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REPRESENTATIONS OF GUIDED MODES OF INTEGRATED-OPTICAL MULTILAYER THIN-FILM WAVEGUIDES

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Представления направляемых мод планарных многослойных тонкопленочных волноводов

Проведен теоретический анализ распространения направляемых (собственных) мод в многослойных диэлектрических волноводах в рамках модели волноводов сравнения. Полагалось, что рассматриваемый волновод образован немагнитными средами, у которых диэлектрические проницаемости вещественны. Использовалась запись волновых уравнений для TE- и TM-мод через поперечные и продольные компоненты полей в декартовых координатах. Решения уравнений волноводных мод записывались через разные фундаментальные системы решений: комплекснозначные и вещественнозначные функции. Для каждой из них выведены соответствующие формы дисперсионных соотношений для TE- и TM-мод трехслойных и четырехслойных волноводов. Реализованы устойчивые методы решения нелинейных трансцендентных алгебраических дисперсионных уравнений и соответствующих систем линейных алгебраических уравнений для вычисления полей волноводных мод. Приведены выражения, позволяющие вычислить толщины отсечек для соответствующих TE- и TM-мод.

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Ayryan E. A. et al. E11-2011-31
Representations of Guided Modes of Integrated-Optical Multilayer Thin-Film Waveguides

We investigate the guided propagation (eigen) modes in the regular multilayer dielectric waveguides. The waveguide involves several nonmagnetic media with real dielectric constants, and the description of the corresponding wave equations is done in terms of transverse and longitudinal field components in Cartesian coordinates. In order to allow comparison with various previous approaches, the solutions of the equations of the guided modes are expressed in terms of both real valued and complex valued fundamental systems of solutions. For each of them we derive the appropriate form of the dispersion relation for the TE and TM modes of three-layer and four-layer waveguides. Stable methods of solving the resulting nonlinear transcendental algebraic dispersion equations and related systems of linear algebraic equations are implemented and used for the calculation of the fields of the waveguide modes.

The investigation has been performed at the Laboratory of Information Technologies, JINR.

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1. INTRODUCTION

This publication represents the first stage of the consideration of the method of comparison of waveguides as a verification way of the method of adiabatic modes. This method was published in [1–9] with applications to solving some specific problems. They presented the results of comparing the results obtained by the method of adiabatic modes with the results of alternative methods of solving specific problems [1–9].

We will consider the material medium, consisting of dielectric subregions that fill in together all three-dimensional space. That means that the dielectric constants of the subdomains are different and real, and the permeability everywhere is equal to the permeability of vacuum.

We consider the environment with zero charges and currents. Scalar Maxwell's equations can be obtained from the vector equations. The boundary conditions for the normal components can also be obtained from the boundary conditions for the tangential component (see, [10–20]). Material equations in this case we believe to be linear. Thus, the electromagnetic field in a space filled with dielectrics in the Gaussian system of units is described by the equations:

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \quad \operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}, \tag{1.1}$$

where $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{E}$, $\mathbf{H}$ are vectors of the electric and magnetic fields, $\mathbf{D}$ — electric displacement vector, $\mathbf{B}$ — the vector of magnetic induction, c — the speed of electromagnetic waves in a vacuum. In this case, the boundary conditions are valid:

$$\left.\mathbf{H}_\tau\right|_1 = \left.\mathbf{H}_\tau\right|_2, \quad \left.\mathbf{E}_\tau\right|_1 = \left.\mathbf{E}_\tau\right|_2, \tag{1.2}$$

and the asymptotic boundary conditions at infinity are:

$$\|\mathbf{E}\| \xrightarrow[|x|\to\infty]{} 0, \quad \|\mathbf{H}\| \xrightarrow[|x|\to\infty]{} 0, \tag{1.3}$$

that ensures the uniqueness of solutions of (1.1)–(1.3).

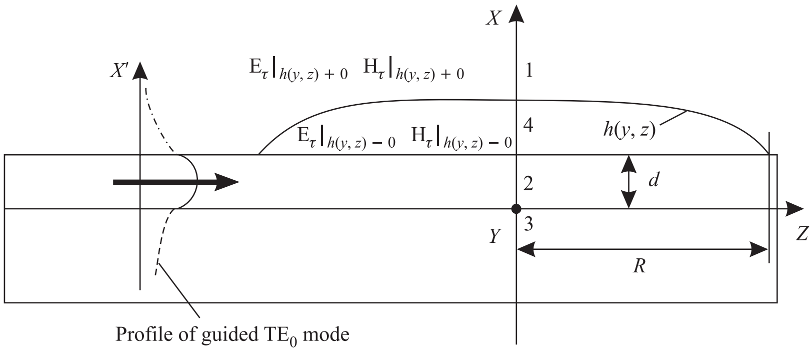

Fig. 1. Cross section of an integrated-optical structure is formed by a regular three-layer waveguide (left panel) and a smoothly irregular four-layer waveguide (on the right side of the figure). The three-layered waveguide is formed by 1–3 media, and four-layer — by 1–4 media. On the figure are indicated by: 1 — framing the environment or cover layer (air) with a refractive index n_c, 2 — waveguide layer with a refractive index n_f, 3 — substrate with a refractive index n_s, and 4 — the second waveguide layer with a refractive index n_l; d is a thickness of the first waveguide layer of integrated-optical structure; h — the thickness of the second waveguide layer

The method of adiabatic modes is to describe the individual guided modes of irregular integrated optical waveguide (see Fig. 1) as:

$$\begin{aligned} \tilde{\mathbf{E}}(x,y,z,t) &= \exp(i\omega t)\frac{\mathbf{E}_v(x;y,z)}{\sqrt{\beta(y,z)}}\exp\left[-ik_0\int^{y,z}\beta(y',z')\,ds\,(y',z')\right], \\ \tilde{\mathbf{H}}(x,y,z,t) &= \exp(i\omega t)\frac{\mathbf{H}_v(x;y,z)}{\sqrt{\beta(y,z)}}\exp\left[-ik_0\int^{y,z}\beta(y',z')\,ds\,(y',z')\right], \end{aligned} \tag{1.4}$$

where $\beta(y,z) = \sqrt{\beta_y^2(y,z)+\beta_z^2(y,z)}$ is a length (norm) of two-dimensional vector field $\boldsymbol{\beta}(y,z) = (\beta_y(y,z), \beta_z(y,z))^t$, composed of partial derivatives of eikonal $\beta_y(y,z) = \partial\phi/\partial y$, $\beta_z(y,z) = \partial\phi/\partial z$, as well as $\beta_y = k_y/k_0$, $\beta_z = k_z/k_0$. The eikonal (phase) $\phi(y,z) = k_0 \int^{y,z} \beta(y',z')\,ds\,(y',z')$ is calculated by integrating along the rays, after the dispersion relation and the isolated computation of rays and wave fronts in the horizontal plane [1, 2, 9], where $ds = \sqrt{dy^2+dz^2}$ is a ray length.

Substituting (1.4) into Maxwell's equations (1.1) leads to a system of ordinary differential equations of second order for the longitudinal components of vector-functions $\mathbf{E}_v(x;y,z)$ and $\mathbf{H}_v(x;y,z)$:

$$\frac{\partial^2 E_z}{\partial x^2} + \chi^2 E_z = -p_y \chi_z^2 \frac{\partial}{\partial y}\left(\frac{1}{\chi_z^2}\right) E_z - \frac{1}{i\omega\varepsilon}\left(\chi_z^2 p_z \frac{\partial}{\partial y}\left(\frac{1}{\chi_z^2}\right)\right)\frac{\partial H_z}{\partial x} \quad (1.5)$$

for quasi TM modes and

$$\frac{\partial^2 H_z}{\partial x^2} + \chi^2 H_z = -p_y \chi_z^2 \frac{\partial}{\partial y}\left(\frac{1}{\chi_z^2}\right) H_z + \frac{1}{i\omega\mu}\left(\chi_z^2 p_z \frac{\partial}{\partial y}\left(\frac{1}{\chi_z^2}\right)\right)\frac{\partial E_z}{\partial x} \quad (1.6)$$

for quasi-TE modes.

For transverse and vertical components of the vector-valued functions $\mathbf{E}_v(x;y,z)$ and $\mathbf{H}_v(x;y,z)$ we thus obtain the analytical expressions in terms of E_z, H_z and their derivatives with respect to the vertical argument x:

$$\chi_z^2 H_y = \frac{\partial^2 H_z}{\partial z \partial y} - \varepsilon \frac{\partial^2 E_z}{\partial t \partial x} \text{ and } \chi_z^2 E_x = \frac{\partial^2 E_z}{\partial z \partial x} - \mu \frac{\partial^2 H_z}{\partial t \partial y} \quad (1.7)$$

for quasi-TM modes and

$$\chi_z^2 H_x = \frac{\partial^2 H_z}{\partial z \partial x} + \varepsilon \frac{\partial^2 E_z}{\partial t \partial y} \text{and } \chi_z^2 E_y = \frac{\partial^2 E_z}{\partial z \partial y} + \mu \frac{\partial^2 H_z}{\partial t \partial x} \quad (1.8)$$

for quasi-TE modes. Here we use the notations $\chi_z^2 = k_0^2 \varepsilon\mu + p_z p_z + \partial p_z/\partial z$, $\chi^2 = \chi_z^2 + p_y p_y + \partial p_y/\partial y$, $p_y = -ik_0\beta_y - (2\beta)^{-1}\partial\beta/\partial y$, $p_z = -ik_0\beta_z - (2\beta)^{-1}\partial\beta/\partial z$. In the case of a smoothly irregular integrated optical waveguide, «the condition of quasi-classicality» is performed [1, 2, 9]

$$\delta = \max\left|\nabla_{y,\,z}\beta\right|(k_0\beta^2)^{-1} \ll 1, \quad (1.9)$$

that allows us to solve the problem (1.5)–(1.8) by asymptotic method in the dimensionless small parameter δ.

The study of approximation of zero and first order problem (1.5)–(1.8), and comparison of our approximations with the models of other authors describing smoothly irregular waveguides [6, 8, 9], have led us to the formation of a hierarchy of matrix models describing the propagation of guided modes in a smoothly irregular waveguides and to the necessity of their detailed study. The most inaccurate model of this hierarchy is the matrix model of comparison waveguides [6, 8, 9]. In fact it is a model of regular planar multilayer thin film waveguides with variable thickness of the layers. To establish the essential characteristics of the models of propagating modes and other objects of the study, it is sufficient to study details of the three- and four-layer planar dielectric waveguides.

In various books [11–18] and papers [21–37] for integrated optics various forms of reducing Maxwell's equations to different systems of ordinary differential equations for TE and TM modes are used. At the same time, various forms of the conditions for the solvability of these ordinary differential equations, called dispersion relations, are used. Bearing in mind the subsequent using of the model of regular planar multilayer thin-film dielectric waveguides with variable thickness of the layers to describe irregular waveguides, as well as further comparison of obtained in their framework analytical and numerical results with those of other authors, we consider here all possible systems of ordinary differential equations for TE and TM modes, as well as all forms of the dispersion equations. After that, we show that every expression of [11–37] coincides with one of our expressions, and show by numerical calculations that different forms of expressions describe quantitatively equally guided (waveguide) modes of regular planar multilayer thin-film dielectric waveguides with variable layer thicknesses.

2. THE REDUCED ORDINARY DIFFERENTIAL EQUATIONS FOR GUIDED MODES

We shall describe the electromagnetic field with complex amplitudes to simplify the calculations [1]. We will consider the material medium, consisting of dielectric subregions that fill in together all three-dimensional space. The latter means, that the dielectric constants of the subdomains are different and real, and the permeability is everywhere equal to the permeability of vacuum. It follows that in the absence of external currents and charges, induced currents and charges are equal to zero.

In equations (1.1): $\varepsilon = \varepsilon_r \varepsilon_0$ is the permittivity of the medium; $\mu = \mu_r \mu_0$ is the permeability of the medium; ε_r, μ_r are relative permittivity and permeability, respectively (in the nonmagnetic medium $\mu_r = 1$ is assumed); ε_0 and μ_0 are dielectric and magnetic constants of vacuum, respectively; $\omega\sqrt{\mu\varepsilon} = nk_0$, n is the index of refraction of the medium (here and further of the layer under consideration in multilayered dielectric structure); $k_0 = 2\pi/\lambda_0$, ω is the cyclic frequency of the electromagnetic field; $\mathbf{E}$, $\mathbf{H}$ are the vectors of the electric and magnetic fields.

Assume also that all subdomains are endless and are limited by planes parallel to the plane yOz, so that further $\varepsilon = \varepsilon(x), \mu = 1$ (see Fig. 2).

Waveguide is formed by media 1–3. The figure indications are: 1 is a framing medium or cover layer (air) with refractive index n_c; 2 is a waveguide layer (film) with a refractive index n_f; 3 is a substrate with refractive index n_s; d is the thickness of the waveguide layer. Film and substrate are homogeneous in the x and z directions, the substrate is usually much thicker than the film.

We consider the propagation of monochromatic polarized electromagnetic radiation in the above three-layer dielectric regular system (see Fig. 2) (assuming an energy source located infinitely far away from the area under consideration). Under the conditions of the transverse resonance (also known as the quanti-

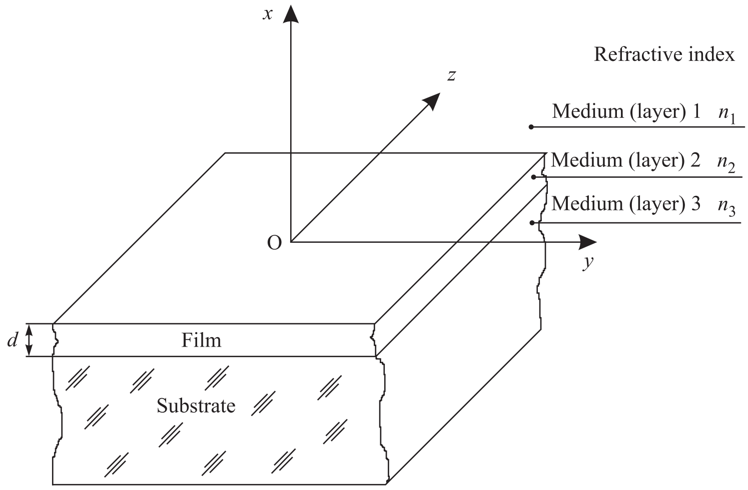

Fig. 2. Scheme of a flat three-layer dielectric waveguide

zation conditions of Bohr–Sommerfeld [11]), the dielectric system is a regular three-layer dielectric waveguide capable of supporting guided waveguide modes. These conditions are reduced to the implementation of the relevant dispersion relations in the early work on integrated optics called the characteristic equations (see [11–17]).

Maxwell's equations in Cartesian coordinates in the GHS are as follows:

$$\begin{aligned}
&\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\varepsilon}{c}\frac{\partial E_x}{\partial t}, \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu}{c}\frac{\partial H_x}{\partial t},\\
&\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{\varepsilon}{c}\frac{\partial E_y}{\partial t}, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu}{c}\frac{\partial H_y}{\partial t},\\
&\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{\varepsilon}{c}\frac{\partial E_z}{\partial t}, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu}{c}\frac{\partial H_z}{\partial t}.
\end{aligned} \tag{2.1}$$

We seek after solutions of equations (2.1) of the class of infinite (twice) differentiable functions of four variables, harmonically time-dependent and invariant under translation along the axis Oy, so that they satisfy relations $\dfrac{\partial}{\partial y} \equiv 0$, $\dfrac{\partial H_i}{\partial t} = -i\omega H_i$, $\dfrac{\partial E_j}{\partial t} = -i\omega E_j$. Consideration of these relationships reduces the system (2.1) as follows:

$$
\begin{aligned}
-\frac{\partial H_y}{\partial z} &= -\frac{i\omega\varepsilon}{c}E_x, \quad -\frac{\partial E_y}{\partial z} = \frac{i\omega\mu}{c}H_x, \\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= -\frac{i\omega\varepsilon}{c}E_y, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = \frac{i\omega\mu}{c}H_y, \\
\frac{\partial H_y}{\partial x} &= -\frac{i\omega\varepsilon}{c}E_z, \quad \frac{\partial E_y}{\partial x} = \frac{i\omega\mu}{c}H_z.
\end{aligned} \tag{2.2}
$$

The system of equations (2) splits into two independent subsystems for two different polarizations (they correspond to two types of guided modes: TE modes and TM modes). In view of $k_0 = \frac{\omega}{c}$, the subsystems are as follows:

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = ik_0\mu H_y, \tag{2.3}$$

$$E_x = \frac{1}{ik_0\varepsilon}\frac{\partial H_y}{\partial z}, \tag{2.4}$$

$$E_z = -\frac{1}{ik_0\varepsilon}\frac{\partial H_y}{\partial x} \tag{2.5}$$

and

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = ik_0\varepsilon E_y, \tag{2.6}$$

$$H_x = -\frac{1}{ik_0\mu}\frac{\partial E_y}{\partial z}, \tag{2.7}$$

$$H_z = \frac{1}{ik_0\mu}\frac{\partial E_y}{\partial x}. \tag{2.8}$$

Substitution of (2.4) and (2.5) into (2.3) leads them to the form of Helmholtz equation:

$$\frac{1}{ik_0}\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon}\frac{\partial H_y}{\partial z}\right) + \frac{1}{ik_0}\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial H_y}{\partial x}\right) = ik_0\mu H_y. \tag{2.9}$$

Similarly, substitution of (2.7) and (2.8) into (2.6) leads them to the form of Helmholtz equation:

$$-\frac{1}{ik_0\mu}\frac{\partial^2 E_y}{\partial z^2} - \frac{1}{ik_0\mu}\frac{\partial^2 E_y}{\partial x^2} = -ik_0\varepsilon E_y. \tag{2.10}$$

We transform two equations to the standard form:

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + k_0^2\varepsilon\mu\right)E_y = 0, \tag{2.11}$$

$$\left(\frac{\partial^2}{\partial z^2} + \varepsilon \frac{\partial}{\partial x} \left(\frac{1}{\varepsilon} \frac{\partial}{\partial x} \right) + k_0^2 \varepsilon \mu \right) H_y = 0. \tag{2.12}$$

Equation (2.11) is obtained from the wave equation for TE mode with the leading component E_y, while Eq. (2.12) is obtained from the wave equation for TM mode with the leading component H_y, so in the theory of planar waveguides (see [11–17]), both of these equations are usually called the wave equations.

