E5-2012-38

P. E. Zhidkov

ON WEAK SOLUTIONS OF THE INITIAL VALUE PROBLEM FOR THE EQUATION $u_{tt} = a(x,t)u_{xx} + f(t,x,u,u_t,u_x)$

Submitted to «Nonlinear Analysis: Theory, Methods and Applications»

*E-mail: zhidkov@theor.jinr.ru

Жидков П. Е. Е5-2012-38
О слабых решениях задачи с начальными данными для уравнения
$u_{tt} = a(x,t)u_{xx} + f(t,x,u,u_t,u_x)$

Для уравнения вида, указанного в заголовке, предполагается, грубо говоря, что $a(\cdot,t) \in C(\mathbb{R}; W_2^1) \cap L_\infty(\mathbb{R}; W_\infty^1) \cap C^1(\mathbb{R}; L_2)$ и $a_t(\cdot,t) \in L_\infty(\mathbb{R}; L_\infty)$ и что существуют $0 < a_1 < a_2$ и $a_3 > 0$ такие, что $a_1 \leqslant a(x,t) \leqslant a_2$ и $|\nabla a(x,t)| \leqslant a_3$ для любых $x, t \in \mathbb{R}$. Функция f предполагается непрерывно дифференцируемой и удовлетворяющей условию $f(t,x,0,r,s) \equiv 0$. Предполагается, что начальные данные принадлежат $(W_2^1 \cap W_\infty^1) \times (L_2 \cap L_\infty)$. Доказаны существование и единственность локального слабого $(W_2^1 \cap W_\infty^1)$-решения. Кроме того, в специальном случае $f(t,x,u,u_t,u_x) = -|u|^{q-1}u$, $q \geqslant 1$ доказано существование глобального слабого решения.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2012

Zhidkov P. E. E5-2012-38
On Weak Solutions of the Initial Value Problem for the Equation
$u_{tt} = a(x,t)u_{xx} + f(t,x,u,u_t,u_x)$

For the equation of the kind indicated in the title, it is assumed roughly speaking that $a(\cdot,t) \in C(\mathbb{R}; W_2^1) \cap L_\infty(\mathbb{R}; W_\infty^1) \cap C^1(\mathbb{R}; L_2)$ and $a_t(\cdot,t) \in L_\infty(\mathbb{R}; L_\infty)$ and that there exist $0 < a_1 < a_2$ and $a_3 > 0$ such that $a_1 \leqslant a(x,t) \leqslant a_2$ and $|\nabla a(x,t)| \leqslant a_3$ for any $x, t \in \mathbb{R}$. The function f is assumed to be continuously differentiable and satisfying $f(t,x,0,r,s) \equiv 0$. The initial data are assumed to be in $(W_2^1 \cap W_\infty^1) \times (L_2 \cap L_\infty)$. The existence and uniqueness of a local weak $(W_2^1 \cap W_\infty^1)$-solution is proved. In addition, in the special case $f(t,x,u,u_t,u_x) = -|u|^{q-1}u$, $q \geqslant 1$ the existence of a global weak solution is proved.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 2012

1. INTRODUCTION. STATEMENTS OF THE MAIN RESULTS

In several last decades, a large number of publications was devoted to studies of the existence and uniqueness of solutions for semilinear wave equations in the case when these equations are autonomous (i.e., when their coefficients do not depend on time t). For an information on this subject, see, for example, monograph [6] and the references therein. Another classical field of investigations consists of the same questions for quasilinear hyperbolic equations. It is known that in this case, generally speaking, an initial value problem has a local sufficiently smooth solution which is not global (that is, it cannot be continued onto the entire real line $t \in \mathbb{R}$). There is a number of basis results on this subject (see, for example, [1–4] and the references therein). A general theory of hyperbolic equations, mainly of linear ones, is presented, for example, in the recent book [5].

