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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

COUPLING RUNNING THROUGH THE LOOKING-GLASS OF DIMENSIONAL REDUCTION

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The dimensional reduction, in a form of transition from four to two dimensions, was used in the 90s of the past century in a context of the HE Regge scattering. Recently, it has got a new impetus in quantum gravity where it opens the way to renormalizability and finite short-distance behaviour. We consider a QFT model $g\varphi^4$ with running coupling defined in both domains of different dimensionality; the $\bar{g}(q^2)$ evolutions being duly correlated at the reduction scale $q \sim M$. Beyond this scale, in the deep UV 2-dimensional region, the running coupling does not increase any more. Instead, it slightly decreases and tends to a finite value $\bar{g}_2(\infty) < \bar{g}_2(M^2)$ from above. As a result, the global evolution picture looks quite peculiar and proposes a base for the modified scenario of gauge couplings behavior with UV fixed points provided by dimensional reduction instead of leptoquarks.

Редукция размерностей в виде перехода от четырех к двум измерениям была использована в 1990-х гг. в контексте реджевского рассеяния. Недавно она получила новую жизнь в квантовой гравитации, где с ее помощью приходят к перенормируемости и конечному ультрафиолетовому поведению. Мы рассматриваем квантовую модель $g\varphi^4$ с бегущим взаимодействием $\bar{g}(q^2)$ в двух областях различной размерности, эволюция которого должным образом сопряжена на масштабе редукции $q \sim M$. За этим масштабом, в глубокой УФ двумерной области, эффективная функция связи более не возрастает и стремится к конечному значению $\bar{g}_2(\infty) < \bar{g}_2(M^2)$ сверху. В итоге полученная картина эволюции выглядит весьма своеобразно и может представлять интерес в качестве основы модифицированного сценария Великого объединения взаимодействий с редукцией размерностей как основы конечного УФ поведения вместо лептокварков.

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Dedicated to the memory
of Albert Tavkhelidze

1. INTRODUCTION

1.1. Motivation. Initial motive for considering the topic mentioned in the title is related to the hot quest of the Higgs particle still escaping from direct observation. Anticipating the possibility that no Higgs peak will be observed in the «window» (140 ± 25) GeV, we try to investigate the *Ginzburg–Landau–Higgs* possibility, with Higgs field being a classical field with nonzero constant component ~ 250 GeV sufficient for the mass production in the current version of SM. In other words, we are inclined to use the fact that the Higgs mechanism for creating masses could work when the Higgs field is not a quantum one being rather an analog of the Ginzburg–Landau two-component order parameter from the theory of superconductivity[1].

However, changing a quantum Higgs for the classical external field yields the trouble of renormalization in the EW sector of SM. Having no intention to enter this complicated problem we prefer to postpone its solution and look for some temporary practical remedy for the time being[2]. The possible way is to involve an invariant regularization procedure, like the Pauli–Villars one. To this goal, it is more intriguing to exploit a transition from the four-dimensional manifold to the one with a *smaller* number of dimensions $d < 4$ at high enough energy or small distance. Technically, this could provide us with artificial cutoff with only one additional parameter, the range of reduction.

The trick with changing the number of dimensions is a frequent one in current literature (on superstrings, etc.) devoted to the HE behavior. This transition à lá Kaluza–Klein to a larger number of dimensions $d > 4$ inevitably confronts us with the nonrenormalizability. Instead, we consider another, a rather opposite possibility, the *dimensional reduction* (DR).

To explore some practical aspects of DR, as a first test-flight to terra incognita, we turn here to a limited subject, the issue of transferring the renormalization-invariant running coupling $\bar{g}(q^2)$ through the region of reduction and relating its behavior in two domains with different dimensionality.

1.2. Dimensional Reduction. The dimensional reduction was used first about 15 years ago[3] as a pragmatic tool in the analysis of the HE Regge scattering. This line of reasoning was refreshed [4] quite recently in the context of the LHC physics with a more explicit emphasis on the DR physical implementation.

