} = 1, \quad n = 1, \ldots, 3,$$

and $\mathcal{J}_a$, $\mathcal{K}_a$ are corresponding representations of $SU(2) \times SU(2)$ group by $N \times N$ matrices.

Acknowledgements. The authors thank A. Dorokhov for the useful discussion. One of the authors (V. Voronyuk) is indebted to Prof. R. Kragler for his help and to DAAD for the financial support.

REFERENCES

1. *t'Hooft G.* // Phys. Rev. D. 1976. V. 14. P. 3432;
   Belavin A. A. et al. // Phys. Lett. B. 1975. V. 59. P. 85.
2. *Actor A.* // Rev. Mod. Phys. 1979. V. 51. P. 461.
3. *Bernard C. et al.* // Phys. Rev. D. 1977. V. 16. P. 2967.
4. *De Alfaro V., Fubini S., Furlan G.* // Phys. Lett. B. 1976. V. 65. P. 163.
5. *Bernreuther W.* // Phys. Rev. D. 1977. V. 16. P. 3609.
6. *Bär O., Wiese U.-J.* // Nucl. Phys. B. 2001. V. 609. P. 225.
7. *Manjavidze J.* // Sov. J. Nucl. Phys. 1987. V. 45. P. 442;
   Manjavidze J. // J. Math. Phys. 2000. V. 41. P. 5710;
   Manjavidze J., Sissakian A. // J. Math. Phys. 2001. V. 42. P. 4158;
   Manjavidze J., Sissakian A. // Phys. Rep. 2001. V. 346. P. 1.
8. *Castell L.* // Phys. Rev. D. 1972. V. 6. P. 536.
9. *Schechter M.* // Phys. Rev. D. 1977. V. 16. P. 3015.

Received on October 17, 2005.