In the present paper, we consider the problem

$$u_{tt} = a(x,t)u_{xx} + b(x,t)u + c(x,t)u_t + d(x,t)u_x + \\ + f(t,x,u,u_t,u_x), \quad u = u(x,t),\ (x,t) \in \mathbb{R}^2, \quad (1)$$

$$u(\cdot,0) = u_0(\cdot) \in W_2^1(\mathbb{R}) \cap W_\infty^1(\mathbb{R}),\ u_t(\cdot,0) = u_1(\cdot) \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R}). \quad (2)$$

Hereafter, all the quantities we deal with are real, $a(\cdot,t)$ is, speaking not quite precisely, in $C(\mathbb{R}; W_2^1(\mathbb{R})) \cap L_\infty(\mathbb{R}; W_\infty^1(\mathbb{R})) \cap C^1(\mathbb{R}; L_2(\mathbb{R}))$ and $a_t(\cdot,t) \in L_\infty(\mathbb{R}; L_\infty)$ and, for simplicity, f is assumed to be smooth and, in addition, $f(t,x,0,u_t,u_x) \equiv 0$ (we shall give precise definitions in the following). We assume that equation (1) is uniformly hyperbolic, that is, that for any bounded interval $I \subset \mathbb{R}$ there exist $0 < a_1 < a_2$ such that

$$a_1 \leqslant a(x,t) \leqslant a_2 \quad (3)$$

for any $t \in I$ fixed for almost all $x \in \mathbb{R}$. It is known that the methods used usually for autonomous semilinear hyperbolic problems of this type in our case do not apply (on the theory of autonomous problems of this type, see, for example, [6]). One of the reasons of this is that the Strichartz-type estimates exploiting often in the autonomous case are not known for problem (1)–(2). Our results for this problem may be considered as a step in the way of proving the existence and uniqueness of a weak solution for a quasilinear second-order hyperbolic problem (for example, when we have a coefficient $a_1 = a_1(u(x,t))$ in place of $a(x,t)$), if such a solution exists at all. For this aim, we shall consider lower regularity coefficients in equation (1), though if one assumes that these coefficients are sufficiently smooth and bounded with their derivatives, then our proofs become essentially simpler. The quasilinear equations have a lot of applications in physics, in particular, in the theory of nonlinear waves, in the elasticity theory, etc. We have to note in addition that with lemmas 10 and 11 we establish the existence and uniqueness of a local smooth solution of problem (1)–(2) when the initial data (u_0, u_1) and the coefficients in the equation are sufficiently smooth and bounded. However, in the present paper, our aim is to prove the existence and uniqueness just of a weak solution of this problem which we shall establish with theorems 1 and 2. The author of this work believes that the existence and uniqueness of a local smooth solution of equations (1) and (2) is a technical result that can be easily obtained by the methods developed earlier for quasilinear equations (on this subject, see, for example, $[1,2,4]$). It seems to be essential to note that the maximal intervals on which smooth and weak solutions can be continued (in the case of smooth initial data) simply coincide with each other (see lemmas 10 and 11 in the following).