The method of separation of variables leads to a factorization of solutions, resulting in:

$$E_y(x, y, z, t) = E_y(x) \exp\{i(\omega t \pm k_0 \beta z)\}. \tag{2.13}$$

Similarly, the solution of Eq. (2.12) has the form:

$$H_y(x, y, z, t) = H_y(x) \exp\{i(\omega t \pm k_0 \beta z)\}. \tag{2.14}$$

In this case, from (2.7) and (2.8) we get:

$$E_x = \frac{1}{ik_0 \varepsilon} \frac{\partial H_y}{\partial z} = \frac{ik_0 \beta}{ik_0 \varepsilon} H_y = \frac{\beta}{\varepsilon} H_y, \tag{2.15}$$

$$E_z = -\frac{1}{ik_0 \varepsilon} \frac{\partial H_y}{\partial x}. \tag{2.16}$$

Similarly, from (2.4) and (2.5) we get:

$$H_x = -\frac{ik_0 \beta}{ik_0 \mu} E_y = -\frac{\beta}{\mu} E_y, \tag{2.17}$$

$$H_z = \frac{1}{ik_0 \mu} \frac{dE_y}{dx}. \tag{2.18}$$

For the solutions of the form (2.13), (2.14), Eqs. (2.11) and (2.12) take the form:

$$\frac{d^2 E_y}{dx^2} + k_0^2 \left(\varepsilon \mu - \beta^2 \right) E_y(x) = 0, \tag{2.19}$$

$$\varepsilon \frac{d}{dx} \left(\frac{1}{\varepsilon} \frac{dH_y}{dx} \right) + k_0^2 \left(\varepsilon \mu - \beta^2 \right) H_y(x) = 0. \tag{2.20}$$

Thus, Eqs. (2.19) and (2.20) for the transverse components E_y and H_y determine the TE- and TM-polarized solutions of Maxwell's Eqs. (2.1). Expressions (2.16) and (2.18) define a second pair of tangential components required for the formulation of boundary conditions (1.2). Expressions (2.15) and (2.17) will be required to write the energy conservation law.

Equivalent reduction of Maxwell's Eqs. (2.1) to a pair of equations for the longitudinal components E_z and H_z, can be obtained as follows. Let us represent the solutions of the form

$$\mathbf{E}(x, y, z, t) = \mathbf{E}_v(x) \exp\left\{ i(\omega t \pm k_0 \beta z) \right\}, \tag{2.21}$$

$$\mathbf{H}(x,y,z,t) = \mathbf{H}_v(x)\exp\{i(\omega t \pm k_0\beta z)\} \tag{2.22}$$

in the form:

$$\mathbf{E} = \vec{E}_\perp + E_z\vec{e}_z, \quad \mathbf{H} = \vec{H}_\perp + H_z\vec{e}_z.$$

Let us represent also a three-dimensional operator $\vec{\nabla}$ in the form:

$$\begin{aligned}\vec{\nabla} &= \vec{\nabla}_\perp + \vec{e}_z\frac{\partial}{\partial z} = \vec{\nabla}_\perp - ik_0\beta\vec{e}_z,\\ \vec{\nabla}_\perp &= \vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} = \vec{e}_x\frac{\partial}{\partial x}.\end{aligned} \tag{2.23}$$

Let us put $\chi^2 = k_0^2(\varepsilon\mu - \beta^2)$, then taking into account the condition $\partial/\partial y \equiv 0$, the system of Maxwell's equations after additional application of $\vec{\nabla}\times$ splits into two subsystems. Subsystem for the TM modes, recorded in the longitudinal components, takes the form:

$$\begin{aligned}\vec{\nabla}_\perp^2 E_z + \chi^2 E_z &= 0, H_z \equiv 0,\\ \vec{H}_\perp &= \frac{i\omega\varepsilon}{\chi^2}[(\vec{\nabla}_\perp E_z)\times\vec{e}_z],\\ \vec{E}_\perp &= -\left(\frac{ik_0\beta}{\chi^2}\right)\vec{\nabla}_\perp E_z.\end{aligned}$$

Subsystem for the TE modes, recorded in the longitudinal components, takes the form:

$$\begin{aligned}\vec{\nabla}_\perp^2 H_z + \chi^2 H_z &= 0, E_z \equiv 0,\\ \vec{H}_\perp &= -\left(\frac{ik_0\beta}{\chi^2}\right)\vec{\nabla}_\perp H_z,\\ \vec{E}_\perp &= -\frac{i\omega\mu}{\chi^2}[(\vec{\nabla}_\perp H_z)\times\vec{e}_z].\end{aligned}$$

If we decompose longitudinal and transverse components further by coordinates, the above subsystems of equations turn to the following form:
for TM modes

$$\begin{aligned}\frac{d^2E_z}{dx^2} + k_0^2(\varepsilon\mu - \beta^2)E_z &= 0, \quad H_z = 0,\\ E_x &= -\left(\frac{ik_0\beta}{\chi^2}\right)\frac{dE_z}{dx}, \quad E_y = 0,\\ H_y &= -\left(\frac{i\omega\varepsilon}{\chi^2}\right)\frac{dE_z}{dx}, \quad H_x = 0,\end{aligned} \tag{2.24}$$

and for TE modes

$$\begin{aligned}
&\frac{d^2H_z}{dx^2} + k_0^2(\varepsilon\mu - \beta^2)H_z = 0, \quad E_z = 0, \\
&H_x = -\left(\frac{ik_0\beta}{\chi^2}\right)\frac{dH_z}{dx}, \quad H_y = 0, \\
&E_y = \left(\frac{i\omega\mu}{\chi^2}\right)\frac{dH_z}{dx}, \quad E_x = 0.
\end{aligned} \tag{2.25}$$

3. THE METHOD OF SOLUTION OF THE REDUCED ODE FOR TE AND TM MODES OF DECOMPOSITION OF THE FUNDAMENTAL SYSTEMS OF SOLUTIONS

Shown in Fig. 2 stratified along the axis Ox the dielectric medium admits a square integrable distributions $E_j(x)$, $H_j(x)$; $j = x, y, z$ if the asymptotic behavior of these distributions tends to zero with the distance along the axis Ox of the waveguide layers tending to infinity. For this purpose it is necessary that the dielectric constants of the plates — the substrate ε_s and the covering layer ε_c — were lower than dielectric constants ε_{f1} and ε_{fm} of the adjacent waveguide layers, in case the structure consists of the m waveguide layers and two parietal. In this case, the internal waveguide layers may have a dielectric constant not higher than ε_s and ε_c assuming inside antiwaveguide modes.

Comment. Equations (2.19) and (2.20) for the transverse components, and similar Eqs. (2.24) and (2.25) for the longitudinal components, are equal to (within signs) one-dimensional quantum mechanical Schrödinger equation for the potential well of stepwise or for a system of potential wells separated by an internal barrier (analog to antiwaveguide layer). Function of potential energy for such a problem has a mirror symmetric form for the distribution of dielectric constants of the layers of the planar waveguide.

Solutions (2.21) and (2.22) describe the forward and backward waves propagating along the axis Oz with the speed β times less than the speed of light. Square-integrable distributions $E_j(x)$, $H_j(x)$; $j = x, y, z$ are solutions of the Eqs. (2.19) and (2.20) or Eqs. (2.24) and (2.25), corresponding to the discrete spectrum of values β that lie in the range of $n_s = \sqrt{\varepsilon_s}$ (and while $n_s \geqslant n_c$ and $\varepsilon_s \geqslant \varepsilon_c$) to a maximal value $n_{\max} = \max\limits_{1\leqslant k\leqslant m} n_{fk} = \sqrt{\varepsilon_{\max}}$, $\varepsilon_{\max} = \max\limits_{1\leqslant k\leqslant m} \varepsilon_{fk}$. For each fixed β_j we introduce the notation $\gamma_c^j = k_0\sqrt{\beta_j^2 - \varepsilon_c}$, $\gamma_s^j = k_0\sqrt{\beta_j^2 - \varepsilon_s}$, as well as $\gamma_k^j = k_0\sqrt{\beta_j^2 - \varepsilon_{fk}}$ for those sectors that have $\beta_j^2 - \varepsilon_{fk} \geqslant 0$. If $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, we introduce the notation $\chi_k^j = k_0\sqrt{\varepsilon_{fk} - \beta_j^2}$.

3.1. Expressions for the TE Modes Through the Transverse Electric Field Component. Let us write explicit expressions of the components E_y^k of the electric fields of the TE modes with a coefficient of phase delay β_j through the general solution of Eq. (2.19) for all dielectric layers with permittivities $\varepsilon_c, \varepsilon_{f1}, \ldots, \varepsilon_{fk}, \ldots, \varepsilon_{fm}, \varepsilon_s$, as well as explicit expressions of the components H_z^k of the magnetic fields of the TE modes with the help of Eq. (2.18), necessary to record the tangential boundary conditions at the interfaces of layers.

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \geqslant 0$, the general solution of Eq. (2.19) has the form:

$$E_k^j = A_k^+ \exp\{\gamma_k^j x\} + A_k^- \exp\{-\gamma_k^j x\}. \tag{3.1}$$

In the parietal layers of the overall solution, only those terms should be kept, which satisfy the asymptotic boundary conditions at infinity (1.3), so that

$$E_s^j = A_s^+ \exp\{\gamma_s^j x\}, \quad E_c^j = A_c^- \exp\{-\gamma_c^j x\}. \tag{3.2}$$

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, the general solution of Eq. (2.19) has the form:

$$E_k^j = A_k^+ \exp\{i\chi_k^j x\} + A_k^- \exp\{-i\chi_k^j x\}. \tag{3.3}$$

Respectively, in those layers in which $\beta_j^2 - \varepsilon_{fk} \geqslant 0$, the components H_z^k have the form:

$$H_k^j = \frac{\gamma_k^j}{ik_0}\left(A_k^+ \exp\{\gamma_k^j x\} - A_k^- \exp\{-\gamma_k^j x\}\right). \tag{3.4}$$

In the parietal layers the expressions of the components H_z have the form:

$$H_s^j = \frac{\gamma_s^j}{ik_0} A_s^+ \exp\{\gamma_s^j x\}, \;\; H_c^j = -\frac{\gamma_c^j}{ik_0} A_c^- \exp\{-\gamma_c^j x\}. \tag{3.5}$$

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, expressions of the components H_z^k are as follows:

$$H_k^j = \frac{\chi_k^j}{k_0}\left(A_k^+ \exp\{i\chi_k^j x\} - A_k^- \exp\{-i\chi_k^j x\}\right). \tag{3.6}$$

In addition to providing a general solution of (2.19) in the form of (3.3) one can use the other two equivalent representations:

$$E_k^j = A_k^c \cos\{\chi_k^j x\} + A_k^s \sin\{\chi_k^j x\}, \tag{3.7}$$

then the expressions of the components H_z^k have the form:

$$H_k^j = -\frac{\chi_k^j}{ik_0}\left(A_k^c \sin\{\chi_k^j x\} - A_k^s \cos\{\chi_k^j x\}\right). \tag{3.8}$$

And also if

$$E_k^j = C_k \cos\{\chi_k^j x + \phi_k\}, \tag{3.9}$$

then the expressions of the components H_z^k have the form:

$$H_k^j = -\frac{\chi_k^j}{ik_0} C_k \sin\{\chi_k^j x + \phi_k\}. \tag{3.10}$$

Using expressions (3.7) and (3.8) allows the numerical calculations for dielectric waveguides using real arithmetic, which will be seen later. Using expressions (3.9) and (3.10) allows us to write dispersion relations in the most widespread trigonometric form.

3.2. Expressions for TM Modes through the Transverse Magnetic Field Component. We now write the explicit expressions of the components H_y^k of the magnetic fields of TM modes with a coefficient of phase delay β_j through the general solution of Eq. (2.20) for all dielectric layers with permittivities $\varepsilon_c, \varepsilon_{f1}, \ldots, \varepsilon_{fk}, \ldots, \varepsilon_{fm}, \varepsilon_s$, as well as explicit expressions of the components of the electric fields of TM modes using Eq. (2.16), necessary to record the tangential boundary conditions at the interfaces.

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \geqslant 0$, the general solution of Eq. (2.20) has the form:

$$H_k^j = B_k^+ \exp\{\gamma_k^j x\} + B_k^- \exp\{-\gamma_k^j x\}. \tag{3.11}$$

In the parietal layers of the overall solution, only those terms should be kept, which satisfy the asymptotic boundary conditions at infinity (1.3), so that

$$H_s^j = B_s^+ \exp\{\gamma_s^j x\}, \;\; H_c^j = B_c^- \exp\{-\gamma_c^j x\}. \tag{3.12}$$

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, the general solution of equation (2.20) has the form:

$$H_k^j = B_k^+ \exp\{i\chi_k^j x\} + B_k^- \exp\{-i\chi_k^j x\}. \tag{3.13}$$

Accordingly, in those layers, in which $\beta_j^2 - \varepsilon_{fk} \geqslant 0$, the expressions of the components E_z^k are as follows:

$$E_k^j = -\frac{\gamma_k^j}{ik_0\varepsilon_k}\left(B_k^+ \exp\{\gamma_k^j x\} - B_k^- \exp\{-\gamma_k^j x\}\right). \tag{3.14}$$

In the parietal layers the expressions of the components E_z are as follows:

$$E_s^j = -\frac{\gamma_s^j}{ik_0\varepsilon_s} B_s^+ \exp\left\{\gamma_s^j x\right\}, \;\; E_c^j = \frac{\gamma_c^j}{ik_0\varepsilon_c} B_c^- \exp\{-\gamma_c^j x\}. \tag{3.15}$$

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, expressions of the components E_z^k are as follows:

$$E_k^j = -\frac{\chi_k^j}{k_0\varepsilon_k}\left(B_k^+ \exp\{i\chi_k^j x\} - B_k^- \exp\{-i\chi_k^j x\}\right). \tag{3.16}$$

In addition to providing a general solution of (2.20) in the form of (3.13), one can use the other two equivalent representations:

$$H_k^j = B_k^c \cos\{\chi_k^j x\} + B_k^s \sin\{\chi_k^j x\}, \tag{3.17}$$

then the expressions of the components E_z^k are as follows:

$$E_k^j = \frac{\chi_k^j}{ik_0\varepsilon_k}\left(B_k^c \sin\{\chi_k^j x\} - B_k^s \cos\{\chi_k^j x\}\right), \tag{3.18}$$

and also if

$$H_k^j = D_k \cos\{\chi_k^j x + \psi_k\}, \tag{3.19}$$

then the expressions of the components E_z^k are as follows:

$$E_k^j = \frac{\chi_k^j}{ik_0\varepsilon_k} D_k \sin\{\chi_k^j x + \psi_k\}. \tag{3.20}$$

Using expressions (3.17) and (3.18) allows the numerical calculations for dielectric waveguides using real arithmetic, which will be seen later. Using expressions (3.19) and (3.20) allows us to write dispersion relations in the most widespread trigonometric form.

3.3. Expressions for the TE Modes in Terms of Longitudinal Magnetic Field Component. In the same way as was done in 3.2. for the fields of TM modes through the transverse components of magnetic fields H_y^k, write explicit expressions for the fields of TE modes in terms of longitudinal components $\tilde{H}_z^k$ of magnetic fields of the general solution of equation (2.25) with a coefficient of phase delay β_j for the layers with permittivities $\varepsilon_c, \varepsilon_{f1}, \ldots, \varepsilon_{fk}, \ldots, \varepsilon_{fm}, \varepsilon_s$, as well as explicit expressions of the components $\tilde{E}_y^k$ of electric fields.

In those layers, where $\beta_j^2 - \varepsilon_{fk} \geqslant 0$, a common solution $\tilde{H}_z^k$ to Eq. (2.25) has the form:

$$\tilde{H}_k^j = \tilde{B}_k^+ \exp\{\gamma_k^j x\} + \tilde{B}_k^- \exp\{-\gamma_k^j x\}. \tag{3.21}$$

In the parietal layers the solutions have the form:

$$\tilde{H}_s^j = \tilde{B}_s^+ \exp\{\gamma_s^j x\}, \;\; \tilde{H}_c^j = \tilde{B}_c^- \exp\{-\gamma_c^j x\}. \tag{3.22}$$

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, the general solution of Eq. (2.25) has the form:

$$\tilde{H}_k^j = \tilde{B}_k^+ \exp\{i\chi_k^j x\} + \tilde{B}_k^- \exp\{-i\chi_k^j x\}. \tag{3.23}$$

Accordingly, in those layers, in which $\beta_j^2 - \varepsilon_{fk} \geqslant 0$, the expressions the components $\tilde{E}_y^k = \left(\frac{i\omega\mu_k}{\chi_k^2}\right)\frac{d\tilde{H}_z^k}{dx}$ have the form:

$$\tilde{E}_k^j = i\omega\left(\frac{\mu_k}{\gamma_k^j}\right)\left(\tilde{B}_k^+ \exp\{\gamma_k^j x\} - \tilde{B}_k^- \exp\{-\gamma_k^j x\}\right). \tag{3.24}$$

In the parietal layers the components $\tilde{E}_y^j$ have the form:

$$\tilde{E}_s^j = -i\omega\left(\frac{\mu_s}{\gamma_s^j}\right)\tilde{B}_s^+\exp\{\gamma_s^j x\}, \quad \tilde{E}_c^j = i\omega\left(\frac{\mu_c}{\gamma_c^j}\right)\tilde{B}_c^-\exp\{-\gamma_c^j x\}. \quad (3.25)$$

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, expressions of the components $\tilde{E}_y^j$ are as follows:

$$\tilde{E}_k^j = -\omega\left(\frac{\mu_k}{\chi_k^j}\right)\left(\tilde{B}_k^+\exp\{i\chi_k^j x\} - \tilde{B}_k^-\exp\{-i\chi_k^j x\}\right). \quad (3.26)$$

General solutions of Eq. (2.25) for the components $\tilde{H}_z^k$ can also be represented as (3.18):

$$\tilde{H}_k^j = \tilde{B}_k^c\cos\{\chi_k^j x\} + \tilde{B}_k^s\sin\{\chi_k^j x\}, \quad (3.27)$$

then the expressions for the components $\tilde{E}_y^j$ take the form:

$$\tilde{E}_k^j = -i\omega\left(\frac{\mu_k}{\chi_k^j}\right)\left(\tilde{B}_k^c\sin\{\chi_k^j x\} - \tilde{B}_k^s\cos\{\chi_k^j x\}\right) \quad (3.28)$$

And also if

$$\tilde{H}_k^j = \tilde{D}_k\cos\{\chi_k^j x + \tilde{\psi}_k\}, \quad (3.29)$$

then the expressions for the components $\tilde{E}_y^j$ take the form:

$$\tilde{E}_k^j = -i\omega\left(\frac{\mu_k}{\chi_k^j}\right)\tilde{D}_k\sin\{\chi_k^j x + \tilde{\psi}_k\}. \quad (3.30)$$

3.4. Expressions for TM Modes in Terms of Longitudinal Electric Field Components. In the same way as was done in Subsec. 3.1 for the fields of TE modes by transverse components E_y^k, we write explicit expressions for the fields of TM modes in terms of longitudinal components $\tilde{E}_z^k$ of electric fields with a coefficient of phase delay β_j of the general solutions of (2.24) for layers with permittivities $\varepsilon_c, \varepsilon_{f1}, \ldots, \varepsilon_{fk}, \ldots, \varepsilon_{fm}, \varepsilon_s$, as well as explicit expressions of the components $\tilde{H}_y^k$ of magnetic fields.