In the last decade it became quite popular in the theory of quantum gravity. Here, a class of models has been devised by Horava (see, e.g., paper [5] and references therein) with asymptotic anisotropy between space and time dimensions in the short distance UV limit[4].

Our attitude does not imply any modification of the special relativistic concept of the time. We just have in mind some smooth reduction of the spatial topological dimensions. Probably, this scenario is akin to another approach formulated recently [8] for the quantum field living in the fractal space-time.

Agreement on DR. One can discuss the mechanism of dimensional reduction either in the space-time terms or in the energy-momentum ones using the presumptive assumption that *reduction at the space-time scale* $x_{\mathbf{DR}} \sim 1/M_{\mathbf{DR}}$ *is, in a sense, equivalent to the reduction at the energy-momentum scale* $p_{\mathbf{DR}} \sim M_{\mathbf{DR}}$.

This tentative agreement will be used below to compare the Lagrangian (space-time) approach with the direct *ad hoc* modification of the momentum integration of Feynman

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[1] For details see our recent overview [1].

[2] Here, one can rely upon the well-known examples: in the early 1930s, Dirac was brave enough [2] to use cutoff on the proton mass in discussing the momentum dependence of electron charge $e(q)$, a prototype of the QED running coupling.

In the 1950s and 1960s, after the devising of the renormalization procedure, phenomenologists still widely and fruitfully used hardly nonrenormalizable 4-fermion interaction *à lá Fermi* for analyzing weak interactions. They postponed the problem solution till reaching the so-called «unitary limit» at $W_{\rm cm} \sim 100$ GeV.

[3] See paper [3] and references therein.

[4] This activity, in turn, was motivated by remarkable observation made by Ambjorn, Loll and others [6] in the causal dynamical triangulation approach to quantum gravity on lattice. See also an overview [7].

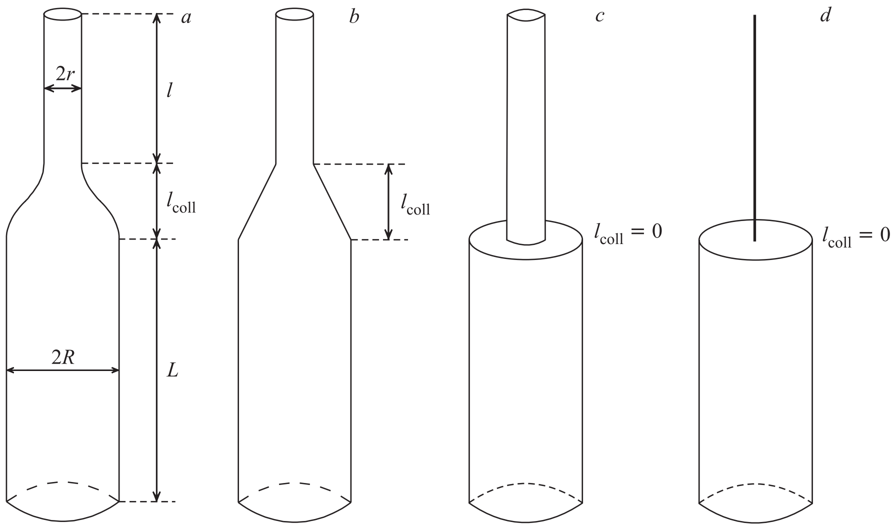

Fig. 1. *a*) Wine bottle with a long body and a long neck; *b*) bottle with a conical collar; *c*) bottle with a horizontal collar; *d*) bottle with a horizontal collar and a thread-like neck

integrals. In the course of the first approach we use a sharp conjunction as an approximation to a softer mechanism of a continuous DR in the second one.

Classical Illustration. To illustrate the idea of approximation, imagine a wine bottle (e.g., posed vertically) like one presented in Fig. 1, a. It consists of the main cylindrical body B with a relatively large radius R and a length L. The bottle's neck N of length l and smaller radius r is connected with the main part by a «collar» — a narrowing transition region C of a varying radius and a short length l_{coll}.