Now, we introduce some *notation*. For $p \in [1,\infty)$, by $L_p = L_p(\mathbb{R})$ and $W_p^1 = W_p^1(\mathbb{R})$ we denote the standard Lebesgue and Sobolev spaces taken respectively with the norms $\|g\|_{L_p} = \left\{\int_{\mathbb{R}} |g(x)|^p\right\}^{1/p}$, if $p < +\infty$, $\|g\|_{L_\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}} |g(x)|$, and $\|g\|_{W_p^1} = \|g\|_{L_p} + \|g'\|_{L_p}$. For $p \in (1,\infty)$, we denote by p' the positive number such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Let Δ be the closure of the operator $\left(-\dfrac{d^2}{dx^2}\right)$, taken first with the domain $C_0^\infty(\mathbb{R})$ of infinitely differentiable finite functions in $\mathbb{R}$, in L_2. It is well known that Δ is a self-adjoint positive operator in L_2. For $p \in (1,\infty)$, by $W_p^{-1} = W_p^{-1}(\mathbb{R})$ we denote the Banach space being the completion of $C_0^\infty(\mathbb{R})$ taken with the norm $\|g\|_{W_p^{-1}} = \|(\Delta + \mathrm{Id})^{-1/2} g\|_{L_p}$, where Id denotes the identity. It is known that the space $W_{p'}^{-1}$ is dual to W_p^1 in the sense that for any linear bounded functional φ in W_p^1 there exists a unique $g' \in W_{p'}^{-1}$ such that $\varphi(g) = (g', g)_{L_2}$ for any $g \in W_p^1$ where $(\cdot,\cdot)_{L_2}$ denotes the standard scalar product in L_2 corresponding to the norm that we took in this space (carefully, one should define the expression $(g', g)_{L_2}$ by a limit procedure; this procedure is known now). Conversely, the space W_p^1 is dual to $W_{p'}^{-1}$ in a similar sense. For a set $A \subset \mathbb{R}^d$, $d \geqslant 1$ is integer, by $C(A)$ we denote the space of continuous bounded functions in A, taken with the uniform norm. For an open set $A \subset \mathbb{R}^d$, $d \geqslant 1$ is integer, $C^\infty(A)$ denotes the linear space of functions infinitely differentiable in A. Let also $\Omega \subset \mathbb{R}^d$, where $d \geqslant 1$ is integer, be an open set. We shall write $g \in L_{p,\mathrm{loc}}(\Omega)$, $g \in W^1_{p,\mathrm{loc}}(\Omega)$, etc., if for any $x \in \Omega$ there exists its open bounded neighborhood $O = O(x) \subset \Omega$ such that $g \in L_p(O)$, $g \in W^1_p(O)$, etc. We shall write $g_n \to g$ in $L_{p,\mathrm{loc}}(\Omega)$, $W^1_{p,\mathrm{loc}}(\Omega)$, etc., if for any $x \in \Omega$ there exists its neighborhood $O = O(x) \subset \Omega$ such that $g_n \to g$ in $L_p(O)$ (resp. in $W^1_p(O)$, etc.).

For an interval $I \in \mathbb{R}$ and a Banach space B with the norm denoted by $\|\cdot\|_B$, by $C^k(I;B)$, where $k = 0,1,2$, we denote the Banach spaces of functions from I into B continuous, continuously differentiable and twice continuously differentiable, respectively, and bounded, bounded with their first derivatives and bounded with their first and second derivatives, respectively, taken with the norms $\|g(\cdot)\|_{C^k(I;B)} = \sum_{l=0}^{k} \sup_{t\in I} \|g^{(l)}(t)\|_B$, $k = 0,1,2$, respectively. In addition, we introduce the Banach space $L_b(I;B)$ that consists of bounded functions $g : I \to B$ such that the function $\|g(t)\|_B$ of the argument t is measurable and that equipped with the norm $\|g(\cdot\|_{L_b(I;B)} = \sup\limits_{t\in I} \|g(t)\|_B$.

Let $I \subset \mathbb{R}$ be an interval. For the simplicity of our notation, we denote $X := X(I) = C(I;W^1_2) \cap L_b(I;W^1_\infty)$ and $Z := Z(I) = C(I;W^{-1}_2)$. We also denote by $Y = Y(I)$ the space of continuously differentiable functions $u(\cdot) : I \to L_2$ bounded together with their derivatives $u'_t(t)$ and such that u and $u'_t(t)$ belong in addition to $L_b(I;L_\infty)$; the space $Y(I)$ is equipped with the norm $\|u(\cdot)\|_{Y(I)} = \|u(\cdot)\|_{C^1(I;L_2)} + \|u(\cdot)\|_{L_b(I;L_\infty)} + |u'(\cdot)\|_{L_b(I;L_\infty)}$. Sometimes we shall denote by $C, C_1, C_2, C', C'', \cdots$ positive constants not speaking especially what they are do not depend on if it is clear from the context.

Our main assumptions are as follows.