In those layers in which $\beta_j^2 - \varepsilon_{fk} \geqslant 0$, common solution E_z^k to Eq. (2.24) has the form:

$$\tilde{E}_k^j = \tilde{A}_k^+\exp\{\gamma_k^j x\} + \tilde{A}_k^-\exp\{-\gamma_k^j x\}. \quad (3.31)$$

In the parietal layers, the solutions have the form:

$$\tilde{E}_s^j = \tilde{A}_s^+\exp\{\gamma_s^j x\}, \quad \tilde{E}_c^j = \tilde{A}_c^-\exp\{-\gamma_c^j x\}. \quad (3.32)$$

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, the general solution of Eq. (2.24) has the form:

$$\tilde{E}_k^j = \tilde{A}_k^+ \exp\{i\chi_k^j x\} + \tilde{A}_k^- \exp\{-i\chi_k^j x\}. \tag{3.33}$$

Accordingly, in those layers, in which $\beta_j^2 - \varepsilon_{fk} \geqslant 0$, the expressions of the components $\tilde{H}_y^k = -\left(\dfrac{i\omega\varepsilon_k}{\chi_k^2}\right)\dfrac{d\tilde{E}_z^k}{dx}$ are as follows:

$$\tilde{H}_k^j = -ik_0\left(\frac{\varepsilon_k\gamma_k^j}{\chi_k^2}\right)\left(\tilde{A}_k^+ \exp\{\gamma_k^j x\} - \tilde{A}_k^- \exp\{-\gamma_k^j x\}\right). \tag{3.34}$$

In the parietal layers, the expressions of the components $\tilde{H}_y^j$ are as follows:

$$\tilde{H}_s^j = -ik_0\left(\frac{\varepsilon_s\gamma_s^j}{\chi_s^2}\right)\tilde{A}_s^+ \exp\{\gamma_s^j x\}, \quad \tilde{H}_c^j = ik_0\left(\frac{\varepsilon_c\gamma_c^j}{\chi_c^2}\right)\tilde{A}_c^- \exp\{-\gamma_c^j x\}. \tag{3.35}$$

In those layers, in which $\beta_j^2 - \varepsilon_{fk} \leqslant 0$, expressions of the components $\tilde{H}_y^j$ are as follows:

$$\tilde{H}_k^j = k_0\left(\frac{\varepsilon_k}{\chi_k^j}\right)\left(\tilde{A}_k^+ \exp\{i\chi_k^j x\} - \tilde{A}_k^- \exp\{-i\chi_k^j x\}\right). \tag{3.36}$$

General solutions of Eq. (2.24) for the components $\tilde{E}_z^k$ can be represented as (3.7):

$$\tilde{E}_k^j = \tilde{A}_k^c \cos\{\chi_k^j x\} + \tilde{A}_k^s \sin\{\chi_k^j x\}, \tag{3.37}$$

then the expressions for the components $\tilde{H}_y^k$ take the form:

$$\tilde{H}_k^j = i\omega\left(\frac{\varepsilon_k}{\chi_k^j}\right)\left(\tilde{A}_k^c \sin\{\chi_k^j x\} - \tilde{A}_k^s \cos\{\chi_k^j x\}\right). \tag{3.38}$$

And also if

$$\tilde{E}_k^j = \tilde{C}_k \cos\{\chi_k^j x + \tilde{\phi}_k\}, \tag{3.39}$$

then the expressions for the components $\tilde{H}_y^k$ take the form:

$$\tilde{H}_k^j = i\omega\left(\frac{\varepsilon_k}{\chi_k^j}\right)\tilde{C}_k \sin\{\chi_k^j x + \tilde{\phi}_k\}. \tag{3.40}$$

4. BOUNDARY EQUATIONS FOR THREE-LAYER WAVEGUIDES

4.1. TE Modes, Expressed through the Transverse Component E_y. The boundaries between layers of three-layer planar waveguide cross the axis Ox at a point $x = a_1$ between the substrate and the waveguide layer and at a point $x = a_2$ between the waveguide layer and covering layer. On these boundaries for the TE mode, which is expressed through a transverse component E_y, the boundary conditions have the form: $E_{ys}(a_1) = E_{yf}(a_1)$ and $H_{zs}(a_1) = H_{zf}(a_1)$ at a point $x = a_1$ and also $E_{yf}(a_2) = E_{yc}(a_2)$ and $H_{zf}(a_2) = H_{zc}(a_2)$ at a point $x = a_2$. Let us express them in terms of amplitude coefficients $A_k^{\pm}$:

$$A_s^+ \exp\{\gamma_s^j a_1\} = A_1^+ \exp\{i\chi_1^j a_1\} + A_1^- \exp\{-i\chi_1^j a_1\},$$

$$\frac{\gamma_s^j}{ik_0} A_s^+ \exp\{\gamma_s^j a_1\} = \frac{\chi_1^j}{k_0}\left(A_1^+ \exp\{i\chi_1^j a_1\} - A_1^- \exp\{-i\chi_1^j a_1\}\right),$$

$$A_1^+ \exp\{i\chi_1^j a_2\} + A_1^- \exp\{-i\chi_1^j a_2\} = A_c^- \exp\{-\gamma_c^j a_2\},$$

$$\frac{\chi_1^j}{k_0}\left(A_1^+ \exp\{i\chi_1^j a_2\} - A_1^- \exp\{-i\chi_1^j a_2\}\right) = -\frac{\gamma_c^j}{ik_0} A_c^- \exp\{-\gamma_c^j a_2\}.$$

The result is a homogeneous system of linear algebraic equations (SLAE) $\mathbf{M}_{TE}^{\perp 4}(\beta)$ for the unknown amplitude coefficients A_s^+, A_1^+, A_1^+, A_c^-, whose solution gives us its values in the expressions (3.2), (3.3) and (3.5), (3.6). Notation $\mathbf{M}_{TE}^{\perp 4}(\beta)$ emphasizes that the system is obtained from the boundary equations for the TE mode, which is expressed through the transverse ($\perp$) component E_y, has dimension 4 and its matrix elements depend on β. Homogeneous SLAE is nontrivial solvable under the condition of vanishing of its determinant, this condition in integrated optics is called the dispersion relation. The dispersion relation gives the dependence of the phase retardation β of the thickness of the waveguide layer $d = a_2 - a_1$ (see Fig. 3).

If, instead of expressions (3.3) and (3.6), we use the expressions (3.7) and (3.8), the boundary conditions for the tangential components take the form:

$$A_s^+ \exp\{\gamma_s^j a_1\} = A_1^c \cos\{\chi_1^j a_1\} + A_1^s \sin\{\chi_1^j a_1\},$$

$$\frac{\gamma_s^j}{ik_0} A_s^+ \exp\{\gamma_s^j a_1\} = -\frac{\chi_1^j}{ik_0}\left(A_1^c \sin\{\chi_1^j a_1\} - A_1^s \cos\{\chi_1^j a_1\}\right),$$

$$A_1^c \cos\{\chi_1^j a_2\} + A_1^s \sin\{\chi_1^j a_2\} = A_c^- \exp\{-\gamma_c^j a_2\},$$

$$-\frac{\chi_1^j}{ik_0}\left(A_1^c \sin\{\chi_1^j a_2\} - A_1^s \cos\{\chi_1^j a_2\}\right) = -\frac{\gamma_c^j}{ik_0} A_c^- \exp\{-\gamma_c^j a_2\}.$$

We have obtained a homogeneous SLAE $\mathbf{M}_{TE}^{\perp 4Re}(\beta)$ with real matrix elements, therefore, if the condition $\det\{\mathbf{M}_{TE}^{\perp 4Re}(\beta)\} = 0$ is fulfilled, one can find its real-valued solution A_s^+, A_1^c, A_1^s, A_c^-. The dispersion equation for this is a real transcendental algebraic equation for β, its solutions are presented on Fig. 4.

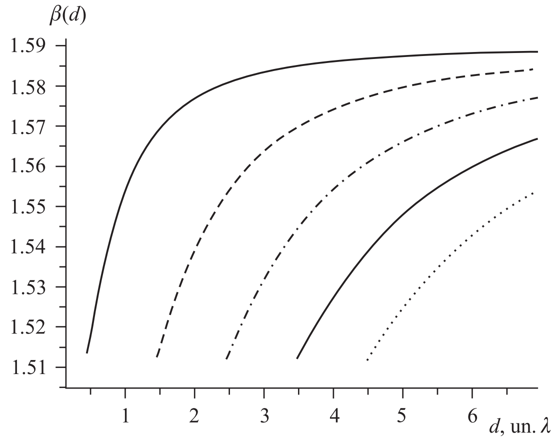

Fig. 3. Graphs of the dispersion relations for the first five TE modes of a three-layer polystyrene waveguide obtained by complex-valued functions of the fundamental system of solutions

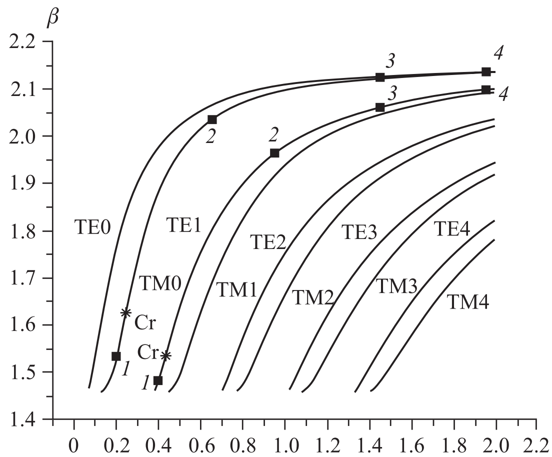

Fig. 4. Graphs of the dispersion relations for the first five TE modes of a three-layer polystyrene waveguide obtained using real-valued functions of the fundamental system of solutions

4.2. TM Modes, Expressed through the Transverse Component H_y. At the same boundaries for the TM mode, which is expressed through a transverse component H_y, the boundary conditions are fulfilled: $H_{ys}(a_1) = H_{yf}(a_1)$ and $E_{zs}(a_1) = E_{zf}(a_1)$ at a point $x = a_1$ and also $H_{yf}(a_2) = H_{yc}(a_2)$ and

$E_{zf}(a_2) = E_{zc}(a_2)$ at a point $x = a_2$. Let us express them in terms of amplitude coefficients $B_k^{\pm}$:

$$B_s^+ \exp\{\gamma_s^j a_1\} = B_1^+ \exp\{i\chi_1^j a_1\} + B_1^- \exp\{-i\chi_1^j a_1\},$$

$$\frac{\gamma_s^j}{ik_0\varepsilon_s} B_s^+ \exp\{\gamma_s^j a_1\} = \frac{\chi_1^j}{k_0\varepsilon_1}\left(B_1^+ \exp\{i\chi_1^j a_1\} - B_1^- \exp\{-i\chi_1^j a_1\}\right),$$

$$B_1^+ \exp\{i\chi_1^j a_2\} + B_1^- \exp\{-i\chi_1^j a_2\} = B_c^- \exp\{-\gamma_c^j a_2\},$$

$$\frac{\chi_1^j}{k_0\varepsilon_1}\left(B_1^+ \exp\{i\chi_1^j a_2\} - B_1^- \exp\{-i\chi_1^j a_2\}\right) = -\frac{\gamma_c^j}{ik_0\varepsilon_c} B_c^- \exp\{-\gamma_c^j a_2\}.$$

The result is a homogeneous system of linear algebraic equations (SLAE) $\mathbf{M}_{TM}^{\perp 4}(\beta)$ for the unknown amplitude coefficients B_s^+, B_1^+, B_1^+, B_c^-, whose solution gives us its values in the expressions (3.12), (3.13) and (3.15), (3.16). Notation $\mathbf{M}_{TM}^{\perp 4}(\beta)$ emphasizes that the system is obtained from the boundary equations for the TM mode, which is expressed through the transverse ($\perp$) component H_y, has dimension 4 and its matrix elements depend on β. Homogeneous SLAE is nontrivial solvable under the condition of vanishing of its determinant, this condition in integrated optics is called the dispersion relation. The dispersion relation gives the dependence of the phase retardation β of the thickness of the waveguide layer $d = a_2 - a_1$.

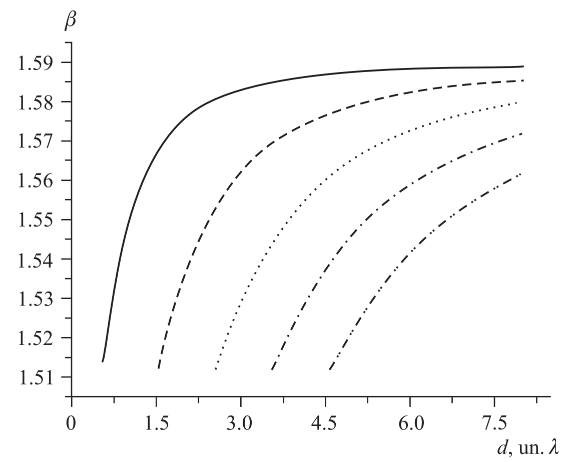

Fig. 5. Graphs of the dispersion relations for the first five TM modes of a three-layer polystyrene waveguide obtained using complex-valued functions of the fundamental system of solutions

If, instead of expressions (3.13) and (3.16) we use the expressions (3.17) and (3.18), the boundary conditions for the tangential components take the form:

$$B_s^+ \exp\{\gamma_s^j a_1\} = B_1^c \cos\{\chi_1^j a_1\} + B_1^s \sin\{\chi_1^j a_1\},$$

$$-\frac{\gamma_s^j}{ik_0\varepsilon_s} B_s^+ \exp\{\gamma_s^j a_1\} = \frac{\chi_1^j}{ik_0\varepsilon_1}\left(B_1^c \sin\{\chi_1^j a_1\} - B_1^s \cos\{\chi_1^j a_1\}\right),$$

$$B_1^c \cos\{\chi_1^j a_2\} + B_1^s \sin\{\chi_1^j a_2\} = B_c^- \exp\{-\gamma_c^j a_2\},$$

$$\frac{\chi_1^j}{ik_0\varepsilon_1}\left(B_1^c \sin\{\chi_1^j a_2\} - B_1^s \cos\{\chi_1^j a_2\}\right) = \frac{\gamma_c^j}{ik_0\varepsilon_c} B_c^- \exp\{-\gamma_c^j a_2\}.$$

We have obtained a homogeneous SLAE $\mathbf{M}_{TM}^{\perp 4Re}(\beta)$ with real matrix elements, therefore, if the condition $\det\{\mathbf{M}_{TM}^{\perp 4Re}(\beta)\} = 0$ is fulfilled, one can find its real-valued solution B_s^+, B_1^c, B_1^s, B_c^-. The dispersion equation for this is a real transcendental algebraic equation for β.

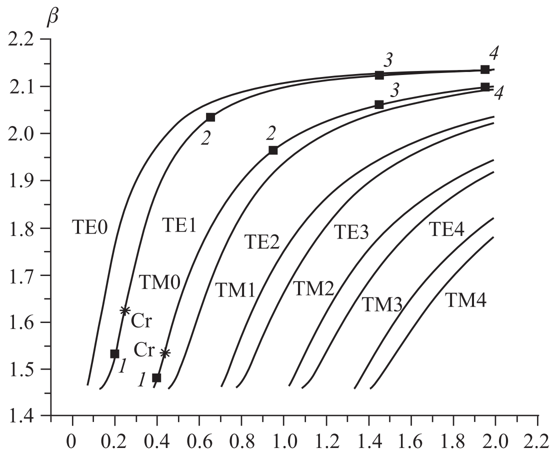

Fig. 6. Graphs of the dispersion relations for the first five TM modes of a three-layer polystyrene waveguide obtained using real-valued functions of the fundamental system of solutions

4.3. TE Modes, Expressed in Terms of the Longitudinal Component H_z. The boundaries between layers of three-layer planar waveguide still cross the axis Ox at a point $x = a_1$ between the substrate and the waveguide layer and at a point $x = a_2$ between the waveguide layer and cladding layer. On these boundaries for the TE mode, which is expressed through a longitudinal component $\tilde{H}_z$, the boundary conditions have the form: $\tilde{H}_{zs}(a_1) = \tilde{H}_{zf}(a_1)$ and $\tilde{E}_{ys}(a_1) = \tilde{E}_{yf}(a_1)$

at a point $x = a_1$ and also $\tilde{H}_{zf}(a_2) = \tilde{H}_{zc}(a_2)$ and $\tilde{E}_{yf}(a_2) = \tilde{E}_{yc}(a_2)$ at a point $x = a_2$. Let us express them in terms of amplitude coefficients $\tilde{B}_k^{\pm}$:

$$\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{B}_1^+ \exp\{i\chi_1^j a_1\} + \tilde{B}_1^- \exp\{-i\chi_1^j a_1\},$$

$$-i\omega\left(\frac{\mu_s}{\gamma_s^j}\right)\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = -\omega\left(\frac{\mu_1}{\chi_1^j}\right)(\tilde{B}_1^+ \exp\{i\chi_1^j a_1\} - \tilde{B}_1^- \exp\{-i\chi_1^j a_1\}),$$

$$\tilde{B}_1^+ \exp\{i\chi_1^j a_2\} + \tilde{B}_1^- \exp\{-i\chi_1^j a_2\} = \tilde{B}_c^- \exp\{-\gamma_c^j a_2\},$$

$$-\omega\left(\frac{\mu_1}{\chi_1^j}\right)(\tilde{B}_1^+ \exp\{i\chi_1^j a_2\} - \tilde{B}_1^- \exp\{-i\chi_1^j a_2\}) = i\omega\left(\frac{\mu_c}{\gamma_c^j}\right)\tilde{B}_c^- \exp\{-\gamma_c^j a_2\}.$$

The result is a homogeneous system of linear algebraic equations (SLAE) $\tilde{\mathbf{M}}_{TM}^{\|4}(\beta)$ for the unknown amplitude coefficients $\tilde{B}_s^+$, $\tilde{B}_1^+$, $\tilde{B}_1^+$, $\tilde{B}_c^-$, whose solution gives us its values in the expressions (3.21)–(3.23) and (3.24)–(3.26). Notation $\tilde{\mathbf{M}}_{TM}^{\|4}(\beta)$ emphasizes that the system is obtained from the boundary equations for the TM mode, which is expressed through the longitudinal component $\tilde{H}_z$, has dimension 4 and its matrix elements depend on β. Homogeneous SLAE is nontrivial solvable under the condition of vanishing of its determinant. The dispersion relation gives the dependence of the phase retardation β of the thickness of the waveguide layer $d = a_2 - a_1$ (see Fig. 5).

