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ON THE LORENTZ GROUP $SO(3,1)$, GEOMETRICAL SUPERSYMMETRIC ACTION FOR PARTICLES, AND SQUARE ROOT OPERATORS

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In this work the problem of the square root quantum operators is analyzed from the theoretical group point of view. To this end, we considered the relativistic geometrical action of a particle in the superspace in order to quantize it and to obtain the spectrum of physical states with the Hamiltonian remaining in the natural square root form. The generators of group $SO(3,1)$ are introduced and the quantization of this model is performed completely. The obtained spectrum of physical states and the Fock construction for the physical states from the Hamiltonian operator in square root form was proposed, explicitly constructed and compared with the spectrum and Fock construction obtained from the Hamiltonian in the standard form (i.e., quadratic in momenta). We show that the only states that the square root Hamiltonian can operate with correspond to the representations with the lowest weights $\lambda = 1/4$ and $\lambda = 3/4$ with four possible (nontrivial) fractional representations for the group decomposition of the spin structure.

С точки зрения теоретического группового подхода анализируется проблема операторов квадратного корня. Для этого мы рассматриваем релятивистское геометрическое действие для частицы в суперпространстве, проводим ее квантование и получаем спектр физических состояний с гамильтонианом, сохраняющим естественное представление через квадратные корни. Вводятся генераторы группы $SO(3,1)$, и квантование модели полностью завершается. Полученный спектр физических состояний и фоковское построение физических состояний из оператора Гамильтона через квадратные корни сравниваются с фоковским построением из гамильтониана в стандартной форме (т. е. квадратичного по импульсам). Показано, что только состояния, с которыми можно оперировать в гамильтониане, выраженном через квадратные корни, соответствуют представлениям с низшими весами $\lambda = 1/4$ и $\lambda = 3/4$ с четырьмя возможными дробными представлениями для группового разложения спиновой структуры.

INTRODUCTION AND SUMMARY

The problem of the square root operator in theoretical physics, in particular in Quantum Mechanics and QFT is well known [6]. Several attempts to avoid the problem of locality and quantum interpretation of Hamiltonian as a square root operator were described in the literature: pseudodifferential operators, several expansions of the fractional-exponential operator, etc. [5]. The main characteristic of all these attempts is to eliminate the square root of the Hamiltonian. In this manner, the set of operators into the square root operates freely on the physical states, paying the price to lose locality and quantum interpretation of the spectrum of a well-formulated field theory.

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Recently [15–17], several works have appeared where the problem of the quantization procedure and the square root operators was carefully analyzed. In these papers it was demonstrated for different simple problems (harmonic oscillator, massive particle on hyperboloid, etc.) that the spectrum changes drastically if the Hamiltonian operator has the square root form or does not: the explicit computation of the Casimir operator of the symmetry group puts this difference in evidence.

In this work, strongly motivated for the several fundamental reasons described above, we considered the simple model of superparticle of Volkov and Pashnev [1], that is type G4 in the description of Casalbuoni [2, 3], in order to quantize it and to obtain the spectrum of physical states with the Hamiltonian remaining in the natural square root form. To this end, we used the Hamiltonian formulation described by Lanczos in [7] and the inhomogeneous Lorentz group as a representation for the obtained physical states [12–14]. The quantization of this model is performed completely and the obtained spectrum of physical states, with the Hamiltonian operator in its square root form, is compared with the spectrum obtained with the Hamiltonian in the standard form (i.e., quadratic in momenta). We show that the only states that the square root Hamiltonian can operate with correspond to the representations with the lowest weights $\lambda_{1,2} = 1/4$ and $\lambda_{1,2} = 3/4$. In this manner, we also show that the superparticle relativistic actions as of Ref. [1] are a good geometrical and natural candidate to describe quartionic states [9–11] (semions). The plan of this paper is as follows: in order to make this work self-contained, in Secs. 1, 2, and 3 we borrow from reference [18] the geometrical description, the Hamiltonian treatment and quantization of the superparticle model. Section 4 is describes the process of quantization and the obtaining of the mass spectrum of the superparticle model under consideration; and finally, some conclusions and remarks are given.

1. THE SUPERPARTICLE MODEL

In the superspace the coordinates are given not only by the spacetime x_μ coordinates, but also for anticommuting spinors θ^α and $\overline{\theta}^{\dot{\alpha}}$. The resulting metric [1, 4, 18] must be invariant to the action of the Poincare group, and also invariant to the supersymmetry transformations:

$$x'_\mu = x_\mu + i\left(\theta^\alpha (\sigma)_{\alpha\dot{\beta}} \overline{\xi}^{\dot{\beta}} - \xi^\alpha (\sigma)_{\alpha\dot{\beta}} \overline{\theta}^{\dot{\beta}}\right);\ \theta'^\alpha = \theta^\alpha + \xi^\alpha;\ \ \overline{\theta'}^{\dot{\alpha}} = \overline{\theta}^{\dot{\alpha}} + \overline{\xi}^{\dot{\alpha}}.$$

The simplest super-interval that obeys the requirements of invariance given above is the following:

$$ds^2 = \omega^\mu \omega_\mu + a\omega^\alpha \omega_\alpha - a^* \omega^{\dot{\alpha}} \omega_{\dot{\alpha}}, \tag{1}$$

where (to simplify notation from here we avoid the contracted indexes between the spin-tensors $(\sigma)_{\alpha\dot{\beta}}$ and the anticommuting spinors θ^α and $\overline{\theta}^{\dot{\alpha}}$, as usual)

$$\omega_\mu = dx_\mu - i\left(d\theta\ \sigma_\mu \overline{\theta} - \theta\ \sigma_\mu d\overline{\theta}\right); \quad \omega^\alpha = d\theta^\alpha; \quad \omega^{\dot{\alpha}} = \overline{d\theta}^{\dot{\alpha}}$$

are the Cartan forms of the group of supersymmetry [4].