(A1) *Let for any bounded interval $I \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$ the function $a(\cdot,t)$ be in $C(I;W^1_2(x_0-1,x_0+1)) \cap C^1(I;L_2(x_0-1,x_0+1))$, estimates (3) hold for any $t \in I$ fixed for almost all $x \in \mathbb{R}$ and $a'_t(\cdot,t), a'_x(\cdot,t) \in L_b(I;L_\infty)$.*

(A2) *Let for any bounded interval $I \subset \mathbb{R}$ the coefficients b, c and d be in $L_b(I;L_\infty)$ and let for any $x_0 \in \mathbb{R}$ each of them belong to $C(I;L_2(x_0-1,x_0+1))$.*

(A3) *Let the function f be continuously differentiable and for any $R > 0$ there exist $C > 0$ such that*

$$|f(z_1,z_2,z_3,z_4,z_5)| \leqslant C|z_3|$$

and

$$|f'_{z_3}(z_1,z_2,z_3,z_4,z_5)| + |f'_{z_4}(z_1,z_2,z_3,z_4,z_5)| + |f'_{z_5}(z_1,z_2,z_3,z_4,z_5)| \leqslant C$$

for any $z = (z_1,z_2,z_3,z_4,z_5) \in \mathbb{R}^5$ satisfying $|(z_3,z_4,z_5)| \leqslant R$ and $|z_1| \leqslant R$.

We accept the following definition of a weak solution of problem (1)–(2).

Definition 1. *Let the above assumptions (A1)–(A3) be valid and let $I \in \mathbb{R}$ be an interval that contains 0. Suppose that a function $u(\cdot,t)$ belongs to $X \cap Y \cap Z$. Observe that the operator in the right-hand side of (1) maps this function u*

into a function that belongs to $C(I;W_2^{-1})$ *(see lemma 1 in what follows for a justification). We say that this function* $u(\cdot,t)$ *is a weak solution (or a* $(W_2^1\cap W_\infty^1)$*-solution) of problem (1)–(2) if* $u(\cdot,0)=u_0(\cdot),\ u_t(\cdot,0)=u_1(\cdot)$ *in the senses of* W_2^1 *and* L_2*, respectively, and if equality (1) is valid in the sense of the space* Z.

Now we can establish our main results. They are as follows.

Theorem 1. *Under the assumptions (A1), (A2) and (A3) for any* $D>0$ *and* u_0,u_1 *satisfying* $\|u_0\|_{W_2^1\cap W_\infty^1}\leqslant D$ *and* $\|u_1\|_{L_2\cap L_\infty}\leqslant D$ *there exist* $T>0$ *that depends only on* D *and a unique weak solution* $u(\cdot,t)$ *of problem (1)–(2) in the interval of time* $I:=[-T,T]$*. This weak solution can be uniquely continued on a maximal interval* $(-T_1,T_2)$ *of time t (here* $T_1,T_2>0$*) such that either* $T_1=-\infty$ *(resp.* $T_2=+\infty$*) or* $\limsup\limits_{t\to -T_1+0}[\|u(\cdot,t)\|_{W_2^1\cap W_\infty^1}+\|u_t(\cdot,t)\|_{L_2\cap L_\infty}]=+\infty$ *(resp.* $\limsup\limits_{t\to T_2-0}[\|u(\cdot,t)\|_{W_2^1\cap W_\infty^1}+\|u_t(\cdot,t)\|_{L_2\cap L_\infty}]=+\infty$*). A weak solution depends continuously on the initial data* (u_0,u_1) *in the sense that for any compact interval* I *on which a given weak solution can be continued for any initial data sufficiently close to* (u_0,u_1) *in* $(W_2^1\cap W_\infty^1)\times(L_2\cap L_\infty)$ *the corresponding solution of equations (1) and (2) can be continued on the interval* I *and the correspondence* $(u_0,u_1)\to u(\cdot,t)$ *as a map from* $(W_2^1\cap W_\infty^1)\times(L_2\cap L_\infty)$ *into* $X\cap Y$ *is continuous. If in addition the initial data* (u_0,u_1) *are compactly supported, then the support of our weak solution* $u(x,t)$*, which is regarded here as a function of the argument* x*, is bounded uniformly with respect to* t *in any compact interval on which this solution* $u(x,t)$ *can be continued.*

In the next Sec. 2, we shall describe more precisely the behavior of the support in time t of a weak solution of problem (1)–(2) in the case when this weak solution is finite.