Calculations of dispersion curves, made using the expressions obtained through the waveguide equation for the longitudinal components, coincided with the calculations of dispersion curves for the same waveguide modes, obtained through the transverse components.

If, instead of expressions (3.23) and (3.26) we use the expressions (3.27) and (3.28), the boundary conditions for the tangential components take the form:

$$\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{B}_1^c \cos\{\chi_1^j a_1\} + \tilde{B}_1^s \sin\{\chi_1^j a_1\},$$

$$-i\omega\left(\frac{\mu_s}{\gamma_s^j}\right)\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = -i\omega\left(\frac{\mu_1}{\chi_1^j}\right)\left(\tilde{B}_1^c \sin\{\chi_1^j a_1\} - \tilde{B}_1^s \cos\{\chi_1^j a_1\}\right),$$

$$\tilde{B}_1^c \cos\{\chi_1^j a_2\} + \tilde{B}_1^s \sin\{\chi_1^j a_2\} = \tilde{B}_c^- \exp\{-\gamma_c^j a_2\},$$

$$-\omega\left(\frac{\mu_1}{\chi_1^j}\right)\left(\tilde{B}_1^c \sin\{\chi_1^j a_2\} - \tilde{B}_1^s \cos\{\chi_1^j a_2\}\right) = i\omega\left(\frac{\mu_c}{\gamma_c^j}\right)\tilde{B}_c^- \exp\{-\gamma_c^j a_2\}.$$

We have obtained a homogeneous SLAE $\tilde{\mathbf{M}}_{TE}^{\|4Re}(\beta)$ with real matrix elements, therefore, if the condition $\det\{\tilde{\mathbf{M}}_{TE}^{\|4Re}(\beta)\} = 0$ is fulfilled, one can find its real-valued solution $\tilde{B}_s^+$, $\tilde{B}_1^c$, $\tilde{B}_1^s$, $\tilde{B}_c^-$. The dispersion equation for this is a real transcendental algebraic equation for β, its solutions are presented on Fig. 6.

In this case calculations of dispersion curves, made using the expressions obtained through the waveguide equation for the longitudinal components, coincided with the calculations of dispersion curves for the same waveguide modes, obtained through the transverse components.

4.4. TM Modes, Expressed in Terms of the Longitudinal Component E_z. At the same boundaries for the TM mode, which is expressed through the longitudinal component E_z, the boundary conditions have the form: $E_{zs}(a_1) = E_{zf}(a_1)$ and $H_{ys}(a_1) = H_{yf}(a_1)$ at the point $x = a_1$, and also $E_{zf}(a_2) = E_{zc}(a_2)$ and $H_{yf}(a_2) = H_{yc}(a_2)$ at the point $x = a_2$. Let us express them in terms of amplitude coefficients $\tilde{A}_k^{\pm}$:

$$\tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{A}_1^+ \exp\{i\chi_1^j a_1\} + \tilde{A}_1^- \exp\{-i\chi_1^j a_1\},$$

$$i\omega\left(\frac{\varepsilon_s}{\gamma_s^j}\right)\tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = \omega\left(\frac{\varepsilon_1}{\chi_1^j}\right)\left(\tilde{A}_1^+ \exp\{i\chi_1^j a_1\} - \tilde{A}_1^- \exp\{-i\chi_1^j a_1\}\right),$$

$$\tilde{A}_1^+ \exp\{i\chi_1^j a_2\} + \tilde{A}_1^- \exp\{-i\chi_1^j a_2\} = \tilde{A}_c^- \exp\{-\gamma_c^j a_2\},$$

$$\omega\left(\frac{\varepsilon_1}{\chi_1^j}\right)(\tilde{A}_1^+ \exp\{i\chi_1^j a_2\} - \tilde{A}_1^- \exp\{-i\chi_1^j a_2\}) = -i\omega\left(\frac{\varepsilon_c}{\gamma_c^j}\right)\tilde{A}_c^- \exp\{-\gamma_c^j a_2\}.$$

The result is a homogeneous system of linear algebraic equations (SLAE) $\tilde{\mathbf{M}}_{TM}^{\|4}(\beta)$ for the unknowns $\tilde{A}_s^+$, $\tilde{A}_1^+$, $\tilde{A}_1^+$, $\tilde{A}_c^-$ whose solution gives us the values of the unknown amplitude coefficients in the expressions (3.31)–(3.33) and (3.34)–(3.36). Notation $\tilde{\mathbf{M}}_{TM}^{\|4}(\beta)$ emphasizes that the system is obtained from the boundary equations for the TM mode, which is expressed through the longitudinal ($\|$) component E_z, is of dimension 4 and its matrix elements depend on β.

Calculations of the dispersion relations for TM modes, performed using the expressions obtained through the waveguide equation for the longitudinal components, coincided with the calculations of dispersion curves for the same waveguide modes, obtained through the transverse components.

If, instead of expressions (3.33) and (3.36), we use the expressions (3.37) and (3.38), the boundary conditions for the tangential components take the form:

$$\tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{A}_1^c \cos\{\chi_1^j a_1\} + \tilde{A}_1^s \sin\{\chi_1^j a_1\},$$

$$i\omega\left(\frac{\varepsilon_s}{\gamma_s^j}\right)\tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = i\omega\left(\frac{\varepsilon_1}{\chi_1^j}\right)\left(\tilde{A}_1^c \sin\{\chi_1^j a_1\} - \tilde{A}_1^s \cos\{\chi_1^j a_1\}\right),$$

$$\tilde{A}_1^c \cos\{\chi_1^j a_2\} + \tilde{A}_1^s \sin\{\chi_1^j a_2\} = \tilde{A}_c^- \exp\{-\gamma_c^j a_2\},$$

$$i\omega \left(\frac{\varepsilon_1}{\chi_1^j} \right) \left(\tilde{A}_1^c \sin \{\chi_1^j a_2\} - \tilde{A}_1^s \cos \{\chi_1^j a_2\} \right) = -i\omega \left(\frac{\varepsilon_c}{\gamma_c^j} \right) \tilde{A}_c^- \exp \{-\gamma_c^j a_2\}.$$

We have obtained a homogeneous linear algebraic equations $\tilde{\mathbf{M}}_{TM}^{\|4Re}(\beta)$ with real-matrix elements, therefore, if the condition $\det\{\tilde{\mathbf{M}}_{TM}^{\|4Re}(\beta)\} = 0$ is valid, one can find its real-valued solutions $\tilde{A}_s^+$, $\tilde{A}_1^c$, $\tilde{A}_1^s$, $\tilde{A}_c^-$. The dispersion equation in this case is a real transcendental algebraic equation for β.

And in this case also calculations of dispersion curves for TM modes, obtained through the longitudinal components, coincided with the calculations of dispersion curves for the same waveguide modes, obtained through the transverse components.

5. BOUNDARY EQUATIONS FOR FOUR-LAYER WAVEGUIDES

5.1. TE Modes, Expressed through the Transverse Component E_y. The boundaries between layers of a four-planar waveguide cross the axis Ox at a point $x = a_1$ between the substrate and the first waveguide layer, at a point $x = a_2$ between the first waveguide layer and the second waveguide layer and at a point $x = a_3$ between the second waveguide layer and the covering. On these boundaries for the TE mode, which is expressed through a transverse component E_y, the boundary conditions are valid: $E_{ys}(a_1) = E_{yf}(a_1)$ and $H_{zs}(a_1) = H_{zf}(a_1)$ at a point $x = a_1$, $E_{y1}(a_2) = E_{y2}(a_2)$ and $H_{z1}(a_2) = H_{z2}(a_2)$ at a point $x = a_2$, and also $E_{y2}(a_3) = E_{yc}(a_3)$ and $H_{z2}(a_3) = H_{zc}(a_3)$ at a point $x = a_3$. Let us express them in terms of amplitude coefficients $A_k^{\pm}$:

$$A_s^+ \exp \{\gamma_s^j a_1\} = A_1^+ \exp \{i\chi_1^j a_1\} + A_1^- \exp \{-i\chi_1^j a_1\},$$

$$\frac{\gamma_s^j}{ik_0} A_s^+ \exp \{\gamma_s^j a_1\} = \frac{\chi_1^j}{k_0} \left(A_1^+ \exp \{i\chi_1^j a_1\} - A_1^- \exp \{-i\chi_1^j a_1\} \right),$$

$$A_1^+ \exp \{i\chi_1^j a_2\} + A_1^- \exp \{-i\chi_1^j a_2\} = A_2^+ \exp \{i\chi_2^j a_2\} + A_2^- \exp \{-i\chi_2^j a_2\},$$

$$\frac{\chi_1^j}{k_0} \left(A_1^+ \exp \{i\chi_1^j a_2\} - A_1^- \exp \{-i\chi_1^j a_2\} \right) = \\ = \frac{\chi_2^j}{k_0} \left(A_2^+ \exp \{i\chi_2^j a_2\} - A_2^- \exp \{-i\chi_2^j a_2\} \right),$$

$$A_2^+ \exp \{i\chi_2^j a_3\} + A_2^- \exp \{-i\chi_2^j a_3\} = A_c^- \exp \{-\gamma_c^j a_3\},$$

$$\frac{\chi_2^j}{k_0} \left(A_2^+ \exp \{i\chi_2^j a_3\} - A_2^- \exp \{-i\chi_2^j a_3\} \right) = -\frac{\gamma_c^j}{ik_0} A_c^- \exp \{-\gamma_c^j a_3\}.$$

The result is a homogeneous system of linear algebraic equations (SLAE) $\mathbf{M}_{TE}^{\perp 6}(\beta)$ for the unknowns A_s^+, , A_1^+, A_1^+, A_2^+, A_2^+, A_c^- whose solution gives us the values of the unknown amplitude coefficients in the expressions (3.2), (3.3) and (3.5), (3.6). Notation $\mathbf{M}_{TE}^{\perp 6}(\beta)$ emphasizes that the system is obtained from the boundary equations for the TE mode, which is expressed through the transverse ($\perp$) component E_y, is of dimension 6 and its matrix elements depend on β. Homogeneous SLAE is nontrivial solvable under the condition of vanishing of its determinant. This dispersion relation gives the dependence of the phase retardation β of the thickness of the first waveguide layer $d = a_2 - a_1$ and the thickness of the second waveguide layer $h = a_3 - a_2$.

If, instead of expressions (3.3) and (3.6), we use the expressions (3.7) and (3.8), the boundary conditions for the tangential components take the form:

$$A_s^+ \exp\{\gamma_s^j a_1\} = A_1^c \cos\{\chi_1^j a_1\} + A_1^s \sin\{\chi_1^j a_1\},$$

$$\frac{\gamma_s^j}{ik_0} A_s^+ \exp\{\gamma_s^j a_1\} = -\frac{\chi_1^j}{ik_0}\left(A_1^c \sin\{\chi_1^j a_1\} - A_1^s \cos\{\chi_1^j a_1\}\right),$$

$$A_1^c \cos\{\chi_1^j a_2\} + A_1^s \sin\{\chi_1^j a_2\} = A_2^c \cos\{\chi_2^j a_2\} + A_2^s \sin\{\chi_2^j a_2\},$$

$$-\frac{\chi_1^j}{ik_0}\left(A_1^c \sin\{\chi_1^j a_2\} - A_1^s \cos\{\chi_1^j a_2\}\right) = \\ = -\frac{\chi_2^j}{ik_0}\left(A_2^c \sin\{\chi_2^j a_2\} - A_2^s \cos\{\chi_2^j a_2\}\right),$$

$$A_2^c \cos\{\chi_2^j a_3\} + A_2^s \sin\{\chi_2^j a_3\} = A_c^- \exp\{-\gamma_c^j a_3\},$$

$$-\frac{\chi_2^j}{ik_0}\left(A_2^c \sin\{\chi_2^j a_3\} - A_2^s \cos\{\chi_2^j a_3\}\right) = -\frac{\gamma_c^j}{ik_0} A_c^- \exp\{-\gamma_c^j a_3\}.$$

We have obtained a homogeneous SLAE $\mathbf{M}_{TE}^{\perp 6Re}(\beta)$ with real matrix elements, therefore, if the condition $\det\{\mathbf{M}_{TE}^{\perp 6Re}(\beta)\} = 0$ is valid, one can find its real-valued solution A_s^+, A_1^c, A_1^s, A_2^c, A_2^s, A_c^-. The dispersion equation for this is a real transcendental algebraic equation for β.

5.2. TM Modes, Expressed through the Transverse Component H_y. At the same boundaries for the TM mode, which is expressed through a transverse component H_y, the boundary conditions have the form: $H_{ys}(a_1) = H_{yf}(a_1)$ and $E_{zs}(a_1) = E_{zf}(a_1)$ at a point $x = a_1$, $H_{y1}(a_2) = H_{y2}(a_2)$ and $E_{z1}(a_2) = E_{z2}(a_2)$ at a point $x = a_2$, and also $H_{y2}(a_3) = H_{yc}(a_3)$ and $E_{z2}(a_3) = E_{zc}(a_3)$ at a point $x = a_3$. Let us express them in terms of amplitude coefficients $B_k^\pm$:

$$B_s^+ \exp\{\gamma_s^j a_1\} = B_1^+ \exp\{i\chi_1^j a_1\} + B_1^- \exp\{-i\chi_1^j a_1\},$$

$$\frac{\gamma_s^j}{ik_0\varepsilon_s}B_s^+\exp\{\gamma_s^j a_1\} = \frac{\chi_1^j}{k_0\varepsilon_1}\left(B_1^+\exp\{i\chi_1^j a_1\} - B_1^-\exp\{-i\chi_1^j a_1\}\right),$$

$$B_1^+\exp\{i\chi_1^j a_2\} + B_1^-\exp\{-i\chi_1^j a_2\} = B_2^+\exp\{i\chi_2^j a_2\} + B_2^-\exp\{-i\chi_2^j a_2\},$$

$$\frac{\chi_1^j}{k_0\varepsilon_1}\left(B_1^+\exp\{i\chi_1^j a_2\} - B_1^-\exp\{-i\chi_1^j a_2\}\right) = \\ = \frac{\chi_2^j}{k_0\varepsilon_2}\left(B_2^+\exp\{i\chi_2^j a_2\} - B_2^-\exp\{-i\chi_2^j a_2\}\right),$$

$$B_2^+\exp\{i\chi_2^j a_3\} + B_2^-\exp\{-i\chi_2^j a_3\} = B_c^-\exp\{-\gamma_c^j a_3\},$$

$$\frac{\chi_2^j}{k_0\varepsilon_2}\left(B_2^+\exp\{i\chi_2^j a_3\} - B_2^-\exp\{-i\chi_2^j a_3\}\right) = -\frac{\gamma_c^j}{ik_0\varepsilon_c}B_c^-\exp\{-\gamma_c^j a_3\}.$$

The result is a homogeneous system of linear algebraic equations $\mathbf{M}_{TM}^{\pm 6}(\beta)$ for the unknown B_s^+, B_1^+, B_1^+, B_2^+, B_2^+, B_c^-, whose solution gives us the values of the unknown amplitude coefficients in the expressions (3.12), (3.13) and (3.15), (3.16). Homogeneous SLAE is nontrivial solvable under the condition of vanishing of its determinant, this condition gives the dependence of the phase retardation β of TM mode on the thickness of the waveguide layers: $d = a_2 - a_1$ and $h = a_3 - a_2$.

Graph of the dispersion curve for the TM mode repeats the features of the graphic of the dispersion curve for the TE mode shown in Fig. 7.

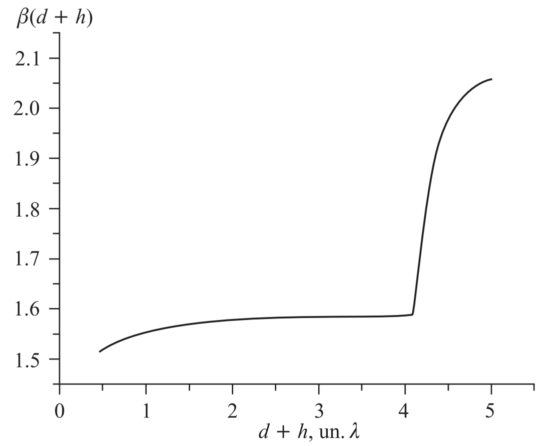

Fig. 7. Dispersion curve of three-layer $d = 0 \div 4(\lambda)$ and four-layer $d = 4\lambda$, $h = 0 \div 1(\lambda)$ planar regular waveguide, calculated for TE mode

If, instead of expressions (3.13) and (3.16) we use the expressions (3.17) and (3.18), the boundary conditions for the tangential components take the form:

$$B_s^+ \exp\{\gamma_s^j a_1\} = B_1^c \cos\{\chi_1^j a_1\} + B_1^s \sin\{\chi_1^j a_1\},$$

$$-\frac{\gamma_s^j}{ik_0\varepsilon_s} B_s^+ \exp\{\gamma_s^j a_1\} = \frac{\chi_1^j}{ik_0\varepsilon_1}\left(B_1^c \sin\{\chi_1^j a_1\} - B_1^s \cos\{\chi_1^j a_1\}\right),$$

$$B_1^c \cos\{\chi_1^j a_2\} + B_1^s \sin\{\chi_1^j a_2\} = B_2^c \cos\{\chi_2^j a_2\} + B_2^s \sin\{\chi_2^j a_2\},$$

$$\frac{\chi_1^j}{ik_0\varepsilon_1}\left(B_1^c \sin\{\chi_1^j a_2\} - B_1^s \cos\{\chi_1^j a_2\}\right) =$$

$$= \frac{\chi_2^j}{ik_0\varepsilon_2}\left(B_2^c \sin\{\chi_2^j a_2\} - B_2^s \cos\{\chi_2^j a_2\}\right),$$

$$B_2^c \cos\{\chi_2^j a_3\} + B_2^s \sin\{\chi_2^j a_3\} = B_c^- \exp\{-\gamma_c^j a_3\},$$

$$\frac{\chi_2^j}{ik_0\varepsilon_2}\left(B_2^c \sin\{\chi_2^j a_3\} - B_2^s \cos\{\chi_2^j a_3\}\right) = \frac{\gamma_c^j}{ik_0\varepsilon_c} B_c^- \exp\{-\gamma_c^j a_3\}.$$

We have obtained a homogeneous SLAE $\mathbf{M}_{TM}^{\perp 6Re}(\beta)$ with real matrix elements, therefore, if the condition $\det\{\mathbf{M}_{TM}^{\perp 6Re}(\beta)\} = 0$ is valid, one can find its real-valued solution B_s^+, B_1^c, B_1^s, B_2^c, B_2^s, B_c^-. The dispersion equation in this case is also a real transcendental algebraic equation for β.

In this case, the timetable of the dispersion curve for the TM mode repeats the features of the graph of the dispersion curve for the TE mode shown in Fig. 8.