The spinorial indexes are related as follows:

$$\theta^\alpha = \varepsilon^{\alpha\beta}\theta_\beta; \quad \theta_\alpha = \theta^\beta \varepsilon_{\beta\alpha}; \quad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}; \quad \varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}; \quad \varepsilon_{12} = \varepsilon^{12} = 1$$

and in an analogous manner for the spinors with punctuated indexes. The complex constants a and a^* in the line element (1) are arbitrary. This arbitrarity for the choice of a and a^* are constrained by the invariance and reality of the interval (1).

As we have extended our manifold to include fermionic coordinates, it is natural to extend also the concept of trajectory of point particle to the superspace. To do this, we take the coordinates $x(\tau)$, $\theta(\tau)$ and $\overline{\theta}^{\dot{\alpha}}(\tau)$ depending on the evolution parameter τ. Geometrically, the function action that will describe the world-line of the superparticle is

$$S = -m \int_{\tau 1}^{\tau 2} d\tau \sqrt{\overset{\circ}{\omega}_\mu \overset{\circ}{\omega}^\mu + a\dot{\theta}^\alpha \dot{\theta}_\alpha - a^* \dot{\overline{\theta}}^{\dot{\alpha}} \dot{\overline{\theta}}_{\dot{\alpha}}} = \int_{\tau 1}^{\tau 2} d\tau L\left(x, \theta, \overline{\theta}\right), \tag{2}$$

where $\overset{\circ}{\omega}_\mu = \dot{x}_\mu - i\left(\dot{\theta}\, \sigma_\mu \overline{\theta} - \theta\, \sigma_\mu \dot{\overline{\theta}}\right)$ and the upper point means derivative with respect to the parameter τ, as usual.

The momenta, canonically conjugated to the coordinates of the superparticle, are

$$\begin{aligned} \mathcal{P}_\mu &= \partial L/\partial x^\mu = \left(m^2/L\right) \overset{\circ}{\omega}_\mu, \\ \mathcal{P}_\alpha &= \partial L/\partial \dot{\theta}^\alpha = i\mathcal{P}_\mu \left(\sigma^\mu\right)_{\alpha\dot{\beta}} \overline{\theta}^{\dot{\beta}} + \left(m^2 a/L\right) \dot{\theta}_\alpha, \\ \mathcal{P}_{\dot{\alpha}} &= \partial L/\partial \dot{\overline{\theta}}^{\dot{\alpha}} = i\mathcal{P}_\mu \theta^\alpha \left(\sigma^\mu\right)_{\alpha\dot{\alpha}} - \left(m^2 a/L\right) \dot{\overline{\theta}}_{\dot{\alpha}}. \end{aligned} \tag{3}$$

It is difficult to study this system in the Hamiltonian formalism framework because of the constraints and the nullification of the Hamiltonian. As the action (2) is invariant under reparametrizations of the evolution parameter

$$\tau \to \widetilde{\tau} = f(\tau),$$

one way to overcome this difficulty is to make the dynamic variable x_0 the time. For this, it is sufficient to use the chain rule of derivatives (with special care of the anticommuting variables)[1] and to write the action in the form

$$S = -m \int_{\tau 1}^{\tau 2} \dot{x}_0 d\tau \sqrt{\left[1 - iW^0_{,0}\right]^2 - \left[x^i_{,0} - W^i_{,0}\right]^2 + \dot{x}_0^{-2} \left(a\dot{\theta}_\alpha \dot{\theta}^\alpha - a^* \dot{\overline{\theta}}_{\dot{\alpha}} \dot{\overline{\theta}}^{\dot{\alpha}}\right)},$$

where the $W^\mu_{,0}$ was defined by

$$\begin{aligned} \overset{\circ}{\omega}{}^0 &= \dot{x}^0 \left[1 - iW^0_{,0}\right], \\ \overset{\circ}{\omega}{}^i &= \dot{x}^0 \left[x^i_{,0} - iW^i_{,0}\right], \end{aligned}$$

[1] We take the Berezin convention for the Grassmannian derivatives: $\delta F(\theta) = \dfrac{\partial F}{\partial \theta} \delta\theta$.

whence $x_0(\tau)$ turns out to be the evolution parameter

$$S = -m \int_{x_0(\tau_1)}^{x_0(\tau_2)} dx_0 \sqrt{\left[1 - iW^0_{,0}\right]^2 - \left[x^i_{,0} - W^i_{,0}\right]^2 + a\dot{\theta}^{\alpha}\dot{\theta}_{\alpha} - a^* \dot{\overline{\theta}}^{\dot{\alpha}} \dot{\overline{\theta}}_{\dot{\alpha}}} \equiv \int dx_0 L.$$

Physically, this parameter (we call it the dynamical parameter) is the time measured by an observer's clock in the rest frame.

Therefore, the invariance of a theory with respect to the invariance of the coordinate evolution parameter means that one of the dynamic variables of the theory ($x_0(\tau)$ in this case) becomes the observed time with the corresponding nonzero Hamiltonian

$$H = \mathcal{P}_\mu \dot{x}^\mu + \Pi^\alpha \dot{\theta}_\alpha + \Pi^{\dot{\alpha}} \dot{\theta}_{\dot{\alpha}} - L = \sqrt{m^2 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a}\Pi^\alpha \Pi_\alpha - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}, \tag{4}$$

where

$$\Pi_\alpha = \mathcal{P}_\alpha + i\,\mathcal{P}_\mu \left(\sigma^\mu\right)_{\alpha\dot{\beta}} \overline{\theta}^{\dot{\beta}},$$
$$\Pi_{\dot{\alpha}} = \mathcal{P}_{\dot{\alpha}} - i\mathcal{P}_\mu \theta^\alpha \left(\sigma^\mu\right)_{\alpha\dot{\alpha}}.$$

That gives the well-known mass shell condition and losing, from the quantum point of view, the operability of the Hamiltonian.