Theorem 2. *Let assumption (A1) be valid,* $b\equiv c\equiv d\equiv 0$ *for simplicity and* $f(t,x,u,u_t,u_x)=-|u|^{q-1}u$*, where* $q\geqslant 1$ *is a constant. Then, an arbitrary weak solution of problem (1)–(2) given by theorem 1 is global, that is, it can be uniquely continued on the entire real line* $t\in\mathbb{R}$.

The function f in theorem 2 above is a standard model nonlinearity used in the literature for many times.

In the next Sec. 2 we shall prove theorem 1 and in Sec. 3 — theorem 2.

Using this occasion, the author wants to thank his colleagues for their support without which this paper cannot appear.

2. LOCAL WELL-POSEDNESS. PROOF OF THEOREM 1

In this Section, we accept that $I_0\ni 0$ is a bounded open interval and $I=\overline{I}_0$ is its closure. We divide our proof of theorem 1 in several lemmas.

Lemma 1. $Lg:=-a(\cdot,t)g_{xx}$ *is a bounded linear operator from* $C(I;W_2^1)\cap L_b(I;W_\infty^1)$ *in* $C(I;W_2^{-1})$.

Proof. Since as is known, Δ is a bounded linear operator from W_p^1 in W_p^{-1} for any $p \in (1, \infty)$ and $t \in \mathbb{R}$ fixed, the proof easily follows from our assumptions.□

Consider the equations for the characteristics of equation (1):

$$\frac{d}{dt}\overline{X}_1(t) = \sqrt{a(\overline{X}_1(t), t)}, \quad t \in I, \tag{4}$$

$$\frac{d}{dt}\overline{X}_2(t) = -\sqrt{a(\overline{X}_2(t), t)}, \quad t \in I. \tag{5}$$

We supply equations (4) and (5) with the following initial data:

$$\overline{X}_i(0) = d \in \mathbb{R}, \quad i = 1, 2, \tag{6}$$

where d is a parameter. Since the right-hand sides in (4) and (5) are continuous, for any $d \in \mathbb{R}$ each of the sets of equations (4), (5) and (4), (6) has a local solution $\overline{X}_1(t; d)$ and $\overline{X}_2(t; d)$, respectively, where we indicate explicitly that these solutions depend on d. A simple corollary of assumption (A1) is that each of these two solutions is unique. Since the function a is bounded, each of these two solutions is global, that is, it can be uniquely continued on the entire real line $t \in \mathbb{R}$.

Now, we introduce the functions $\chi(x, t)$ and $\eta(x, t)$ by setting $\chi(\overline{X}_1(t; d), t) \equiv d$ and $\eta(\overline{X}_2(t; d), t) \equiv d$, where t and d run over the entire real line. Clearly, for any $(x, t) \in \mathbb{R}$, the quantities $\chi(x, t)$ and $\eta(x, t)$ are well defined.

Lemma 2. *Let I be a compact interval. There exist the derivatives $\overline{X}_{iy}(t; y)$, $\overline{X}_{ity}(t; y)$ and $\overline{X}_{itt}(t; y)$ and they belong to $L_b(I; L_\infty)$. In addition, there exist $0 < c_1 < C_1$ such that*

$$c_1 \leqslant \overline{X}_{iy}(t; y) \leqslant C_1 \quad for\ any\ t \in I\ fixed\ for\ almost\ all\ x \in \mathbb{R},$$

and for any $x_0 \in \mathbb{R}$ one has: $\overline{X}_{iy}(t; \cdot), \overline{X}_{ity}(t; \cdot), \overline{X}_{itt}(t; \cdot) \in C(I; L_2(x_0 - 1, x_0 + 1))$.