Suppose now that there is a mathematical equation(s) describing some process on the two-dimensional external surface S_2 of the bottle. One can have in mind a stationary boundary value problem for the second-order partial differential equation(s), as well as some dynamical process like wave propagation with radiation or heat conductivity and so on.

A number of problems with exact analytical solutions on the surfaces $S_{R,L}$ and $S_{r,l}$ of cylindrical parts can be found. Many of them could be solved for the whole two-dimensional manifold $S_2 = S_{R,L} + S_{r,l} + S_{\text{coll}}$ for simple enough smooth forms of the junction collar region C.

Of particular interest are nonstationary processes, like a solitary wave propagating upward due to some short-time perturbation at the lower edge of the bottom.

It will be instructive to study several issues: the dependence of solution details on the form of the surface S_{coll} of the collar region; transition to a sharp change of radius (to a horizontal collar) as $l_{\text{coll}} \to 0$, Fig. 1, c; the limiting case $r \to 0$, that is a transition from the 2-dimensional surface of the neck $S_{r,l}$ to the 1-dimensional linear manifold $S_{0,l} \to L_l$, Fig. 1, *d*.

In the further analysis, along with smooth transition in Subsec. 2.1, we shall use «hard conjunction» in Subsec. 2.2 of two regions with different dimensions. The first one resembles Fig. 1, a in the limit $r \to 0$, while the second one is an analog of a system presented in Fig. 1, *d*.

2. EFFECTIVE $\bar{g}$ FOR THE φ^4 MODEL IN VARIOUS DIMENSIONS

Take the one-component scalar massive quantum field $\varphi(x)$ with the self-interaction Lagrangian

$$L = T - V, \quad V(m, g; \varphi) = \frac{m^2}{2}\varphi^2 + \frac{4\pi^{d/2}M^{4-d}}{9} g_d \varphi^4, \quad g > 0 \tag{1}$$

in parallel in four ($d = 4$) and two ($d = 2$) dimensions. Limit ourselves to the one-loop approximation level for $\bar{g}$ that corresponds to the only Feynman diagram contribution, the first correction to the 4-vertex function, see Fig. 2.

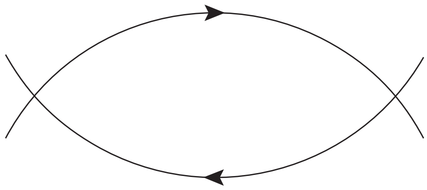

Fig. 2. One-loop vertex diagram

Its contribution I enters into the effective running coupling as follows:

$$\bar{g}(q^2) = \frac{g_i}{1 - g_i I(q^2; m^2, m_i^2)}. \tag{2}$$

2.1. Smooth DR in the Momentum Picture. We start with reduction of dimensions in Feynman integral by modifying metrics in the momentum space

$$dk = d^4k \to d_M k = \frac{d^4k}{1 + k^2/M^2}, \quad k^2 = \mathbf{k}^2 - k_0^2. \tag{3}$$

In particular, for the one-loop integral (Fig. 2) one gets

$$I\left(\frac{q^2}{m^2}\right) = \frac{i}{\pi^2}\int \frac{dk}{(m^2 + k^2)[m^2 + (k+q)^2]} \to$$
$$\to \frac{i}{\pi^2}\int \frac{d_M k}{(m^2 + k^2)[m^2 + (k+q)^2]} = J(\kappa; \mu),$$

with $\kappa = q^2/4m^2$, $\mu = M^2/m^2$, $q^2 = \mathbf{q}^2 - q_0^2$. The integral J can be calculated explicitly. We give its asymptotics. In the «deep 4-dimensional» region $m^2 \ll q^2 \ll M^2$ one gets an «intermediate» logarithmic behavior with M playing the role of the Pauli–Villars regulator. Meanwhile, in the «deep 2-dimensional» region $q^2 \gg M^2 \gg m^2$, the UV limit is finite. In usual normalization