In the work [1], where this type of superparticle action was explicitly presented, the problem of nullification of Hamiltonian was avoided in the standard form. This means that the analog to a mass shell condition (4) in superspace was introduced by means of a multiplier (einbein) to obtain a new Hamiltonian

$$H = \frac{\varkappa}{2}\left\{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)\right\}. \tag{5}$$

With this Hamiltonian it is clear that in order to perform the quantization of the superparticle the problems dissapear: $\mathcal{P}_0$ is restored into the new Hamiltonian, and the square root is eliminated. The full spectrum from this Hamiltonian was obtained in [1] where the quantum Hamiltonian referred to the center of mass was

$$H_{\rm cm} = m^2 - M^2 + \frac{2^{3/2}M}{|a|}\left[1 - \left(\sigma_0\right)_{\alpha\dot{\beta}} \overline{s}^{\dot{\beta}} s^\alpha\right] \tag{6a}$$

with the mass distribution of the physical states being the following: two scalar supermultiplets $M_{1s} = \frac{2^{1/2}}{|a|} + \sqrt{\frac{2}{|a|} + m^2}$ and $M_{2s} = \sqrt{\frac{2}{|a|} + m^2} - \frac{2^{1/2}}{|a|}$; and one vector supermultiplet $M_v = m$. The Fock construction in the center of mass for Eq. (6a) (Hamiltonian quadratic in momenta) consists of the following vectors:

$$\begin{array}{lll} S_1 = |0\rangle\, \mathrm{e}^{iMt}, & \Xi_{1\alpha} = \overline{d}_{\dot\alpha}\, |0\rangle\, \mathrm{e}^{iMt}, & P_1 = \overline{d}^{\dot\beta}\overline{d}_{\dot\beta}\, |0\rangle\, \mathrm{e}^{iMt}, \\ \Xi_{2\alpha} = \overline{s}_{\dot\alpha}\, |0\rangle\, \mathrm{e}^{iMt}, & V_{\alpha\beta} = \overline{s}_{\dot\alpha}\overline{d}_{\dot\beta}\, |0\rangle\, \mathrm{e}^{iMt}, & \Xi_{3\alpha} = \overline{s}_{\dot\alpha}\overline{d}^{\dot\beta}\overline{d}_{\dot\beta}\, |0\rangle\, \mathrm{e}^{iMt}, \\ P_2 = \overline{s}^{\dot\alpha}\overline{s}_{\dot\alpha}\, |0\rangle\, \mathrm{e}^{iMt}, & \Xi_{4\alpha} = \overline{d}_{\dot\alpha}\overline{s}^{\dot\beta}\overline{s}_{\dot\beta}\, |0\rangle\, \mathrm{e}^{iMt}, & \\ S_2 = \overline{d}^{\dot\beta}\overline{d}_{\dot\beta}\overline{s}^{\dot\alpha}\overline{s}_{\dot\alpha}\, |0\rangle\, \mathrm{e}^{iMt}, & & \end{array} \tag{6b}$$

where operators s_α and d_α acting on the vacuum give zero: $s_\alpha |0\rangle = d_\alpha |0\rangle = 0$.

We will show in this work that it is possible, in order to quantize the superparticle action, to remain the Hamiltonian in the square root form. As is very obvious, in the form of square root the Hamiltonian operator is not linearly proportional with the operator $n_s = \overline{s}^{\dot\beta} s^\alpha$. The Fock construction for the Hamiltonian into the square root form agrees formally with the description given above for Ref. [1], but the operability of this Hamiltonian is over basic states with lowest helicities $\lambda = 1/4, 3/4$. This means that the superparticle Hamiltonian preserving the square root form operates over physical states of particles with fractionary quantum statistics and fractional spin (quartions).

2. HAMILTONIAN TREATMENT IN LANCZO'S FORMULATION

In order to solve our problem from the dynamical and quantum mechanical point of view, we will use the formulation given in [7, 8]. This Hamiltonian formulation for dynamical systems was proposed by C. Lanczos and allows us to preserve the square root form in the new Hamiltonian. We start from expression (4)

$$H = \sqrt{m^2 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot\alpha}\Pi_{\dot\alpha}\right)}$$

if

$$\frac{dt}{d\tau} \equiv \frac{dx^0}{d\tau} = g\left(\mathcal{P}_0,\ \mathcal{P}_i,\ \Pi_\alpha,\ \Pi_{\dot\alpha},\ x_0,\ x_i,\ \theta_\alpha,\ \overline{\theta}_{\dot\alpha}\right)$$

with the arbitrary function g given by

$$g = \frac{\sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + 1/a\Pi^\alpha\Pi_\alpha - 1/a^*\Pi^{\dot\alpha}\Pi_{\dot\alpha}\right)}}{\sqrt{m^2 - \left(\mathcal{P}_i\mathcal{P}^i + 1/a\Pi^\alpha\Pi_\alpha - 1/a^*\Pi^{\dot\alpha}\Pi_{\dot\alpha}\right)} + \mathcal{P}_0}, \tag{7}$$

the new Hamiltonian $\mathcal{H}$ takes the required «square root» form

$$\mathcal{H} \equiv g\left(H + \mathcal{P}_0\right) = \sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot\alpha}\Pi_{\dot\alpha}\right)}, \tag{8}$$

where we shall set $\mathcal{H} = 0$ (now depending on $2n + 2$ canonical variables), and the variable $\mathcal{P}_0$ is clearly identificated by the dynamical expression

$$\frac{d\mathcal{P}_0}{d\tau} = -g\frac{\partial\mathcal{H}}{\partial x^0} \quad \text{or} \quad \frac{d\mathcal{P}_0}{d\tau} = -\frac{\partial\mathcal{H}}{\partial t}. \tag{9}$$

This means that $\mathcal{P}_0 = -H + \text{const}$.