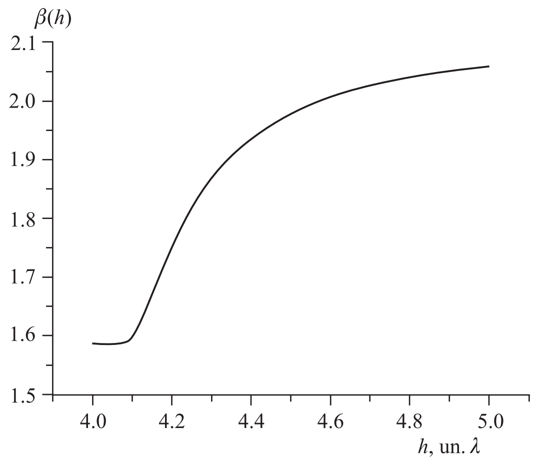

Fig. 8. Dispersion curve of four-layer $d = 4\lambda$, $h = 0 \div 1(\lambda)$ planar regular waveguide, calculated for TE mode

5.3. TE Modes, Expressed in Terms of the Longitudinal Component H_z. The boundaries between layers of three-layer planar waveguide still cross the axis Ox at a point $x = a_1$ between the substrate and the first waveguide layer, at a point $x = a_2$ between the first waveguide layer and the second waveguide layer and at a point $x = a_3$ between the second waveguide layer and cladding layer. On the boundaries for the TE mode, which is expressed through the longitudinal component $\tilde{H}_z$, the boundary conditions have the form: $\tilde{H}_{zs}(a_1) = \tilde{H}_{z1}(a_1)$ and $\tilde{E}_{ys}(a_1) = \tilde{E}_{y1}(a_1)$ at a point $x = a_1$, $\tilde{H}_{z1}(a_2) = \tilde{H}_{z2}(a_2)$ and $\tilde{E}_{y1}(a_2) = \tilde{E}_{y2}(a_2)$ at a point $x = a_2$,and also $\tilde{H}_{z2}(a_3) = \tilde{H}_{zc}(a_3)$ and $\tilde{E}_{y2}(a_3) = \tilde{E}_{yc}(a_3)$ at a point $x = a_3$. Let us express them in terms of amplitude coefficients $\tilde{B}_k^{\pm}$:

$$\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{B}_1^+ \exp\{i\chi_1^j a_1\} + \tilde{B}_1^- \exp\{-i\chi_1^j a_1\},$$

$$-i\omega\left(\frac{\mu_s}{\gamma_s^j}\right)\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = \\ = -\omega\left(\frac{\mu_1}{\chi_1^j}\right)\left(\tilde{B}_1^+ \exp\{i\chi_1^j a_1\} - \tilde{B}_1^- \exp\{-i\chi_1^j a_1\}\right),$$

$$\tilde{B}_1^+ \exp\{i\chi_1^j a_2\} + \tilde{B}_1^- \exp\{-i\chi_1^j a_2\} = \tilde{B}_2^+ \exp\{i\chi_2^j a_2\} + \tilde{B}_2^- \exp\{-i\chi_2^j a_2\},$$

$$-\omega\left(\frac{\mu_1}{\chi_1^j}\right)(\tilde{B}_1^+ \exp\{i\chi_1^j a_2\} - \tilde{B}_1^- \exp\{-i\chi_1^j a_2\}) = \\ = -\omega\left(\frac{\mu_2}{\chi_2^j}\right)\left(\tilde{B}_2^+ \exp\{i\chi_2^j a_2\} - \tilde{B}_2^- \exp\{-i\chi_2^j a_2\}\right),$$

$$\tilde{B}_2^+ \exp\{i\chi_2^j a_3\} + \tilde{B}_2^- \exp\{-i\chi_2^j a_3\} = \tilde{B}_c^- \exp\{-\gamma_c^j a_3\},$$

$$-\omega\left(\frac{\mu_2}{\chi_2^j}\right)\left(\tilde{B}_2^+ \exp\{i\chi_2^j a_3\} - \tilde{B}_2^- \exp\{-i\chi_2^j a_3\}\right) = \\ = i\omega\left(\frac{\mu_c}{\gamma_c^j}\right)\tilde{B}_c^- \exp\{-\gamma_c^j a_3\}.$$

The result is a homogeneous system of linear algebraic equations $\tilde{\mathbf{M}}_{TM}^{\|6}(\beta)$ for the unknown $\tilde{B}_s^+$, $\tilde{B}_1^+$, $\tilde{B}_1^+$, $\tilde{B}_2^+$, $\tilde{B}_2^+$, $\tilde{B}_c^-$, whose solution gives us the values of the unknown amplitude coefficients in the expressions (3.21)–(3.23) and (3.24)–(3.26). Homogeneous SLAE is nontrivial solvable under the condition of vanishing of its determinant, this condition gives the dependence of the phase retardation β of TE mode on the thickness of the waveguide layers: $d = a_2 - a_1$ and $h = a_3 - a_2$.

Calculations of the dispersion curves for TE modes, performed using the expressions obtained through the waveguide equation for the longitudinal components, coincided with the calculations of dispersion curves for the same waveguide modes, obtained through the transverse components.

If, instead of expressions (3.23) and (3.26) we use the expressions (3.27) and (3.28), the boundary conditions for the tangential components take the form:

$$\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{B}_1^c \cos\{\chi_1^j a_1\} + \tilde{B}_1^s \sin\{\chi_1^j a_1\},$$

$$-i\omega\left(\frac{\mu_s}{\gamma_s^j}\right)\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = -i\omega\left(\frac{\mu_1}{\chi_1^j}\right)\left(\tilde{B}_1^c \sin\{\chi_1^j a_1\} - \tilde{B}_1^s \cos\{\chi_1^j a_1\}\right),$$

$$\tilde{B}_1^c \cos\{\chi_1^j a_2\} + \tilde{B}_1^s \sin\{\chi_1^j a_2\} = \tilde{B}_2^c \cos\{\chi_2^j a_2\} + \tilde{B}_2^s \sin\{\chi_2^j a_2\},$$

$$-i\omega\left(\frac{\mu_1}{\chi_1^j}\right)(\tilde{B}_1^c \sin\{\chi_1^j a_2\} - \tilde{B}_1^s \cos\{\chi_1^j a_2\}) =$$

$$= -i\omega\left(\frac{\mu_2}{\chi_2^j}\right)\left(\tilde{B}_2^c \sin\{\chi_2^j a_2\} - \tilde{B}_2^s \cos\{\chi_2^j a_2\}\right),$$

$$\tilde{B}_2^c \cos\{\chi_2^j a_3\} + \tilde{B}_2^s \sin\{\chi_2^j a_3\} = \tilde{B}_c^- \exp\{-\gamma_c^j a_3\},$$

$$-i\omega\left(\frac{\mu_2}{\chi_2^j}\right)\left(\tilde{B}_2^c \sin\{\chi_2^j a_3\} - \tilde{B}_2^s \cos\{\chi_2^j a_3\}\right) = i\omega\left(\frac{\mu_c}{\gamma_c^j}\right)\tilde{B}_c^- \exp\{-\gamma_c^j a_3\}.$$

We have obtained a homogeneous SLAE $\tilde{\mathbf{M}}_{TE}^{\|6Re}(\beta)$ with real matrix elements, therefore, if the condition $\det\{\tilde{\mathbf{M}}_{TE}^{\|6Re}(\beta)\} = 0$ is valid, one can find its real-valued solution $\tilde{B}_s^+$, $\tilde{B}_1^c$, $\tilde{B}_1^s$, $\tilde{B}_2^c$, $\tilde{B}_2^s$, $\tilde{B}_c^-$. The dispersion equation in this case is also a real transcendental algebraic equation for β.

In this case calculations of dispersion curves for TE-modes, obtained through the longitudinal components, coincided with the calculations of dispersion curves for the same waveguide modes, obtained through the transverse components.

5.4. TM Modes, Expressed in Terms of the Longitudinal Component E_z. At the same boundaries for the TM mode, which is expressed through the longitudinal component E_z, the boundary conditions have the form: $E_{zs}(a_1) = E_{z1}(a_1)$ and $H_{ys}(a_1) = H_{y1}(a_1)$ at a point $x = a_1$, $E_{z1}(a_2) = E_{z2}(a_2)$ and $H_{y1}(a_2) = H_{y2}(a_2)$ at a point $x = a_2$, and also $E_{z2}(a_3) = E_{zc}(a_3)$ and $H_{y2}(a_3) = H_{yc}(a_3)$ at a point $x = a_3$. Let us express them in terms of amplitude coefficients $\tilde{A}_k^{\pm}$:

$$\tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{A}_1^+ \exp\{i\chi_1^j a_1\} + \tilde{A}_1^- \exp\{-i\chi_1^j a_1\},$$

$$i\omega\left(\frac{\varepsilon_s}{\gamma_s^j}\right)\tilde{A}_s^+\exp\{\gamma_s^j a_1\}=\omega\left(\frac{\varepsilon_1}{\chi_1^j}\right)\left(\tilde{A}_1^+\exp\{i\chi_1^j a_1\}-\tilde{A}_1^-\exp\{-i\chi_1^j a_1\}\right),$$

$$\tilde{A}_1^+\exp\{i\chi_1^j a_2\}+\tilde{A}_1^-\exp\{-i\chi_1^j a_2\}=\tilde{A}_2^+\exp\{i\chi_2^j a_2\}+\tilde{A}_2^-\exp\{-i\chi_2^j a_2\},$$

$$\omega\left(\frac{\varepsilon_1}{\chi_1^j}\right)\left(\tilde{A}_1^+\exp\{i\chi_1^j a_2\}-\tilde{A}_1^-\exp\{-i\chi_1^j a_2\}\right)=$$
$$=\omega\left(\frac{\varepsilon_2}{\chi_2^j}\right)\left(\tilde{A}_2^+\exp\{i\chi_2^j a_2\}-\tilde{A}_2^-\exp\{-i\chi_2^j a_2\}\right),$$

$$\tilde{A}_2^+\exp\{i\chi_2^j a_3\}+\tilde{A}_2^-\exp\{-i\chi_2^j a_3\}=\tilde{A}_c^-\exp\{-\gamma_c^j a_3\},$$

$$\omega\left(\frac{\varepsilon_2}{\chi_2^j}\right)\left(\tilde{A}_2^+\exp\{i\chi_2^j a_3\}-\tilde{A}_2^-\exp\{-i\chi_2^j a_3\}\right)=$$
$$=-i\omega\left(\frac{\varepsilon_c}{\gamma_c^j}\right)\tilde{A}_c^-\exp\{-\gamma_c^j a_3\}.$$

The result is a homogeneous system of linear algebraic equations (SLAE) $\tilde{\mathbf{M}}_{TM}^{\|6}(\beta)$ for the unknowns $\tilde{A}_s^+$, $\tilde{A}_1^+$, $\tilde{A}_1^+$, $\tilde{A}_2^+$, $\tilde{A}_2^+$, $\tilde{A}_c^-$, whose solution gives us the values of the unknown amplitude coefficients in the expressions (3.31)–(3.33) and (3.34)–(3.36). Notation $\tilde{\mathbf{M}}_{TM}^{\|6}(\beta)$ emphasizes that the system is obtained from the boundary equations for the TM mode, which is expressed through the longitudinal ($\|$) component E_z, is of dimension 6 and its matrix elements depend on β.

Calculations of the dispersion curves for TM modes, performed using the expressions obtained through the waveguide equation for the longitudinal components, coincided with the calculations of dispersion curves for the same waveguide modes, obtained through the transverse components.

If, instead of expressions (3.33) and (3.36) we use the expressions (3.37) and (3.38), the boundary conditions for the tangential components take the form:

$$\tilde{A}_s^+\exp\{\gamma_s^j a_1\}=\tilde{A}_1^c\cos\{\chi_1^j a_1\}+\tilde{A}_1^s\sin\{\chi_1^j a_1\},$$

$$i\omega\left(\frac{\varepsilon_s}{\gamma_s^j}\right)\tilde{A}_s^+\exp\{\gamma_s^j a_1\}=i\omega\left(\frac{\varepsilon_1}{\chi_1^j}\right)\left(\tilde{A}_1^c\sin\{\chi_1^j a_1\}-\tilde{A}_1^s\cos\{\chi_1^j a_1\}\right),$$

$$\tilde{A}_1^c\cos\{\chi_1^j a_2\}+\tilde{A}_1^s\sin\{\chi_1^j a_2\}=\tilde{A}_2^c\cos\{\chi_2^j a_2\}+\tilde{A}_2^s\sin\{\chi_2^j a_2\},$$

$$i\omega\left(\frac{\varepsilon_1}{\chi_1^j}\right)\left(\tilde{A}_1^c\sin\{\chi_1^j a_2\}-\tilde{A}_1^s\cos\{\chi_1^j a_2\}\right)=$$

$$=i\omega\left(\frac{\varepsilon_2}{\chi_2^j}\right)\left(\tilde{A}_2^c\sin\{\chi_2^j a_2\}-\tilde{A}_2^s\cos\{\chi_2^j a_2\}\right),$$

$$\tilde{A}_2^c\cos\{\chi_2^j a_3\}+\tilde{A}_2^s\sin\{\chi_2^j a_3\}=\tilde{A}_c^-\exp\{-\gamma_c^j a_3\},$$

$$i\omega\left(\frac{\varepsilon_2}{\chi_2^j}\right)\left(\tilde{A}_2^c\sin\{\chi_2^j a_3\}-\tilde{A}_2^s\cos\{\chi_2^j a_3\}\right)=-i\omega\left(\frac{\varepsilon_c}{\gamma_c^j}\right)\tilde{A}_c^-\exp\{-\gamma_c^j a_3\}.$$

We have obtained a homogeneous SLAE $\tilde{\mathbf{M}}_{TM}^{\|6Re}(\beta)$ with real matrix elements, therefore, if the condition $\det\{\tilde{\mathbf{M}}_{TM}^{\|6Re}(\beta)\}=0$ is valid, one can find its real-valued solution $\tilde{A}_s^+$, $\tilde{A}_1^c$, $\tilde{A}_1^s$, $\tilde{A}_2^c$, $\tilde{A}_2^s$, $\tilde{A}_c^-$. The dispersion equation in this case is also a real transcendental algebraic equation for β.

In this case also calculations of dispersion curves for TM modes, obtained through the longitudinal components, coincided with the calculations of dispersion curves for the same waveguide modes, obtained through the transverse components.

6. THREE-LAYER WAVEGUIDE DISPERSION RELATIONS IN THE TRIGONOMETRIC FORM

6.1. TE Modes in the Record through the Transverse Components. Solutions in the substrate and cover layer are of the form (3.2) and (3.5), in the waveguide layer solutions have the form (3.9) and (3.10). The boundary conditions for a three-layer waveguide at the points $x=a_1$ and $x=a_2$ are written as:

$$A_s^+\exp\{\gamma_s^j a_1\}=C_1\cos\{\chi_1^j a_1+\phi_1\},\tag{6.1}$$

$$\frac{\gamma_s^j}{ik_0}A_s^+\exp\{\gamma_s^j a_1\}=-\frac{\chi_1^j}{ik_0}C_1\sin\{\chi_1^j a_1+\phi_1\},\tag{6.2}$$

$$C_1\cos\{\chi_1^j a_2+\phi_1\}=A_c^-\exp\{-\gamma_c^j a_2\},\tag{6.3}$$

$$-\frac{\chi_1^j}{ik_0}C_1\sin\{\chi_1^j a_2+\phi_1\}=-\frac{\gamma_c^j}{ik_0}A_c^-\exp\{-\gamma_c^j a_2\}.\tag{6.4}$$

We divide Eq. (6.2) to Eq. (6.1) and obtain

$$\frac{\gamma_s^j}{ik_0}=-\frac{\chi_1^j}{ik_0}\mathrm{tg}\,\{\chi_1^j a_1+\phi_1\}.\tag{6.5}$$

We divide Eq. (6.4) to Eq. (6.3) and we obtain

$$-\frac{\chi_1^j}{ik_0}\mathrm{tg}\,\{\chi_1^j a_2+\phi_1\}=-\frac{\gamma_c^j}{ik_0}. \tag{6.6}$$

Equality (6.5) is equivalent to

$$\mathrm{tg}\,\{\chi_1^j a_1+\phi_1\}=-\frac{\gamma_s^j}{\chi_1^j}\equiv-\mathrm{tg}\,(\phi_1^s), \tag{6.7}$$

and the equality (6.6) is equivalent to

$$\mathrm{tg}\,\{\chi_1^j a_2+\phi_1\}=\frac{\gamma_c^j}{\chi_1^j}\equiv\mathrm{tg}\,(\phi_1^c)\,. \tag{6.8}$$

Introducing the left-hand side of expression (6.8) into the form $\mathrm{tg}\,\{\chi_1^j a_2+\phi_1\}=\mathrm{tg}\,\{\chi_1^j(a_2-a_1)+\chi_1^j a_1+\phi_1\}$ and applying the transformation $\mathrm{tg}\,(A\pm B)=\dfrac{\mathrm{tg}\,(A)\pm\mathrm{tg}\,(B)}{1\mp\mathrm{tg}\,(A)\mathrm{tg}\,(B)}$ to the relations (6.7) and (6.8) several times, we obtain the result relation

$$\mathrm{tg}\,(\chi_f(a_2-a_1))=\mathrm{tg}\,(\phi_1^c+\phi_1^s)\equiv\mathrm{tg}\,\left(\mathrm{arctg}\left(\frac{\gamma_c^j}{\chi_1^j}\right)+\mathrm{arctg}\left(\frac{\gamma_s^j}{\chi_1^j}\right)\right). \tag{6.9}$$

Relation (6.9) is equivalent to the relation $\det\{\mathbf{M}_{TE}^{\perp 4}(\beta)\}=0$ in Subsec. 4.1 and they both hold for all solutions of the roots $\beta_j(d)$ of the dispersion equation (6.9). From relation (6.9) follows the dispersion relation in the form:

$$\chi_f d=\mathrm{arctg}\left(\frac{\gamma_c^m}{\chi_1^m}\right)+\mathrm{arctg}\left(\frac{\gamma_s^m}{\chi_1^m}\right)+m\pi, \tag{6.10}$$

often used in the literature on planar optics [21–28]. Relation (6.10) satisfies every root $\beta_m(d)$ of the dispersion equation for TE modes with the number m of its phase shift $m\pi$.