$$J \to J_i = J\left(\frac{q^2}{4m^2}; \mu\right) - J\left(\frac{m_i^2}{4\,m^2}; \mu\right), \quad m_i \sim m,$$

one has

$$J_i^{[4]}(\kappa; \mu) \sim \ln\left(\frac{q^2}{m_i^2}\right), \quad J_i^{[2]}(\kappa; \mu) \sim \ln\left(\frac{4M^2}{m_i^2}\right) + \frac{M^2}{q^2}\ln\frac{q^2}{M^2}. \tag{4}$$

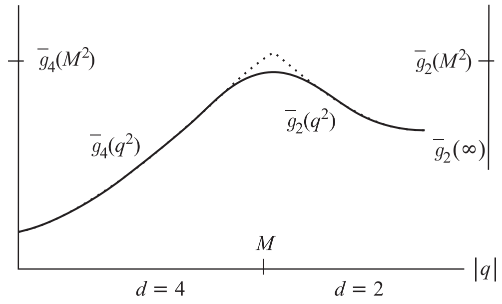

Fig. 3. The effective coupling evolution for the φ^4 model with DR

The first expression is rising, while the second — decreasing. The maximum value of J is attained at the DR scale $q^2 \sim M^2$ and is close to $\ln(M^2/m^2)$. Hence, due to DR, the $\bar{g}(q^2)$ evolution changes drastically. The effective coupling slightly diminishes[1] beyond the reduction scale and tends to a finite value, see Fig. 3.

2.2. Reduction with Lagrangians. Now, along with the *Agreement on DR*, return to the Lagrangian description Eq. (1) in terms of fields. There, under transition to the $d = 2$ case, the field $\varphi_4(x)$ loses its dimensionality $\varphi_4(x) \to \varphi_2(x) \sim M^{-1}\varphi_4(x)$, while the coupling constant (as in Eq. (1)) acquires it: $g_4 \to M^2 g_2$, with some parameter M that we put equal to the DR scale $M = M_{\rm DR}$. Below, the dimensionless constant g_2 will be used.

Invariant Coupling in $d = 2$. In two dimensions, one can as well use finite Dyson transformations, formulate RG invariance and mass-dependent renormalization group[2], define a «massive» running coupling with its explicit one-loop solution [13] (see also § 43.1 in [10][3] and paper [14])

$$\bar{g}^{[2]}(q^2) = \frac{g}{1 - gJ_2(q^2/m^2)}, \quad J_2\left(\frac{q^2}{m^2}\right) = \frac{i}{\pi}\int \frac{M^2 d^2k}{(m^2 + k^2)[m^2 + (k+q)^2]}. \tag{5}$$

Here, J_2 is a finite one-loop contribution from the 4-vertex diagram, Fig. 2, in two dimensions. It is a positive monotonously decreasing function. Asymptotically, $J_2 \sim (M^2/q^2)\ln(q^2/m^2)$, just like in the second Eq. (4). Therefore, two-dimensional effective coupling in the UV limit tends to its limiting fixed value from above.

Hard Conjunction at the Reduction Scale. To obtain the joint picture of coupling evolution, one has to consider a transition from the «low-energy» four-dimensional region $q^2 < M^2$ to the «high-energy» two-dimensional one $q^2 > M^2$.

For the «hard» conjunction[4], the property of continuity $\bar{g}_4(M^2) = \bar{g}_2(M^2)$ yields

$$\bar{g}_4(q^2) = \frac{g_M}{1 - g_M \ln(q^2/M^2)}, \quad q^2 \leqslant M^2 \tag{6}$$

[1]This reverse evolution is not seen in the common massless, pure logarithmic, RG analysis.

[2]As was introduced in the mid-1950s in [9]: see also Ch. VIII in monograph [10] or Ch. IX, § 51 in its 3rd edition [11] and Appendix 9 in textbook [12].