In order to make an analysis of the dynamics of our problem, we can compute the Poisson brackets between all the canonical variables and their conjugate momenta [1–3]

$$\dot{\mathcal{P}}_\mu = \{\mathcal{P}_\mu, \mathcal{H}\}_{\text{pb}} = 0, \tag{10}$$

$$\dot{\theta}^\alpha = \{\theta^\alpha, \mathcal{H}\}_{\text{pb}} = \frac{1}{a}\frac{\Pi^\alpha}{\mathcal{H}}, \tag{11}$$

$$\dot{\overline{\theta}}^{\dot{\alpha}} = \left\{\overline{\theta}^{\dot{\alpha}}, \mathcal{H}\right\}_{\text{pb}} = -\frac{1}{a^*}\frac{\Pi^{\dot{\alpha}}}{\mathcal{H}}, \tag{12}$$

$$\dot{x}_\mu = \{x_\mu, \mathcal{H}\}_{\text{pb}} = \frac{1}{\mathcal{H}}\left\{\mathcal{P}_\mu + \frac{i}{a}\,\Pi^\alpha (\sigma_\mu)_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}} + \frac{i}{a^*}\theta^\alpha (\sigma_\mu)_{\alpha\dot{\beta}}\Pi^{\dot{\beta}}\right\}, \tag{13}$$

$$\dot{\Pi}_\alpha = \{\Pi_\alpha, \mathcal{H}\}_{\text{pb}} = \frac{2i}{a^*\mathcal{H}}\mathcal{P}_{\alpha\dot{\beta}}\Pi^{\dot{\beta}}, \tag{14}$$

$$\dot{\Pi}_{\dot{\alpha}} = \left\{\Pi_{\dot{\alpha}}, \mathcal{H}\right\}_{\text{pb}} = \frac{-2i}{a\mathcal{H}}\Pi^\beta\mathcal{P}_{\beta\dot{\alpha}}, \tag{15}$$

where $\mathcal{P}_{\alpha\dot{\beta}} \equiv \mathcal{P}_\mu\,(\sigma^\mu)_{\alpha\dot{\beta}}$. From the above expressions the set of classical equations to solve is easily seen:

$$\ddot{\Pi}_\alpha = -\left(\frac{4\mathcal{P}^2}{|a|^2\,\mathcal{H}^2}\right)\dot{\Pi}_\alpha, \tag{16}$$

$$\ddot{\Pi}_{\dot{\alpha}} = -\left(\frac{4\mathcal{P}^2}{|a|^2\,\mathcal{H}^2}\right)\dot{\Pi}_{\dot{\alpha}}. \tag{17}$$

Assigning $\dfrac{4\mathcal{P}^2}{|a|^2\,\mathcal{H}^2} \equiv \omega^2$ and having account for $\Pi^+_\alpha = -\Pi_{\dot{\alpha}}$, the solution to equations (16) and (17) takes the form

$$\begin{aligned} \Pi_\alpha &= \xi_\alpha\,\mathrm{e}^{i\omega\tau} + \eta_\alpha\,\mathrm{e}^{-i\omega\tau}, \\ \Pi_{\dot{\alpha}} &= -\overline{\eta}_{\dot{\alpha}}\,\mathrm{e}^{i\omega\tau} - \overline{\xi}_{\dot{\alpha}}\,\mathrm{e}^{-i\omega\tau}. \end{aligned} \tag{18}$$

By means of the substitution of the above solutions into (14) and (15), we find the relation between ξ_α and η_α:

$$\eta_\alpha = \left(\frac{2}{a^*\mathcal{H}\omega}\right)\,\mathcal{P}_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}.$$

From Eqs. (18) and above we obtain

$$\Pi_\alpha = \xi_\alpha\,\mathrm{e}^{i\omega\tau} + \left(\frac{2}{a^*\mathcal{H}\omega}\right)\,\mathcal{P}_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}\,\mathrm{e}^{-i\omega\tau}, \tag{19}$$

$$\Pi_{\dot{\alpha}} = -\left(\frac{2}{a\mathcal{H}\omega}\right) \xi^{\beta}\mathcal{P}_{\beta\dot{\alpha}}\, \mathrm{e}^{i\omega\tau} - \overline{\xi}_{\dot{\alpha}}\, \mathrm{e}^{-i\omega\tau}, \tag{20}$$

where we used the fact that the constant two-component spinors ξ_α verify $\overline{\xi}_{\dot{\alpha}} = \xi^+_\alpha$. Integrating expressions (11) and (12), we obtain explicitly the following:

$$\theta_\alpha = \zeta_\alpha - \frac{i}{a\mathcal{H}\omega}\left[\xi_\alpha\, \mathrm{e}^{i\omega\tau} - \frac{2}{a^*\mathcal{H}\omega}\, \mathcal{P}_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}\, \mathrm{e}^{-i\omega\tau}\right], \tag{21}$$

$$\overline{\theta}_{\dot{\alpha}} = \overline{\zeta}_{\dot{\alpha}} + \frac{i}{a^*\mathcal{H}\omega}\left[-\frac{2}{a\mathcal{H}\omega}\, \xi^{\beta}\mathcal{P}_{\beta\dot{\alpha}}\, \mathrm{e}^{i\omega\tau} + \overline{\xi}_{\dot{\alpha}}\, \mathrm{e}^{-i\omega\tau}\right], \tag{22}$$

where ζ_α and $\overline{\zeta}_{\dot{\alpha}} = \zeta^+_\alpha$ are two-component constant spinors.