6.2. TM Modes in the Record through the Transverse Components. Solutions in the substrate and cover layer are of the form (3.12) and (3.15), and in the waveguide layer solutions have the form (3.19) and (3.20). The boundary conditions for a three-layer waveguide at the points $x=a_1$ and $x=a_2$ are written as:

$$B_s^+\exp\{\gamma_s^j a_1\}=D_1\cos\{\chi_1^j a_1+\psi_1\}, \tag{6.11}$$

$$-\frac{\gamma_s^j}{ik_0\varepsilon_s}B_s^+\exp\{\gamma_s^j a_1\}=\frac{\chi_1^j}{ik_0\varepsilon_1}D_1\sin\{\chi_1^j a_1+\psi_1\}, \tag{6.12}$$

$$D_1 \cos\{\chi_1^j a_2 + \psi_1\} = B_c^- \exp\{-\gamma_c^j a_2\}, \tag{6.13}$$

$$\frac{\chi_1^j}{ik_0\varepsilon_1} D_1 \sin\{\chi_1^j a_2 + \psi_1\} = \frac{\gamma_c^j}{ik_0\varepsilon_c} B_c^- \exp\{-\gamma_c^j a_2\}. \tag{6.14}$$

We divide equality (6.12) to equality (6.11) and transform it to

$$\mathrm{tg}\,\{\chi_1^j a_1 + \psi_1\} = -\frac{\varepsilon_1 \gamma_s^j}{\varepsilon_s \chi_1^j} \equiv -\mathrm{tg}\,(\psi_1^s), \tag{6.15}$$

we divide equality (6.14) to equality (6.13) and transform it to

$$\mathrm{tg}\,\{\chi_1^j a_2 + \psi_1\} = \frac{\varepsilon_1 \gamma_c^j}{\varepsilon_c \chi_1^j} \equiv \mathrm{tg}\,(\psi_1^c). \tag{6.16}$$

Introducing the left-hand side of (6.16) into the form $\mathrm{tg}\,\{\chi_1^j a_2+\psi_1\} = \mathrm{tg}\,\{\chi_1^j(a_2-a_1)+\chi_1^j a_1+\psi_1\}$ and applying the transformation $\mathrm{tg}\,(A\pm B) = \dfrac{\mathrm{tg}\,(A) \pm \mathrm{tg}\,(B)}{1 \mp \mathrm{tg}\,(A)\mathrm{tg}\,(B)}$ to the relations (6.15) and (6.16) several times, we obtain the result relation

$$\mathrm{tg}\,(\chi_f(a_2 - a_1)) = \mathrm{tg}\,(\psi_1^c + \psi_1^s) \equiv \\ \equiv \mathrm{tg}\,\left(\mathrm{arctg}\left(\frac{\varepsilon_1\gamma_c^j}{\varepsilon_c\chi_1^j}\right) + \mathrm{arctg}\left(\frac{\varepsilon_1\gamma_s^j}{\varepsilon_s\chi_1^j}\right)\right). \tag{6.17}$$

Relation (6.17) is equivalent to the relation $\det\{\mathbf{M}_{TM}^{\pm 4}(\beta)\} = 0$ in Subsec. 4.2 and they both hold for all the roots $\beta_j(d)$ of the solutions of dispersion Eq. (6.17). Relation (6.17) implies the dispersion relation in the form:

$$\chi_f d = \mathrm{arctg}\left(\frac{\varepsilon_1\gamma_c^m}{\varepsilon_c\chi_1^m}\right) + \mathrm{arctg}\left(\frac{\varepsilon_1\gamma_s^m}{\varepsilon_s\chi_1^m}\right) + m\pi, \tag{6.18}$$

often used in the literature on planar optics [21–28]. Relation (6.18) satisfies every root $\beta_m(d)$ of the dispersion equation for TM modes with the number m of its phase shift $m\pi$.

6.3. TE Modes in the Record through the Longitudinal Components. Solutions in the substrate and cover layer are of the form (3.22) and (3.25), and in the waveguide layer solutions have the form (3.29) and (3.30). The boundary conditions for a three-layer waveguide at the points $x = a_1$ and $x = a_2$ are written as:

$$\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{D}_1 \cos\{\chi_1^j a_1 + \tilde{\psi}_1\}, \tag{6.19}$$

$$-i\omega\left(\frac{\mu_s}{\gamma_s^j}\right)\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = -i\omega\left(\frac{\mu_1}{\chi_1^j}\right)\tilde{D}_1^c \sin\{\chi_1^j a_1 + \tilde{\psi}_1\}, \tag{6.20}$$

$$\tilde{D}_1 \cos\{\chi_1^j a_2 + \tilde{\psi}_1\} = \tilde{B}_c^- \exp\{-\gamma_c^j a_2\}, \tag{6.21}$$

$$-i\omega \left(\frac{\mu_1}{\chi_1^j}\right) \tilde{D}_1^c \sin\{\chi_1^j a_2 + \tilde{\psi}_1\} = i\omega \left(\frac{\mu_c}{\gamma_c^j}\right) \tilde{B}_c^- \exp\{-\gamma_c^j a_2\}. \tag{6.22}$$

We divide relation (6.20) to relation (6.19) and transform it to

$$\mathrm{tg}\,\{\chi_1^j a_1 + \tilde{\psi}_1\} = -\left(\frac{\chi_1^j \gamma_s^j}{\chi_s^2}\right) \equiv -\mathrm{tg}\,(\tilde{\psi}_1^s), \tag{6.23}$$

then divide relation (6.14) to relation (6.13) and transform it to

$$\mathrm{tg}\,\{\chi_1^j a_2 + \tilde{\psi}_1\} = \left(\frac{\chi_1^j \gamma_c^j}{\chi_c^2}\right) \equiv \mathrm{tg}\,(\tilde{\psi}_1^c). \tag{6.24}$$

Introducing the left-hand side of (6.24) in the form $\mathrm{tg}\,\{\chi_1^j a_2 + \tilde{\psi}_1\} = \mathrm{tg}\,\{\chi_1^j (a_2 - a_1) + \chi_1^j a_1 + \tilde{\psi}_1\}$ and applying the transformation $\mathrm{tg}\,(A \pm B) = \dfrac{\mathrm{tg}\,(A) \pm \mathrm{tg}\,(B)}{1 \mp \mathrm{tg}\,(A)\mathrm{tg}\,(B)}$ to the relations (6.23) and (6.24) several times, we obtain the result relation

$$\mathrm{tg}\,(\chi_f(a_2 - a_1)) = \mathrm{tg}\,(\tilde{\psi}_1^c + \tilde{\psi}_1^s) \equiv \\ \equiv \mathrm{tg}\left(\mathrm{arctg}\left(\frac{\chi_1^j \gamma_c^j}{\chi_c^2}\right) + \mathrm{arctg}\left(\frac{\chi_1^j \gamma_s^j}{\chi_s^2}\right)\right). \tag{6.25}$$

Relation (6.25) is equivalent to the relation $\det\{\mathbf{M}_{TE}^{\|4}(\beta)\} = 0$ in Subsec. 4.3 and they both hold for all solutions of the roots $\beta_j(d)$ of the dispersion equation (6.25). Relation (6.25) implies the dispersion relation in the form:

$$\chi_f d + \mathrm{arctg}\left(\frac{\chi_1^m}{\gamma_c^m}\right) + \mathrm{arctg}\left(\frac{\chi_1^m}{\gamma_s^m}\right) = m\pi, \tag{6.26}$$

often used in the literature on planar optics [21–28]. Relation (6.26) satisfies every root $\beta_m(d)$ of the dispersion relation for TE modes with the number m of its phase shift $m\pi$ (see Fig. 9).

6.4. TM Modes in the Record through the Longitudinal Components. Solutions in the substrate and cover layer are of the form (3.22) and (3.25), and in the waveguide layer, solutions have the form (3.29) and (3.30). The boundary conditions for a three-layer waveguide at the points $x = a_1$ and $x = a_2$ are written as:

$$\tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{C}_1 \cos\{\chi_1^j a_1 + \tilde{\phi}_1\}, \tag{6.27}$$

$$i\omega \left(\frac{\varepsilon_s}{\gamma_s^j}\right) \tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = i\omega \left(\frac{\varepsilon_1}{\chi_1^j}\right) \tilde{C}_1 \sin\{\chi_1^j a_1 + \tilde{\phi}_1\}, \tag{6.28}$$

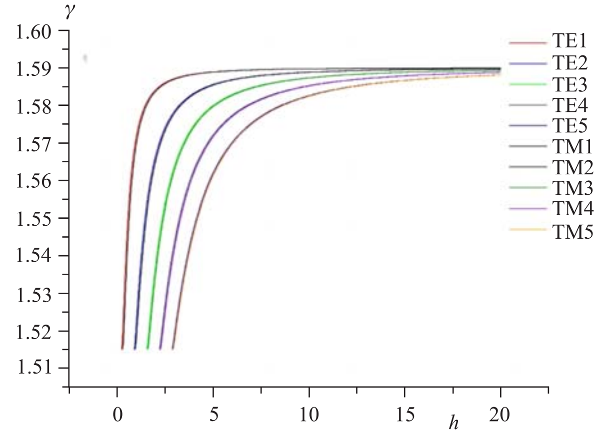

Fig. 9. Dispersion curves for the first five TE and TM modes of polystyrene waveguide on a glass substrate ($n_c = 1.000$, $n_s = 1.515$, $n_f = 1.590$ for $\lambda = 0.633\ \mu$m), calculated using the trigonometric forms of the characteristic equation

$$\tilde{C}_1 \cos\{\chi_1^j a_2 + \tilde{\phi}_1\} = \tilde{A}_c^- \exp\{-\gamma_c^j a_2\}, \tag{6.29}$$

$$i\omega \left(\frac{\varepsilon_1}{\chi_1^j}\right) \tilde{C}_1 \sin\{\chi_1^j a_2 + \tilde{\phi}_1\} = -i\omega \left(\frac{\varepsilon_c}{\gamma_c^j}\right) \tilde{A}_c^- \exp\{-\gamma_c^j a_2\}. \tag{6.30}$$

We divide equality (6.28) to equality (6.27) and transform it to

$$\mathrm{tg}\,\{\chi_1^j a_1 + \tilde{\phi}_1\} = \left(\frac{\varepsilon_s \chi_1^j}{\varepsilon_1 \gamma_s^j}\right) \equiv \mathrm{tg}\,(\tilde{\phi}_1^s), \tag{6.31}$$

then we divide equality (6.30) to equality (6.29) and transform it to

$$\mathrm{tg}\,\{\chi_1^j a_2 + \tilde{\phi}_1\} = -\left(\frac{\varepsilon_c \chi_1^j}{\varepsilon_1 \gamma_c^j}\right) \equiv -\mathrm{tg}(\tilde{\phi}_1^c). \tag{6.32}$$

Introducing the left-hand side of (6.32) in the form $\mathrm{tg}\,\{\chi_1^j a_2 + \tilde{\phi}_1\} = \mathrm{tg}\,\{\chi_1^j (a_2 - a_1) + \chi_1^j a_1 + \tilde{\phi}_1\}$ and applying the transformation $\mathrm{tg}\,(A \pm B) = \dfrac{\mathrm{tg}\,(A) \pm \mathrm{tg}\,(B)}{1 \mp \mathrm{tg}\,(A)\mathrm{tg}\,(B)}$ to the relations (6.31) and (6.32) several times, we obtain the result relation

$$\mathrm{tg}\,(\chi_f (a_2 - a_1)) = \mathrm{tg}\,(\tilde{\phi}_1^c + \tilde{\phi}_1^s) \equiv \\ \equiv \mathrm{tg}\left(-\mathrm{arctg}\left(\frac{\varepsilon_c \chi_1^j}{\varepsilon_1 \gamma_c^j}\right) - \mathrm{arctg}\left(\frac{\varepsilon_s \chi_1^j}{\varepsilon_1 \gamma_s^j}\right)\right). \tag{6.33}$$

Relation (6.33) is equivalent to relation $\det\{\mathbf{M}_{TM}^{\|4}(\beta)\} = 0$ in Subsec. 4.4 and they both hold for all solutions of the roots $\beta_j(d)$ of the dispersion relation (6.33). Relation (6.33) implies the dispersion relation in the form:

$$\chi_f d + \mathrm{arctg}\left(\frac{\varepsilon_c \chi_1^m}{\varepsilon_1 \gamma_c^m}\right) + \mathrm{arctg}\left(\frac{\varepsilon_s \chi_1^m}{\varepsilon_1 \gamma_s^m}\right) = m\pi, \tag{6.34}$$

often used in the literature on planar optics [23–28]. Relation (6.34) satisfies every root $\beta_m(d)$ of the dispersion relation for TM modes with the number m of its phase shift $m\pi$ (see Fig. 10).

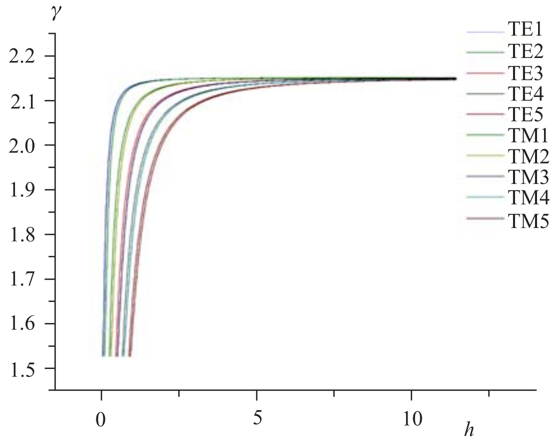

Fig. 10. Dispersion curves for the first five TE and TM modes of tantalum waveguide on a glass substrate ($n_c = 1.000$, $n_s = 2.150$, $n_f = 1.590$ for $\lambda = 0.633$ μm), calculated using the trigonometric forms of the characteristic equation

7. THE DISPERSION RELATIONS OF THE FOUR-LAYER WAVEGUIDE IN THE TRIGONOMETRIC FORM

7.1. TE Modes in the Record through the Transverse Components. Solutions in the substrate and cover layer are of the form (3.2) and (3.5), and in the waveguide layer solutions have the form (3.9) and (3.10). The boundary conditions for a three-layer waveguide in the points $x = a_1$, $x = a_2$ and $x = a_3$ are written as:

$$A_s^+ \exp\{\gamma_s^j a_1\} = C_1 \cos\{\chi_1^j a_1 + \phi_1\}, \tag{7.1}$$

$$\frac{\gamma_s^j}{ik_0} A_s^+ \exp\{\gamma_s^j a_1\} = -\frac{\chi_1^j}{ik_0} C_1 \sin\{\chi_1^j a_1 + \phi_1\}, \tag{7.2}$$

$$C_1 \cos\{\chi_1^j a_2 + \phi_1\} = C_2 \cos\{\chi_2^j a_2 + \phi_2\}, \tag{7.3}$$

$$-\frac{\chi_1^j}{ik_0} C_1 \sin\{\chi_1^j a_2 + \phi_1\} = -\frac{\chi_2^j}{ik_0} C_2 \sin\{\chi_2^j a_2 + \phi_2\}, \tag{7.4}$$

$$C_2 \cos\{\chi_2^j a_3 + \phi_2\} = A_c^- \exp\{-\gamma_c^j a_3\}, \tag{7.5}$$

$$-\frac{\chi_2^j}{ik_0} C_2 \sin\{\chi_2^j a_3 + \phi_2\} = -\frac{\gamma_c^j}{ik_0} A_c^- \exp\{-\gamma_c^j a_3\}. \tag{7.6}$$

We divide equality (7.2) to equality (7.1) and obtain

$$\mathrm{tg}\,\{\chi_1^j a_1 + \phi_1\} = -\frac{\gamma_s^j}{\chi_1^j} \equiv -\mathrm{tg}\,(\phi_1^s), \tag{7.7}$$

we divide equality (7.4) to equality (7.3) and we obtain

$$(\chi_1^j)\,\mathrm{tg}\,\{\chi_1^j a_2 + \phi_1\} = (\chi_2^j)\,\mathrm{tg}\,\{\chi_2^j a_2 + \phi_2\}. \tag{7.8}$$

Then we divide equality (7.6) to equality (7.5) and obtain

$$\mathrm{tg}\,\{\chi_2^j a_3 + \phi_2\} = \frac{\gamma_c^j}{\chi_2^j} \equiv \mathrm{tg}\,(\phi_2^c). \tag{7.9}$$

Representing the left and right sides of (7.8) as

$$\mathrm{tg}\,\{\chi_1^j a_2 + \phi_1\} = \mathrm{tg}\,\{\chi_1^j (a_2 - a_1) + \chi_1^j a_1 + \phi_1\} \tag{7.10}$$

and

$$\mathrm{tg}\,\{\chi_2^j a_2 + \phi_2\} = \mathrm{tg}\,\{\chi_2^j (a_2 - a_3) + \chi_2^j a_3 + \phi_2\}, \tag{7.11}$$

as well as applying the transformation $\mathrm{tg}\,(A \pm B) = \dfrac{\mathrm{tg}\,(A) \pm \mathrm{tg}\,(B)}{1 \mp \mathrm{tg}\,(A)\mathrm{tg}\,(B)}$ several times to the relations (7.7) and (7.9)–(7.11), we obtain the result relation

$$\mathrm{tg}\,(\phi_2^c + \chi_2(a_2 - a_3)) = \frac{\chi_1^j}{\chi_2^j}\,\mathrm{tg}\,(\chi_1(a_2 - a_1) - \phi_1^s). \tag{7.12}$$

Relation (7.12) is equivalent to relation $\det\{\mathbf{M}_{TE}^{\perp 6}(\beta)\} = 0$ in Subsec. 5.1 and they both hold for all the roots $\beta_j(d)$ of the solutions of dispersion relation (7.12). Relation (7.12) implies the dispersion relation in the form:

$$\chi_2^m h = \mathrm{arctg}\left(\frac{\gamma_c^m}{\chi_2^m}\right) - \mathrm{arctg}\left(\frac{\chi_1^m}{\chi_2^m}\mathrm{tg}\left((\chi_1^m d) - \mathrm{arctg}\frac{\gamma_s^m}{\chi_1^m}\right)\right) + m\pi, \tag{7.13}$$

often used in the literature on planar optics [29–32]. Relation (7.13) satisfies every root $\beta_m(d)$ of the dispersion relation for TE modes with the number m of its phase shift $m\pi$.

In works [33–34] the characteristic equations were first solved numerically for real refractive indices, and in [21] for complex refractive indices. Our calculations [3, 6, 9, 37] coincided with the results [21, 33–34].