[3]Unhappily, this piece was omitted in the next edition [11].

[4]In the classical wine-bottle model, this hard conjunction corresponds to Fig. 1, *d*.

and, along with expression (5),

$$\bar{g}_2(q^2) = \frac{g_M}{1 - g_M\,[J_2(q^2/m^2) - J_2(M^2/m^2)]}, \quad q^2 \geqslant M^2 \tag{7}$$

with finite UV limit

$$\bar{g}_2(\infty) = \frac{g_M}{1 + g_M J_2(M^2/m^2)} < g_M. \tag{8}$$

This means that above the reduction scale the effective coupling evolves down to its final UV limit. This behavior corresponds to Fig. 3.

3. DISCUSSION

In the above analysis, one more alternative to the standard Higgs mechanism within the Standard Model was considered. The main idea consists in employing the possibility of reducing the number of dimensions in the far UV limit, the possibility that is intensively discussed now in the context of quantum gravity.

Taking for definiteness the reduction from common four dimensions to two dimensions at some high enough scale $|q| \sim M$, we studied the issue of effective coupling behavior for the φ^4 scalar model (1). Using hard conjunction, by continuity: $\bar{g}_4(M^2) = \bar{g}_2(M^2)$ just at the DR scale, we got the joint picture which is very close to the $\bar{g}$ behavior in the case of continuous DR by changing metric in the momentum space via Eq. (3). The resulting picture is presented in Fig. 3.

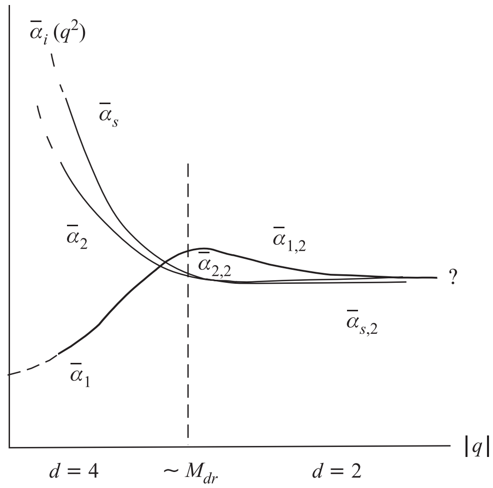

Fig. 4. Modified chart of the hypothetical Grand Unification provided by dimensional reduction instead of leptoquarks

Its essential technical feature is the change of the familiar logarithmic growth of the $\bar{g}_4$ running coupling for the slight decrease of $\bar{g}_2(q^2)$ beyond the reduction scale. There, in the effective 2-dimensional region, the φ_2^4 interaction is super-renormalizable and the one-loop mass-dependent contribution to running coupling is a *decreasing* function. Due to this, the $\bar{g}(q^2)$ evolution changes its pattern. At the infinity it tends to its limiting value from above, as in Eq. (7).

This peculiar property, the reverse running coupling evolution to a fixed point beyond the «distorted mirror» marked by the reduction scale, has a chance to upgrade the Grand Unification scenario with the M value being of the order or even greater than the hypothetical leptoquark scale — see qualitative illustration on Fig. 4.

Indeed, a rough estimate shows that difference between $1/\bar{g}(\infty)$ and $1/g_M$ is of order of unity; being imported to the GUT context, it seems to be numerically sufficient to close the famous dicrepancy triangle. See, e. g., left panel on Fig. 1 in Ref. [15].

The notable observation is that the change of geometry could yield the same final result as an explicit change of dynamics (by adding leptoquark fields, etc.).

Among further quests that are in order, let us put in the first place the problem of examining the possibility of detecting some physical signal «through the looking-glass at scale M» that would provide us with direct evidence on the existence of dimensional reduction of any kind. It could be fruitful to study some classical problems formulated at the end of Sec. 1 to enrich our intuition.

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