Analogically, from expression (13), we obtain x_μ in an explicit form

$$x_\mu = q_\mu - \frac{1}{\mathcal{H}}\left[\mathcal{P}_\mu - \frac{\omega\mathcal{H}}{\mathcal{P}^2}\left(\xi\sigma_\mu\overline{\xi}\right)\right]\tau + \frac{1}{\mathcal{H}\omega}\left[\frac{1}{a}\, \mathrm{e}^{i\omega\tau}\left(\xi\sigma_\mu\overline{\zeta}\right) + \frac{1}{a^*}\, \mathrm{e}^{-i\omega\tau}\left(\zeta\sigma_\mu\overline{\xi}\right)\right] + \\ + \frac{\mathcal{P}_\mu}{2\mathcal{P}^2}\left[\zeta^\alpha\xi_\alpha\, \mathrm{e}^{i\omega\tau} - \overline{\zeta}^{\dot{\alpha}}\overline{\xi}_{\dot{\alpha}}\, \mathrm{e}^{-i\omega\tau}\right]. \tag{23}$$

3. QUANTIZATION

Because of the correspondence between classical and quantum dynamics, the Poisson brackets between coordinates and canonical impulses are transformed into quantum commutators and anticommutators

$$\begin{aligned} &[x_\mu, \mathcal{P}_\mu] = i\,\{x_\mu, \mathcal{P}_\mu\}_{\mathrm{pb}} = -ig_{\mu\nu}, \\ &\{\theta^\alpha, \mathcal{P}_\beta\} = i\,\{\theta^\alpha, \mathcal{P}_\beta\}_{\mathrm{pb}} = -i\delta^\alpha_\beta, \\ &\left\{\theta^{\dot{\alpha}}, \mathcal{P}_{\dot{\beta}}\right\} = i\left\{\theta^{\dot{\alpha}}, \mathcal{P}_{\dot{\beta}}\right\}_{\mathrm{pb}} = -i\delta^{\dot{\alpha}}_{\dot{\beta}} \end{aligned} \tag{24}$$

and the new Hamiltonian (8) operates quantically as follows:

$$\sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}\, |\Psi\rangle = 0, \tag{25}$$

where $|\Psi\rangle$ are the physical states. From the (anti)commutation relations (24) it is possible to obtain easily the commutators between the variables ξ_α, $\overline{\xi}_{\dot{\alpha}}$, ζ_α, $\overline{\zeta}_{\dot{\alpha}}$, q_μ, $\mathcal{P}_\mu$

$$\left\{\xi_\alpha, \overline{\xi}_{\dot{\alpha}}\right\} = -\mathcal{P}_{\alpha\dot{\alpha}}, \quad \left\{\zeta_\alpha, \overline{\zeta}_{\dot{\alpha}}\right\} = -\left(\frac{1}{2\mathcal{P}^2}\right)\mathcal{P}_{\alpha\dot{\alpha}}, \quad [q_\mu, \mathcal{P}_\mu] = -ig_{\mu\nu}. \tag{26}$$

To obtain the physical spectrum we use the relations given by (26) into (25) and the Hamiltonian $\mathcal{H}$ takes the following form:

$$\mathcal{H} = \sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \mathcal{P}_i\mathcal{P}^i - \frac{2^{3/2}\sqrt{(\mathcal{P}_\mu)^2}}{|a|} - \frac{2^{3/2}}{|a|\sqrt{(\mathcal{P}_\mu)^2}}\xi^\alpha\mathcal{P}_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}}. \tag{27}$$

Passing to the center of mass of the system, and defining new operators $s_\alpha = (1/\sqrt{M})\xi_\alpha$, $\overline{s}_{\dot\alpha} = (1/\sqrt{M})\overline{\xi}_{\dot\alpha}$, $d_\alpha = \sqrt{2M}\zeta_\alpha$, $\overline{d}_{\dot\alpha} = \sqrt{2M}\zeta_{\dot\alpha}$, where $M = \mathcal{P}_0$, $\mathcal{H}_{\rm cm}$ is

$$\mathcal{H}_{\rm cm} = \sqrt{m^2 - M^2 + \frac{2^{3/2}M}{|a|}\left[1 - (\sigma_0)_{\alpha\dot\beta}\,\overline{s}^{\dot\beta}s^\alpha\right]}, \tag{28}$$

with

$$\left\{s_\alpha, \overline{s}_{\dot\alpha}\right\} = -(\sigma_0)_{\alpha\dot\alpha}, \quad \left\{d_\alpha, \overline{d}_{\dot\alpha}\right\} = -(\sigma_0)_{\alpha\dot\alpha} \tag{29}$$

being the anticommutation relations of the operators $s_\alpha, \overline{s}_{\dot\alpha}$, d_α, $\overline{d}_{\dot\alpha}$. Now the question is: how does the square-root $\mathcal{H}$ Hamiltonian given by expression (28) operate on a given physical state? The problem of locality and interpretation of the operator like (25) is very well known. Several attemps to avoid these problems were given in the literature [5, 6]: pseudodifferential operators, several expansions of the fractional-exponential operator, etc. The main characteristic of all these attempts is to eliminate the square root of the Hamiltonian. In this manner, the set of operators into the square root operates freely on the physical states, paying the price to lose locality and quantum interpretation of the spectrum of a well-possessed field theory.

Our plan is to take the square root to a bispinor in order to introduce the physical state into the square root Hamiltonian. In the next section, we will perform the square root of a bispinor and obtain the mass spectrum given by the Hamiltonian $\mathcal{H}$.