7.2. TM Modes in the Record through the Transverse Components. Solutions in the substrate and cover layer are of the form (3.12) and (3.15), and in the waveguide layer, solutions have the form (3.19) and (3.20). The boundary conditions for a three-layer waveguide in the points $x = a_1$, $x = a_2$ and $x = a_3$ are written as:

$$B_s^+ \exp\{\gamma_s^j a_1\} = D_1 \cos\{\chi_1^j a_1 + \psi_1\}, \tag{7.14}$$

$$-\frac{\gamma_s^j}{ik_0\varepsilon_s} B_s^+ \exp\{\gamma_s^j a_1\} = \frac{\chi_1^j}{ik_0\varepsilon_1} D_1 \sin\{\chi_1^j a_1 + \psi_1\}, \tag{7.15}$$

$$D_1 \cos\{\chi_1^j a_2 + \psi_1\} = D_2 \cos\{\chi_2^j a_2 + \psi_2\}, \tag{7.16}$$

$$\frac{\chi_1^j}{ik_0\varepsilon_1} D_1 \sin\{\chi_1^j a_2 + \psi_1\} = \frac{\chi_2^j}{ik_0\varepsilon_2} D_2 \sin\{\chi_2^j a_2 + \psi_2\}, \tag{7.17}$$

$$D_2 \cos\{\chi_2^j a_3 + \psi_2\} = B_c^- \exp\{-\gamma_c^j a_3\}, \tag{7.18}$$

$$\frac{\chi_2^j}{ik_0\varepsilon_2} D_2 \sin\{\chi_2^j a_3 + \psi_2\} = \frac{\gamma_c^j}{ik_0\varepsilon_c} B_c^- \exp\{-\gamma_c^j a_3\}. \tag{7.19}$$

We divide equality (7.15) to equality (7.14) and transform it to

$$\mathrm{tg}\,\{\chi_1^j a_1 + \psi_1\} = -\frac{\varepsilon_1\gamma_s^j}{\varepsilon_s\chi_1^j} \equiv -\mathrm{tg}\,(\psi_1^), \tag{7.20}$$

divide equality (7.17) to equality (7.16) and transform it to

$$\left(\frac{\chi_1^j}{\varepsilon_1}\right) \mathrm{tg}\,\{\chi_1^j a_2 + \psi_1\} = \left(\frac{\chi_2^j}{\varepsilon_2}\right) \mathrm{tg}\,\{\chi_2^j a_2 + \psi_2\}. \tag{7.21}$$

Then we divide equality (7.19) to equality (7.18) and transform it to

$$\mathrm{tg}\,\{\chi_2^j a_3 + \psi_2\} = \frac{\varepsilon_2\gamma_c^j}{\varepsilon_c\chi_2^j} \equiv \mathrm{tg}\,(\psi_2^c). \tag{7.22}$$

Introducing the tangents to the left and right sides of (7.21) as

$$\mathrm{tg}\,\{\chi_1^j a_2 + \psi_1\} = \mathrm{tg}\,\{\chi_1^j (a_2 - a_1) + \chi_1^j a_1 + \psi_1\} \tag{7.23}$$

and

$$\mathrm{tg}\,\{\chi_2^j a_2 + \psi_2\} = \mathrm{tg}\,\{\chi_2^j (a_2 - a_3) + \chi_2^j a_3 + \psi_2\} \tag{7.24}$$

and applying the transformation $\mathrm{tg}\,(A \pm B) = \dfrac{\mathrm{tg}\,(A) \pm \mathrm{tg}\,(B)}{1 \mp \mathrm{tg}\,(A)\mathrm{tg}\,(B)}$ to the relations (7.20) and (7.22)–(7.24) several times, we obtain the result relation

$$\mathrm{tg}\,(\psi_2^c + \chi_2(a_2 - a_3)) = \frac{\varepsilon_2 \chi_1^j}{\varepsilon_1 \chi_2^j}\,\mathrm{tg}\,(\chi_1(a_2 - a_1) - \psi_1^s). \tag{7.25}$$

Relation (7.25) is equivalent to relation $\det\{\mathbf{M}_{TM}^{\pm 6}(\beta)\} = 0$ in Subsec. 5.2. and they both hold for all solutions of the roots $\beta_j(d)$ of the dispersion equation (7.25). Relation (7.25) implies the dispersion relation in the form:

$$\chi_2^m h = \mathrm{arctg}\left(\frac{\varepsilon_2 \gamma_c^m}{\varepsilon_c \chi_2^m}\right) + \\ + \mathrm{arctg}\left(\frac{\varepsilon_2 \chi_1^m}{\varepsilon_1 \chi_2^m}\,\mathrm{tg}\left((\chi_1^m d) - \mathrm{arctg}\frac{\varepsilon_1 \gamma_s^m}{\varepsilon_s \chi_1^m}\right)\right) + m\pi, \tag{7.26}$$

often used in the literature on planar optics [29–32]. Relation (7.26) is satisfied for every root $\beta_m(d)$ of the dispersion equation for TM-modes with the number m of its phase shift $m\pi$. In this case our calculations [3, 6, 9, 37] also coincided with the results of studies [21, 33–34].

7.3. TE Modes in the Record through the Longitudinal Components. Solutions in the substrate and cover layer are of the form (3.22) and (3.25), and in the waveguide layer solutions have the form (3.29) and (3.30). The boundary conditions for a three-layer waveguide in the points $x = a_1$, $x = a_2$ and $x = a_3$ are written as:

$$\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{D}_1 \cos\{\chi_1^j a_1 + \tilde{\psi}_1\}, \tag{7.27}$$

$$-i\omega\left(\frac{\mu_s}{\gamma_s^j}\right)\tilde{B}_s^+ \exp\{\gamma_s^j a_1\} = -i\omega\left(\frac{\mu_1}{\chi_1^j}\right)\tilde{D}_1^c \sin\{\chi_1^j a_1 + \tilde{\psi}_1\}, \tag{7.28}$$

$$\tilde{D}_1 \cos\{\chi_1^j a_2 + \tilde{\psi}_1\} = \tilde{D}_2 \cos\{\chi_2^j a_2 + \tilde{\psi}_2\}, \tag{7.29}$$

$$-i\omega\left(\frac{\mu_1}{\chi_1^j}\right)\tilde{D}_1^c \sin\{\chi_1^j a_2 + \tilde{\psi}_1\} = -i\omega\left(\frac{\mu_2}{\chi_2^j}\right)\tilde{D}_2^c \sin\{\chi_2^j a_2 + \tilde{\psi}_2\}, \tag{7.30}$$

$$\tilde{D}_2 \cos\{\chi_2^j a_3 + \tilde{\psi}_2\} = \tilde{B}_c^- \exp\{-\gamma_c^j a_3\}, \tag{7.31}$$

$$-i\omega\left(\frac{\mu_2}{\chi_2^j}\right)\tilde{D}_2^c\sin\{\chi_2^j a_3+\tilde{\psi}_2\}=i\omega\left(\frac{\mu_c}{\gamma_c^j}\right)\tilde{B}_c^-\exp\{-\gamma_c^j a_3\}. \tag{7.32}$$

We divide equality (7.28) to equality (7.27) and transform it to

$$\mathrm{tg}\,\{\chi_1^j a_1+\tilde{\psi}_1\}=\left(\frac{\chi_1^j}{\gamma_s^j}\right)\equiv\mathrm{tg}\,(\tilde{\psi}_1^s), \tag{7.33}$$

divide equality (7.30) to equality (7.29) and transform it to

$$\left(\frac{\mu_1}{\chi_1^j}\right)\mathrm{tg}\,\{\chi_1^j a_2+\tilde{\psi}_1\}=\left(\frac{\mu_2}{\chi_2^j}\right)\mathrm{tg}\,\{\chi_2^j a_2+\tilde{\psi}_2\}, \tag{7.34}$$

then we divide equality (7.32) to equality (7.31) and transform it to

$$\mathrm{tg}\,\{\chi_2^j a_3+\tilde{\psi}_2\}=-\left(\frac{\chi_2^j}{\gamma_c^j}\right)\equiv-\mathrm{tg}\,(\tilde{\psi}_2^c). \tag{7.35}$$

Introducing the tangents to the left and right sides of (7.34) as

$$\mathrm{tg}\,\{\chi_1^j a_2+\tilde{\psi}_1\}=\mathrm{tg}\,\{\chi_1^j(a_2-a_1)+\chi_1^j a_1+\tilde{\psi}_1\} \tag{7.36}$$

and

$$\mathrm{tg}\,\{\chi_2^j a_2+\tilde{\psi}_2\}=\mathrm{tg}\,\{\chi_2^j(a_2-a_3)+\chi_2^j a_3+\tilde{\psi}_2\}, \tag{7.37}$$

and applying the transformation $\mathrm{tg}\,(A\pm B)=\dfrac{\mathrm{tg}\,(A)\pm\mathrm{tg}\,(B)}{1\mp\mathrm{tg}\,(A)\mathrm{tg}\,(B)}$ several times to the relations (7.33) and (7.35)–(7.37), we obtain the result relation

$$\mathrm{tg}\,(-\tilde{\psi}_2^c+\chi_2(a_2-a_3))=\frac{\mu_1\chi_2^j}{\mu_2\chi_1^j}\,\mathrm{tg}\,(\chi_1(a_2-a_1)+\tilde{\psi}_1^s). \tag{7.38}$$

Relation (7.38) is equivalent to relation $\det\left\{\mathbf{M}_{TE}^{\|6}(\beta)\right\}=0$ in Subsec. 5.3, and they both hold for all solutions of the roots $\beta_j(d)$ of the dispersion equation (7.38). Relation (7.38) implies the dispersion relation in the form:

$$\chi_2^m h+\mathrm{arctg}\left(\frac{\chi_2^j}{\gamma_c^j}\right)+\mathrm{arctg}\left(\frac{\chi_2^j}{\chi_1^j}\,\mathrm{tg}\left((\chi_1^m d)+\mathrm{arctg}\,\frac{\chi_1^j}{\gamma_s^j}\right)\right)=m\pi, \tag{7.39}$$

we have not seen in the literature on planar optics. Relation (7.39) is satisfied for every root $\beta_m(d)$ of the dispersion equation for TE modes with the number m of its phase shift $m\pi$.

Calculations made according to the relations (7.39) coincided with the calculations carried out according to the relations (7.13) and with the calculations of [33, 34].

7.4. TM Modes in the Record through the Longitudinal Components. Solutions in the substrate and cover layer are of the form (3.22) and (3.25), and in the waveguide layer solutions have the form (3.29) and (3.30). The boundary conditions for a three-layer waveguide at the points $x = a_1$, $x = a_2$ and $x = a_3$ can be written as:

$$\tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = \tilde{C}_1 \cos\{\chi_1^j a_1 + \tilde{\phi}_1\}, \tag{7.40}$$

$$i\omega \left(\frac{\varepsilon_s}{\gamma_s^j}\right) \tilde{A}_s^+ \exp\{\gamma_s^j a_1\} = i\omega \left(\frac{\varepsilon_1}{\chi_1^j}\right) \tilde{C}_1 \sin\{\chi_1^j a_1 + \tilde{\phi}_1\}, \tag{7.41}$$

$$\tilde{C}_1 \cos\{\chi_1^j a_2 + \tilde{\phi}_1\} = \tilde{C}_2 \cos\{\chi_2^j a_2 + \tilde{\phi}_2\}, \tag{7.42}$$

$$i\omega \left(\frac{\varepsilon_1}{\chi_1^j}\right) \tilde{C}_1 \sin\{\chi_1^j a_2 + \tilde{\phi}_1\} = i\omega \left(\frac{\varepsilon_2}{\chi_2^j}\right) \tilde{C}_2 \sin\{\chi_2^j a_2 + \tilde{\phi}_2\}, \tag{7.43}$$

$$\tilde{C}_2 \cos\{\chi_2^j a_3 + \tilde{\phi}_2\} = \tilde{A}_c^- \exp\{-\gamma_c^j a_3\}, \tag{7.44}$$

$$i\omega \left(\frac{\varepsilon_2}{\chi_2^j}\right) \tilde{C}_2 \sin\{\chi_2^j a_3 + \tilde{\phi}_2\} = -i\omega \left(\frac{\varepsilon_c}{\gamma_c^j}\right) \tilde{A}_c^- \exp\{-\gamma_c^j a_3\}. \tag{7.45}$$

We divide equation (7.41) to equality (7.40) and transform it to

$$\operatorname{tg}\{\chi_1^j a_1 + \tilde{\phi}_1\} = \left(\frac{\varepsilon_s \chi_1^j}{\varepsilon_1 \gamma_s^j}\right) \equiv \operatorname{tg}(\tilde{\phi}_1^s), \tag{7.46}$$

divide equality (7.43) to equality (7.42) and transform it to

$$\left(\frac{\varepsilon_1}{\chi_1^j}\right) \operatorname{tg}\{\chi_1^j a_2 + \tilde{\phi}_1\} = \left(\frac{\varepsilon_2}{\chi_2^j}\right) \operatorname{tg}\{\chi_2^j a_2 + \tilde{\phi}_2\}, \tag{7.47}$$

and divide equality (7.45) to equality (7.44) and transform it to

$$\operatorname{tg}\{\chi_2^j a_3 + \tilde{\phi}_2\} = -\left(\frac{\varepsilon_c \chi_2^j}{\varepsilon_2 \gamma_c^j}\right) \equiv -\operatorname{tg}(\tilde{\phi}_2^c). \tag{7.48}$$

Introducing the tangents to the left and right sides of (7.34) as

$$\operatorname{tg}\{\chi_1^j a_2 + \tilde{\phi}_1\} = \operatorname{tg}\{\chi_1^j (a_2 - a_1) + \chi_1^j a_1 + \tilde{\phi}_1\} \tag{7.49}$$

and

$$\operatorname{tg}\{\chi_2^j a_2 + \tilde{\phi}_2\} = \operatorname{tg}\{\chi_2^j (a_2 - a_3) + \chi_2^j a_3 + \tilde{\phi}_2\} \tag{7.50}$$

and applying the transformation $\mathrm{tg}\,(A \pm B) = \dfrac{\mathrm{tg}\,(A) \pm \mathrm{tg}\,(B)}{1 \mp \mathrm{tg}\,(A)\mathrm{tg}\,(B)}$ several times to the relations (7.49) and (7.50), we obtain the result relation

$$\mathrm{tg}\,(-\tilde{\phi}_2^c + \chi_2(a_2 - a_3)) = \frac{\varepsilon_1 \chi_2^j}{\varepsilon_2 \chi_1^j}\,\mathrm{tg}\,(\chi_1(a_2 - a_1) + \tilde{\phi}_1^s). \tag{7.51}$$

Relation (7.51) is equivalent to relation $\det\{\mathbf{M}_{TM}^{\|6}(\beta)\} = 0$ in Subsec. 5.4, and they both hold for all solutions of the roots $\beta_j(d)$ of the dispersion equation (7.51). Relation (7.51) implies the dispersion relation in the form:

$$\chi_2^m h + \mathrm{arctg}\left(\frac{\chi_2^j \varepsilon_c}{\gamma_c^j \varepsilon_2}\right) - \\ - \mathrm{arctg}\left(\frac{\varepsilon_1 \chi_2^j}{\varepsilon_2 \chi_1^j}\,\mathrm{tg}\left((\chi_1^m d) + \mathrm{arctg}\frac{\chi_1^j \varepsilon_s}{\gamma_s^j \varepsilon_1}\right)\right) = m\pi, \tag{7.52}$$

we have not seen in the literature on planar optics. Relation (7.52) satisfies every root $\beta_m(d)$ of the dispersion relation for TM modes with the number m of its phase shift $m\pi$.

Calculations made according to the relations (7.52), coincided with the calculations carried out according to the relations (7.26), and with the calculations of [33, 34].

8. FIELDS OF GUIDED MODES

In Subsec. 4.1, Figure 4 shows the dispersion curves of the first five TE modes and the first five TM modes of a three-layer polistirol waveguide on a glass substrate. These dependencies are calculated as zeros of the determinant of linear algebraic equations with real matrix elements for undefined fields, the amplitude coefficients and the solutions are real functions $\beta(d)$. Substitution of the calculated value of $\beta(d)$ into the matrix $\mathbf{M}_{TE}^{\perp 4Re}(\beta)$ makes the nontrivial solvability of homogeneous SLAE

$$\mathbf{M}_{TE}^{\perp 4Re}(\beta)\vec{A} = \vec{0}. \tag{7.53}$$

There is a real solution A_s^+, A_1^c, A_1^s, A_c^- of system (7.53). These real coefficients are multiplied by the real-valued functions of the fundamental system of solutions of (3.7), (3.8), as a result we obtain a real-valued amplitude of the vertical distribution of the three nonzero field components of the waveguide modes, whose graphs are shown in Figs. 11, 12.

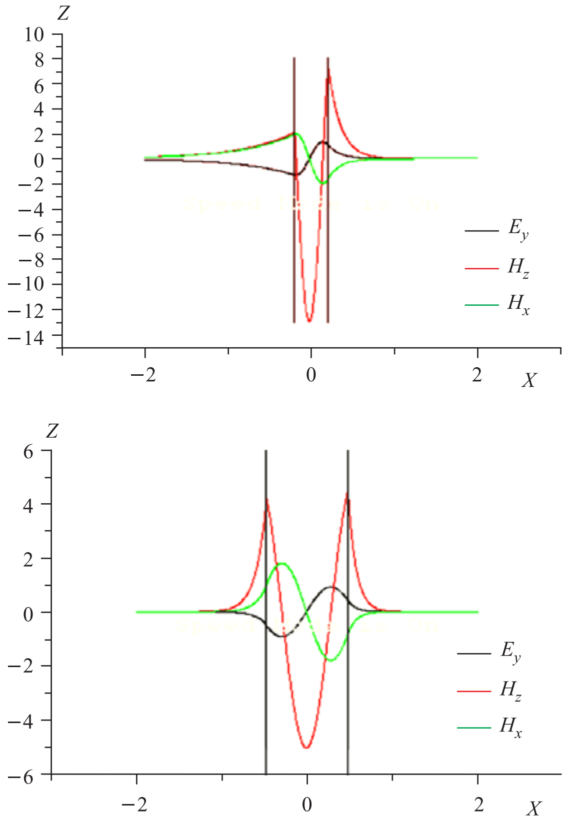

Fig. 11. Vertical distribution of components E_y, H_z, H_x of the fields of waveguide mode TE_1, corresponding to point 1 in Fig. 4, and the fields of waveguide mode TE_1, corresponding to point 2 in Fig. 4

Fields of TM modes are given by a system of linear algebraic equations

$$\mathbf{M}_{TM}^{\perp 4Re}(\beta)\vec{B} = \vec{0}. \tag{7.54}$$

The vanishing of the determinant of this matrix $\det\{\mathbf{M}_{TM}^{\perp 4Re}(\beta)\} = 0$ gives the dispersion curves for TM modes of a three-layer waveguide. Figure 6 shows the first five dispersion curves of three-layer polystyrene waveguide on a glass substrate. After the substitution of computed $\beta(d)$ into a matrix $\mathbf{M}_{TM}^{\perp 4}(\beta)$, system (7.54) admits a real solution B_s^+, B_1^c, B_1^s, B_c^-. These real coefficients are multiplied by the real-valued functions of the fundamental system of solutions of (3.7), (3.8), as a result we obtain a real-valued amplitude of the vertical dis-

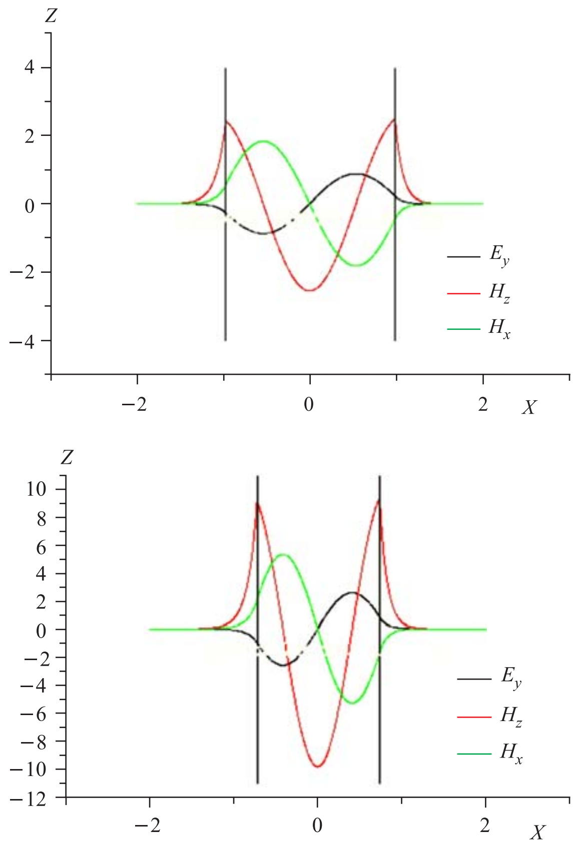

Fig. 12. Vertical distribution of components E_y, H_z, H_x of the fields of waveguide mode TE_1, corresponding to point 3 in Fig. 4, and the fields of waveguide mode TE_1, corresponding to point 4 in Fig. 4

tribution of the three nonzero field components of waveguide TM modes whose graphs are shown in Figs. 13, 14.