4. MASS SPECTRUM AND SQUARE ROOT OF A BISPINOR

The square root from a spinor was extracted in 1965 by S. S. Sannikov from Kharkov (Ukraine) [13] and the analysis of the structure of the Hilbert space containing such «square root» states was worked out by E. C. G. Sudarshan, N. Mukunda and C. C. Chiang in 1981 [21]. Taking the square root from a spinor was performed also by P. A. M. Dirac [14] in 1971.

We know that the group $SL(2,\mathbb{C})$ is locally isomorphous to $SO(3,1)$, and SL (2, R) is locally isomorphous to SO(2, 1). For instance, the generators of the group $SO(3,1)$ for our case can be constructed from the usual operators a, a^+ (or q and p) in the following manner. We start from an irreducible unitary infinite dimensional representation of the Heisenberg–Weyl group, which is realized in the Fock spaces of states of one-dimensional quantum oscillator [10–12]. Creation operators and annihilation operators of these states obey the conventional commutation relations $[a^+, a] = 1$, $[a, a] = [a^+, a^+] = 0$. To describe this representation to the Lorentz group, one may also use the coordinate-momentum realization $\left(q, p = -i\dfrac{\partial}{\partial q}\right)$ of the Heisenberg algebra, which relates to the a, a^+ realization by the formulas

$$a = \frac{q - ip}{\sqrt{2}}, \quad a^+ = \frac{q + ip}{\sqrt{2}}, \tag{30}$$

as usual. Let us introduce the spinors

$$L_\alpha = \begin{pmatrix} a_1 \\ a_1^+ \end{pmatrix}, \quad L_{\dot\alpha} = \begin{pmatrix} a_2 \\ a_2^+ \end{pmatrix}. \tag{31}$$

The commutation relations take the form

$$[L_\alpha, L_\beta] = i\varepsilon_{\alpha\beta}; \quad \left[L_{\dot\alpha}, L_{\dot\beta}\right] = i\varepsilon_{\dot\alpha\dot\beta}; \quad \left[L_{\dot\alpha}, L_\beta\right] = 0. \tag{32}$$

The generators of $SL(2,\mathbb{C})$ are easily constructed [11] from L_α and $L_{\dot\alpha}$

$$\begin{aligned} S_{\alpha\beta} &\equiv iS_{1i}(\sigma^i)_{\alpha\beta} = \frac{1}{4}\left\{L_\alpha, L_\beta\right\}, \\ S_{\dot\alpha\dot\beta} &\equiv iS_{2i}(\sigma^i)_{\dot\alpha\dot\beta} = \frac{1}{4}\left\{L_{\dot\alpha}, L_{\dot\beta}\right\}, \end{aligned} \tag{33}$$

where the index $i = 1, 2, 3$ and the six vectors S_{ai} $(a, b = 1, 2; a \neq b)$, characteristics of the representation of $SL(2,\mathbb{C}) \approx SO(3,1)$, satisfy the commutation relations

$$[S_{ai}, S_{aj}] = -i\varepsilon_{ijk}S_a^k; \quad [S_{bi}, S_{bj}] = -i\varepsilon_{ijk}S_b^k; \quad [S_{ai}, S_{bj}] = 0. \tag{34}$$

Notice that the above construction obeys the described decomposition of $SL(2,\mathbb{C}) \approx SO(3,1)$.

Then the quantities

$$\Phi_\alpha \equiv \langle\Psi|\, L_\alpha \,|\Psi\rangle\,, \quad \overline{\Phi}_{\dot\alpha} \equiv \langle\overline{\Psi}|\, L_{\dot\alpha}\, |\overline{\Psi}\rangle \tag{35}$$

are the two-components of a bispinor

$$\Phi \equiv \left\langle \widehat{\Psi}\right| L \left|\widehat{\Psi}\right\rangle = \begin{pmatrix} \Phi_\alpha \\ \overline{\Phi}_{\dot\alpha} \end{pmatrix},$$

where we define $|\widehat{\Psi}\rangle \equiv \begin{pmatrix} |\Psi\rangle \\ |\overline{\Psi}\rangle \end{pmatrix}$. Notice that $|\Psi\rangle$ and $|\overline{\Psi}\rangle$ are the square roots of each component of this bispinor and can have the same form (given the isomorphism between the generators L_α and $L_{\dot\alpha}$), which is very easy to verify. In terms of q the basic vectors of the representation can be written as [10, 12, 13]

$$\langle q\, |n\rangle = \varphi_n\,(q) = \pi^{-1/4}\,(2^n n!)^{-1/2}\, H_n\,(q)\; \mathrm{e}^{-q^2/2}, \tag{36}$$

$$\int dq \varphi_m^*\,(q)\,\varphi_n\,(q) = \delta_{mn} \tag{37}$$

(where $H_n(q)$ are the Hermite polynomials) and form a unitary representation of $SO\,(3,1)$, and

$$|n\rangle = (n!)^{-1/2}\left(a^+\right)^n |0\rangle \tag{38}$$

are the normalized basic states where the vacuum vector is annihilated by a. The Casimir operator, that is $S_{ai}S_a^i$, has the eigenvalue $\lambda(\lambda-1) = -3/16$ (for each subgroup $ISO(2,1)$ given by Eqs. (33)) and indeed corresponds to the representations with the lowest weights $\lambda = 1/4$ and $\lambda = 3/4$. The wave functions which transform as linear irreducible representation of $ISO(2,1)$, subgroup of $ISO(3,1)$ generated by operators (33) are

$$\Psi_{1/4}\,(x,\theta,q) = \sum_{k=0}^{+\infty} f_{2k}\,(x,\theta)\,\varphi_{2k}\,(q)\,, \tag{39}$$

$$\Psi_{3/4}\left(x,\theta,q\right)=\sum_{k=0}^{+\infty}f_{2k+1}\left(x,\theta\right)\varphi_{2k+1}\left(q\right) \tag{40}$$

(analogically for the $\overline{\Psi}_{1/4}$ and $\overline{\Psi}_{3/4}$ states with contrary helicity). We can easily see that the Hamiltonian $\mathcal{H}$ (28) operates over the states $|\widehat{\Psi}\rangle$, which enter into $\mathcal{H}$ as its square Φ_α and $\Phi_{\dot{\alpha}}$. It is natural to associate, up to a proportional factor, the spinors d_α and $\overline{d}_{\dot{\alpha}}$ with

$$d_\alpha\rightarrow\left(\Phi_{1/4}\right)_\alpha\equiv\left\langle\Psi_{\ 1/4}\right|L_\alpha\left|\Psi_{1/4}\right\rangle,\quad \overline{d}_{\dot{\alpha}}\rightarrow\left(\overline{\Phi}_{1/4}\right)_{\dot{\alpha}}\equiv\left\langle\overline{\Psi}_{1/4}\right|L_{\dot{\alpha}}\left|\overline{\Psi}_{1/4}\right\rangle, \tag{41}$$

and in a similar manner, the spinors s_α and $\overline{s}_{\dot{\alpha}}$ with

$$s_\alpha\rightarrow\left(\Phi_{3/4}\right)_\alpha\equiv\left\langle\Psi_{3/4}\right|L_\alpha\left|\Psi_{3/4}\right\rangle,\quad \overline{s}_{\dot{\alpha}}\rightarrow\left(\overline{\Phi}_{3/4}\right)_{\dot{\alpha}}\equiv\left\langle\overline{\Psi}_{3/4}\right|L_{\dot{\alpha}}\left|\overline{\Psi}_{3/4}\right\rangle. \tag{42}$$

The relations (41) and (42) give a natural link between the spinors $\xi_\alpha\left(\overline{\xi}_{\dot{\alpha}}\right)$ and $\zeta_\alpha\left(\overline{\zeta}_{\dot{\alpha}}\right)$, solutions of the dynamical problem, with the only physical states that can operate freely with the Hamiltonian $\mathcal{H}$: the «square root» states $|\Psi\rangle$, $\left|\overline{\Psi}\right\rangle$ from the bispinor Φ. Notice that there are four (nontrivial) representations for the group decomposition of the bispinor Φ, as follows:

$$\Phi_1=\begin{pmatrix}\Phi_{1/4}\\ \overline{\Phi}_{3/4}\end{pmatrix}\rightarrow\left(1/4,0\right)\oplus\left(0,3/4\right),$$

$$\Phi_2=\begin{pmatrix}\Phi_{3/4}\\ \overline{\Phi}_{1/4}\end{pmatrix}\rightarrow\left(3/4,0\right)\oplus\left(0,1/4\right),$$

$$\Phi_3=\begin{pmatrix}\Phi_{1/4}\\ \overline{\Phi}_{1/4}\end{pmatrix}\rightarrow\left(1/4,0\right)\oplus\left(0,1/4\right),$$

$$\Phi_4=\begin{pmatrix}\Phi_{3/4}\\ \overline{\Phi}_{3/4}\end{pmatrix}\rightarrow\left(3/4,0\right)\oplus\left(0,3/4\right).$$

This result is a consequence of the geometrical Hamiltonian taken in its natural square root form and the Sannikov–Dirac oscillator representation for the generators of the Lorentz group $SO(3,1)$.

Commutation relations (29) obey the Clifford algebra for spinorial creation–annihilation operators. In this manner, the square roots of the operators s_α and d_α in the representation given by the associations (41) and (42) acting on the vacuum give zero, symbolically:

$$\sqrt{s_\alpha}\rightarrow\sqrt{\left(\Phi_{3/4}\right)_\alpha}\left|0\right\rangle=\Psi_{r\ 3/4}\left|0\right\rangle=0,$$

$$\sqrt{d_\alpha}\rightarrow\sqrt{\left(\Phi_{1/4}\right)_\alpha}\left|0\right\rangle=\Psi_{\ r\ 1/4}\left|0\right\rangle=0,$$

where we introduce $r,s,t\ldots$ latin indexes to design the fractionary spin states. The Fock construction in the center of mass of the system consists now, in contrast to the construction (6b), of the following vectors:

$$\widehat{S}_1=\left|0\right\rangle e^{iMt/2},\quad \Xi_{1r}=\overline{\Psi}_{r\ 1/4}\left|0\right\rangle e^{iMt/2},\quad \widehat{P}_1=\overline{\Psi}^{r}_{1/4}\overline{\Psi}_{r\ 1/4}\left|0\right\rangle e^{iMt/2},$$

$$
\begin{gathered}
\Xi_{2r} = \overline{\Psi}_{r\ 3/4}\left|0\right\rangle \mathrm{e}^{iMt/2}, \quad V_{rs} = \overline{\Psi}_{r\ 3/4}\overline{\Psi}_{s\ 1/4}\left|0\right\rangle \mathrm{e}^{iMt/2}, \\
\Xi_{3r} = \overline{\Psi}_{r\ 3/4}\overline{\Psi}^{s}_{1/4}\overline{\Psi}_{s\ 1/4}\left|0\right\rangle \mathrm{e}^{iMt/2}, \\
\\
\widehat{P}_2 = \overline{\Psi}^{r}_{3/4}\overline{\Psi}_{r\ 3/4}\left|0\right\rangle \mathrm{e}^{iMt/2}, \quad \Xi_{4r} = \overline{\Psi}_{r\ 1/4}\overline{\Psi}^{s}_{3/4}\overline{\Psi}_{s\ 3/4}\left|0\right\rangle \mathrm{e}^{iMt/2}, \\
\widehat{S}_2 = \overline{\Psi}^{r}_{1/4}\overline{\Psi}_{r\ 1/4}\overline{\Psi}^{s}_{3/4}\overline{\Psi}_{s\ 3/4}\left|0\right\rangle \mathrm{e}^{iMt/2}.
\end{gathered}
\tag{43}
$$