In case the field of guided modes were expressed through complex-valued functions of the fundamental system of solutions of (3.3) and (3.6), the amplitude coefficients of the fields should be calculated from the SLAE

$$\mathbf{M}_{TE}^{\perp 4}(\beta)\vec{A} = \vec{0}. \tag{7.55}$$

The vanishing of the determinant of this matrix $\det\{\mathbf{M}_{TE}^{\perp 4}(\beta)\} = 0$ gives the dispersion curves of TE modes of three-layer waveguide. Figure 5 shows the first

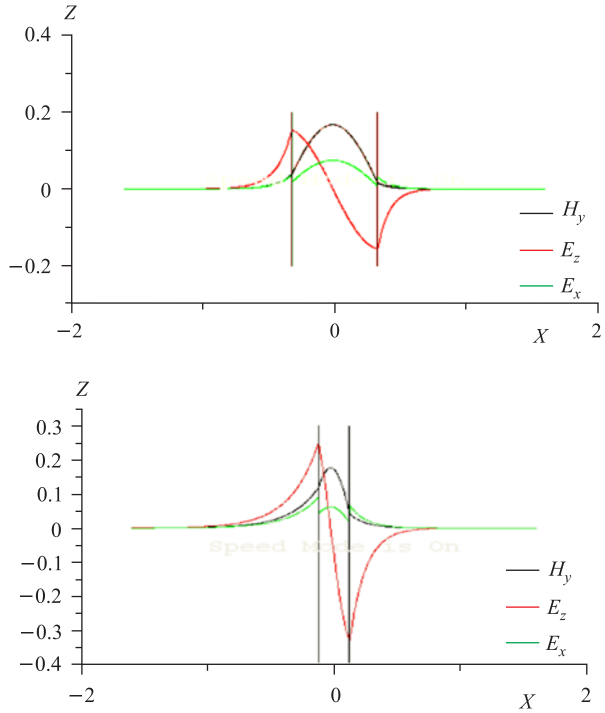

Fig. 13. Vertical distribution of components H_y, E_z, E_x of the fields of waveguide mode TM_0, corresponding to points 1 and 2 in Fig. 6

five dispersion curves of three-layer polystyrene waveguide on a glass substrate. After the substitution of computed $\beta(d)$ into the complex-valued matrix $\mathbf{M}^{\perp 4}_{TE}(\beta)$, system (7.55) admits a complex-valued solution A^+_s, A^+_1, A^+_1, A^-_c. These complex coefficients are multiplied by complex-valued function of the fundamental system of solutions of (3.3) and (3.6), as a result we get a complex-valued amplitude of the vertical field distribution of waveguide TE modes. Figure 15 shows graphs of the real and imaginary parts of the component E_y, and Figure 16 shows graphs of the real and imaginary parts of the component H_z.

In the case where the field of waveguide TM modes of a three-layer waveguide is written in terms of complex-valued function of the fundamental system of solutions of (3.12) and (3.13), the amplitude coefficients of the fields are calculated from the SLAE

$$\mathbf{M}^{\perp 4}_{TM}(\beta)\vec{B} = \vec{0}. \qquad (7.56)$$

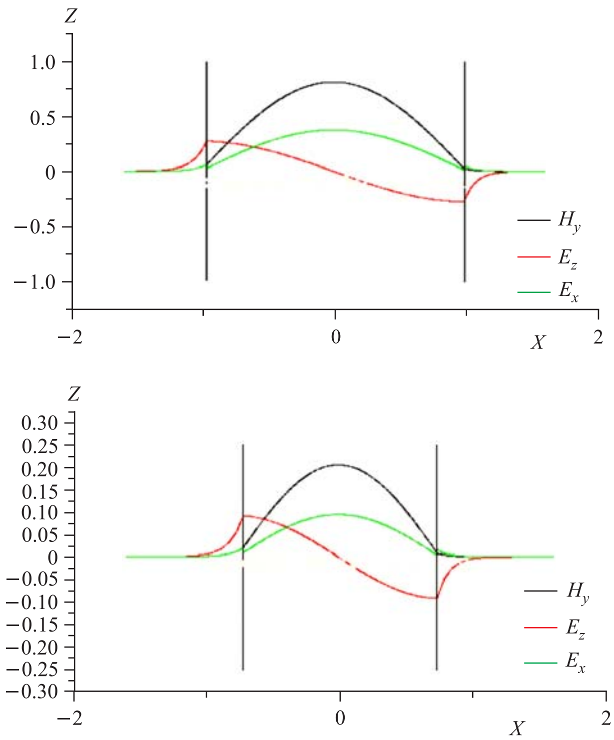

Fig. 14. Vertical distribution of components H_y, E_z, E_x of the fields of waveguide mode TM_0, corresponding to points 3 and 4 in Fig. 6

Dispersion curves of TM modes of a three-layer waveguide are given by the solutions of the characteristic equation $\det\{\mathbf{M}_{TM}^{\perp 4}(\beta)\} = 0$. Figure 7 shows the first five dispersion curves of TM modes, a three-layer polystyrene waveguide on a glass substrate. After the substitution of computed $\beta(d)$ into the complex-valued matrix $\mathbf{M}_{TM}^{\perp 4}(\beta)$, the system (7.56) admits a complex-valued solution B_s^+, B_1^+, B_1^+, B_c^-. These complex coefficients are multiplied by complex-valued functions of the fundamental system of solutions (3.12) and (3.13), as a result we obtain a complex-valued amplitude of the vertical field distribution of waveguide TM modes. Figure 17 shows graphs of the real and imaginary parts of the component H_y, and Figure 18 shows graphs of the real and imaginary parts of the component E_z.

On figures 19.1–19.6 are presented the fields, calculated for the TE_1 mode, in the vicinity of the transition from the first waveguide layer to the other waveguide

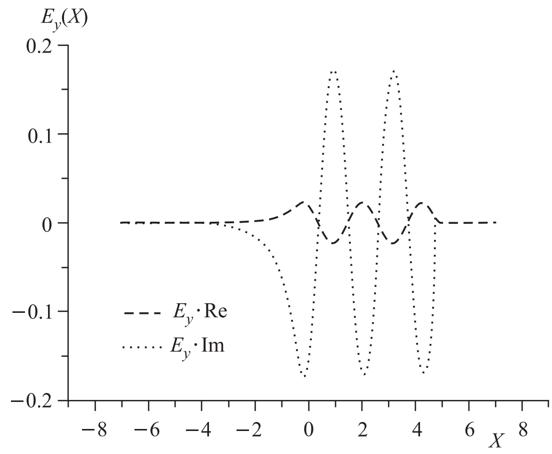

Fig. 15. Graphs of the real and imaginary parts of the component E_y of the waveguide mode TE_4 polystyrene waveguide

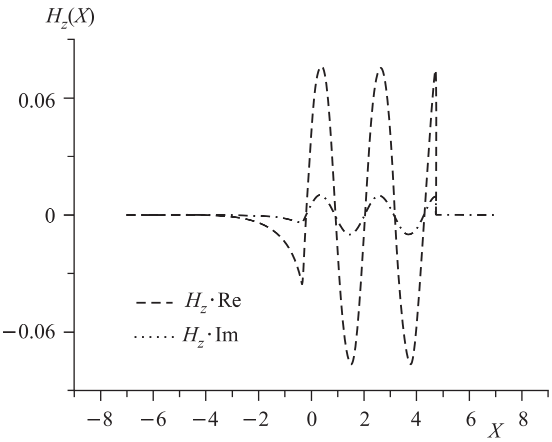

Fig. 16. Graphs of the real and imaginary parts of the component H_z of the waveguide mode TE_4 polystyrene waveguide

layer in the interval from $d = 4\lambda$, $h = 0\lambda$ to $d = 4\lambda$, $h = 0.15\lambda$, corresponding to the dispersion relation of a four-layer regular waveguide.

Field components H_z do not provide additional visual information, so we omit them in this work. A more detailed energy and phase analysis of the

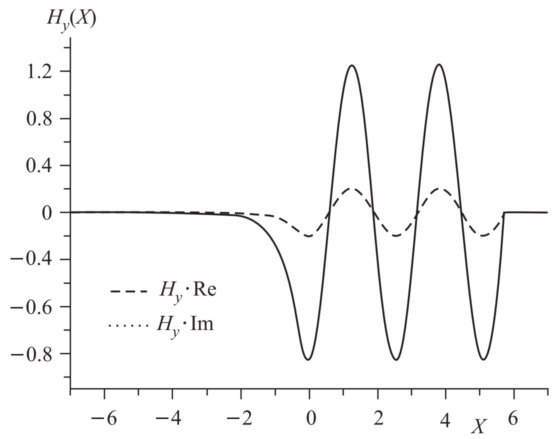

Fig. 17. Graphs of the real and imaginary parts of the component H_y of the waveguide mode TM_4 polystyrene waveguide

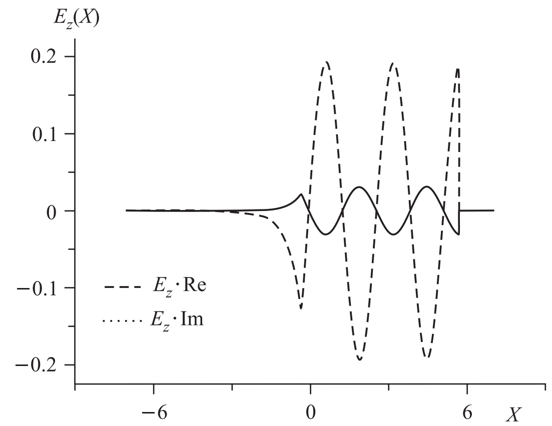

Fig. 18. Graphs of the real and imaginary parts of the component E_z of the waveguide mode TM_4 polystyrene waveguide

evolution of fields in the vicinity $d = 4\lambda$, $h = 0 \div 1(\lambda)$ of the dispersion curve of a three-layer and four-layer planar regular waveguide will be held in one of the following papers in the comparison of different approximate models of the irregular waveguide.

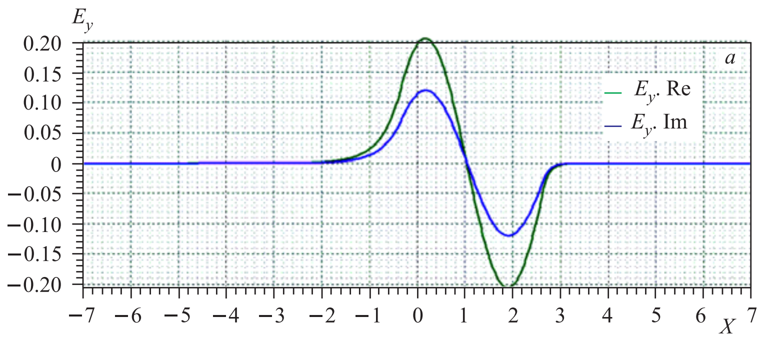

Fig. 19.1. Waveguide layers thicknesses: $d = 4\lambda$, $h = 0.00\lambda$

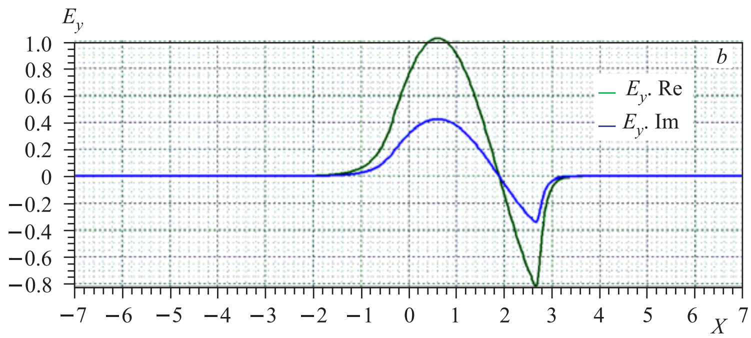

Fig. 19.2. Waveguide layers thicknesses: $d = 4\lambda$, $h = 0.03\lambda$

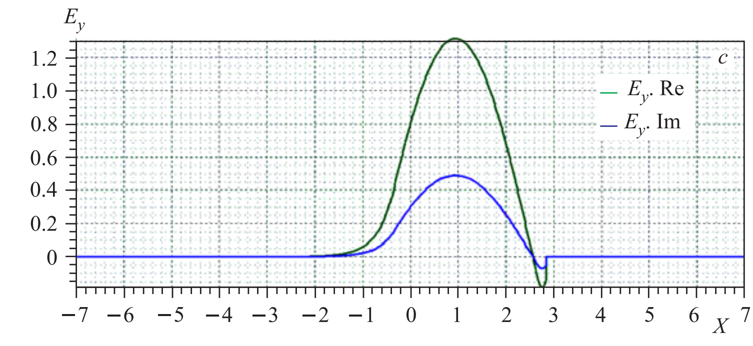

Fig. 19.3. Waveguide layers thicknesses: $d = 4\lambda$, $h = 0.06\lambda$

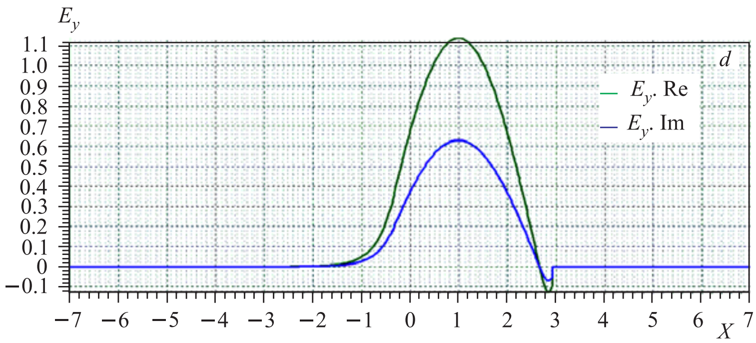

Fig. 19.4. Waveguide layers thicknesses: $d = 4\lambda$, $h = 0.09\lambda$

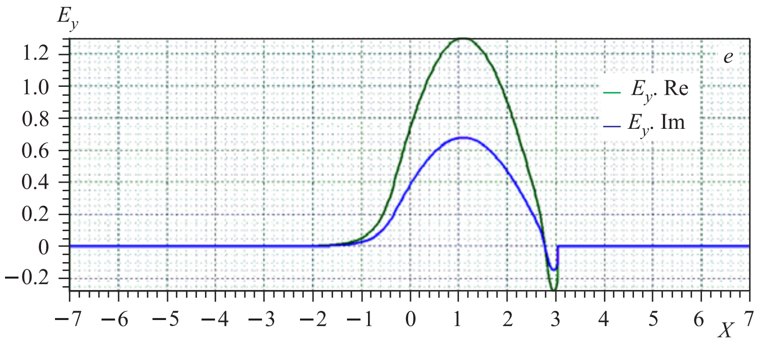

Fig. 19.5. Waveguide layers thicknesses: $d = 4\lambda$, $h = 0.12\lambda$

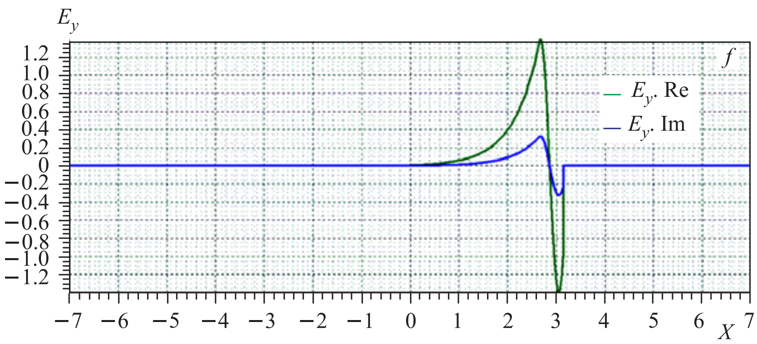

Fig. 19.6. Waveguide layers thicknesses: $d = 4\lambda$, $h = 0.15\lambda$

9. DISCUSSION AND CONCLUSIONS

In most publications on the planar waveguide, dispersion relations (the characteristic equation) are used in the form (6.10), (6.18) and (7.13), (7.26). In [12], the expression of the characteristic equation in the form $\det\{\mathbf{M}(\beta)\} = 0$ is presented and is approved without proper reasoning that it implies the trigonometric form of the dispersion relation.

In most books [11–20] on planar optics, waveguide modes are calculated by the solutions of wave equations for the transverse components of electromagnetic field modes. In [38], a method for calculating the guided modes through the longitudinal components is presented.

In all the cases, described in [11–20, 38–40], characteristic equation of a dielectric planar waveguide is derived from Maxwell's equations without correspondence with field equations. In these cases waveguide modes fields are computed in one way or another, with the normalization of the amplitude on the incident field amplitude or without normalization, using methods that do not ensure the stability of the amplitude coefficients for changes in the parameters of the waveguide.

In the case of numerical simulation of smoothly irregular waveguides in subsequent stages of the problem stated in Section 1, we have to use computational methods for solving systems of linear algebraic equations that are resistant to changes in the parameters of the waveguide. In order the developed for this purpose algorithms and computer programs reproduce the simulation results of regular waveguides, we have at this stage to use the A. N. Tikhonov regularized algorithm for solving systems of linear algebraic equations with imprecise data [6, 9, 41].

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