Notice that the vectors given above are the only states that can operate into the square root operator given by expression (28), and not that constructed directly with the operators s_α and d_α. Schematically, we have, e.g., for Ξ_{4r}, the following operability:

$$
\sqrt{m^2 - M^2 + \frac{2^{3/2}M}{|a|}\left[1 - (\sigma_0)_{\alpha\dot{\beta}}\overline{s}^{\dot{\beta}}s^{\alpha}\right]}\ \overline{\Psi}_{r\ 1/4}\overline{\Psi}^{s}_{3/4}\overline{\Psi}_{s\ 3/4}\left|0\right\rangle \mathrm{e}^{iMt/2} \equiv
$$

$$
\equiv \sqrt{\left[m^2 - M^2 + \frac{2^{3/2}M}{|a|}\left[1 - (\sigma_0)_{\alpha\dot{\beta}}\overline{s}^{\dot{\beta}}s^{\alpha}\right]\right]\overline{d}_{\dot{\gamma}}\overline{s}^{\dot{\beta}}\overline{s}_{\dot{\beta}}\left|0\right\rangle \mathrm{e}^{iMt}}.
$$

From expression (38) and taking into account that the number operator is $\overline{s}^{\dot{\beta}}s^{\alpha} \equiv n_s$, because $\overline{s}^{\dot{\beta}}$ and s^{α} work as creation–annihilation operators, we can easily obtain the mass for the different «square root» or fractionary supermultiplets:

i) $n_s = 0 \to M_{\mathrm{I}} = -\dfrac{2^{1/2}}{|a|} + \sqrt{\dfrac{2}{|a|^2} + m^2}$; fractionary supermultiplet I: $\left(\widehat{S}_1, \Xi_{1r}, \widehat{P}_1\right)$;

ii) $n_s = 1 \to M_{\mathrm{III}} = m$; fractionary supermultiplet II: $(\Xi_{2r}, V_{rs}, \Xi_{3r})$;

iii) $n_s = 2 \to M_{\mathrm{III}} = \sqrt{\dfrac{2}{|a|^2} + m^2} + \dfrac{2^{1/2}}{|a|}$; fractionary supermultiplet III: $\left(\widehat{P}_2, \Xi_{4r}, \widehat{S}_2\right)$.

We emphasize now that the computations and algebraic manipulations given above were made with $\overline{d}_{\dot{\alpha}} \to \left(\Phi_{1/4}\right)_{\dot{\alpha}}$ and $\overline{s}_{\dot{\alpha}} \to \left(\Phi_{3/4}\right)_{\dot{\alpha}}$ (the square of the true states) into the square root Hamiltonian. That means that the physical states are constrained by the the explicit form of the Hamiltonian operator. Notice from expressions (35), (41) and (42) that the physical states for the Hamiltonian in the square root form are one half the number of physical states for the Hamiltonian quadratic in momenta.

Another important point is that the link between the new Hamiltonian $\mathcal{H}$ given by expression (8) and the relativistic Schrödinger equation (e.g., Ref. [20]) can be given through the relation between the conserved currents of the fermionic «square» states and the parastates. This important issue will be analyzed elsewhere [19].

It is interesting to note that the arbitrary c-parameters a and a^* generate a deformation of the usual line element for a superparticle in proper time, and this deformation is responsible, in any meaning, for the multiplets given above. This is not a casuality: one can easily see how the quantum Hamiltonian (28) is modified in the center of mass of the system by the c-parameters a and a^*. The implicancies of this type of superparticle actions with deformations of the quantization will be analyzed in a future paper [19].

CONCLUSIONS

In this work the problem of the square root quantum operators was analyzed considering the simple model of superparticle of Volkov and Pashnev [1]. The quantization of this model was performed completely and the obtained spectrum of physical states, with the Hamiltonian operator in its square root form, was compared with the spectrum obtained with the Hamiltonian in the standard form (i.e., quadratic in momenta). To this end, we used the Hamiltonian formulation described by Lanczos in [7] and the inhomogeneous Lorentz group as a representation for the obtained physical states [12–14] without any other manipulation like the usual quantum equations from the mathematical or operatorial point of view. The Fock construction for these fractionary or «square root states» was proposed, explicitly constructed and compared with the Fock construction given in the reference [1] for the superparticle model with the Hamiltonian in standard form. We have shown that, in contrast to [1], the only states that the square root Hamiltonian can operate correspond to the representations with the lowest weights $\lambda = 1/4$ and $\lambda = 3/4$.

Also we show that there are four possible (nontrivial) fractional representations for the group decomposition of the spin structure from the square root Hamiltonian, instead of (1/2, 0) and (0, 1/2) as in the case when the Hamiltonian is quadratic in momentum (e.g., Ref. [1]). This result is a consequence of the geometrical Hamiltonian taken in its natural square root form and the Sannikov–Dirac oscillator representation for the generators of the Lorentz group $SO(3,1)$.

For instance, we conclude that quantically it is not the same to operate with the square root Hamiltonian as that with its square or other power of this operator; the main problem is that the group theoretical description for the states under which the Hamiltonian operates is sensible to the power of such a Hamiltonian. It is interesting to see that the results presented here for the superparticle are in complete agreement with the results, symmetry group and discussions for nonsupersymmetric examples given in Ref. [15–17]; and seeing that the lowest weights of the states under the square root Hamiltonian can operate, and because no concrete action is known to describe particles with fractionary statistics, superparticle relativistic actions as of [1] can be good geometrical and natural candidates to describe quartionic states [9–12] (semions).

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