Proof. We shall prove this claim only for $\overline{X}_1(t; d)$ because for $\overline{X}_2(t; d)$ it can be made by complete analogy. We have formally

$$\frac{d}{dt}\overline{X}_{1y}(t; y) = a'_x(\overline{X}_1(t, y), t)\overline{X}_{1y}(t; y), \tag{7}$$

$$\overline{X}_{1y}(0; y) = 1. \tag{8}$$

The unique solution of equations (7) and (8) can be represented as follows:

$$\overline{X}'_{1y}(t; y) = e^{\int_0^t a'_x(\overline{X}_1(s,y),s)ds}, \tag{9}$$

therefore

$$\overline{X}_1(t;y) = \int_0^y dr e^{\int_0^t a'_x(\overline{X}_1(s,r),s)ds} + \overline{X}_1(t;0). \tag{10}$$

It is easily seen that one has for the function $\overline{X}_1(t;y)$ in (10):

$$\frac{\partial^2}{\partial t \partial y}\overline{X}_1(t;y) = \frac{\partial^2}{\partial y \partial t}\overline{X}_1(t;y)$$

for any $t \in I$ fixed for almost all $x \in \mathbb{R}$. Therefore, the formal differentiation above is correct and indeed, the derivative $\overline{X}_{1y}$ satisfies equations (7) and (8) and is given by (9). The other part of lemma 2 can be proved by complete analogy.□

Lemma 3. *The derivatives $\chi'_t(x,t)$, $\eta'_t(x,t)$, $\chi'_x(x,t)$, $\eta'_x(x,t)$, $\chi'_{tt}(x,t)$, $\eta_{tt}(x,t)$, $\chi''_{tx}(x,t)$, $\eta''_{tx}(x,t)$, $\chi''_{xx}(t;x)$ and $\eta''_{xx}(t;x)$ are well-defined, each of them belongs to $L_b(I;L_\infty)$ and for any $x_0 \in \mathbb{R}$ each of these functions belongs to $C(I;L_2(x_0-1,x_0+1))$.*

Proof. We establish our proof only for the function χ because for η it can be made by complete analogy. Consider the following two Cauchy problems:

$$\overline{X}_{1t} = \sqrt{a(\overline{X}_1(t),t)}, \quad \overline{X}_1(t_0) = y$$

and

$$\overline{X}_{2t} = -\sqrt{a(\overline{X}_2(t);t)}, \quad \overline{X}_2(t_0) = y$$

and denote by $\overline{X}_i(t_0,t;y)$, $i=1,2$, their solutions, respectively. By analogy with the proof of lemma 2, we have

$$\chi'_y(y,t) = \lim_{\Delta y \to 0}\frac{\chi(y+\Delta y,t)-\chi(y,t)}{\Delta y} =$$
$$= \lim_{\Delta y \to 0}\frac{\overline{X}_1(t,0;y+\Delta y)-\overline{X}_1(t,0;y)}{\Delta y} = \overline{X}_{1y}(t,0;y) \in L_b(I;L_\infty).$$

By analogy,

$$\chi'_t(y,t) = \overline{X}_{1t}(t,0;y) =$$
$$= -\sqrt{a(y,t)}e^{-\int_0^t (2a(\overline{X}_1(t,r;y),r))^{-1/2}a'_x(\overline{X}_1(t,r;y),r)dr} \in L_b(I;L_\infty)$$

and

$$\eta'_t(y,t) = \overline{X}_{2t}(t,0;y) =$$
$$= \sqrt{a(y,t)}e^{-\int_0^t (2a(\overline{X}_2(t,r;y),r))^{-1/2}a'_x(\overline{X}_2(t,r;y),r)dr} \in L_b(I;L_\infty).$$

From this, $\chi_{tt}(\cdot,t)$ and $\chi''_{tx}(\cdot,t)$ are in $L_b(I;L_\infty)$. In addition, each of the latter two quantities for any $x_0\in\mathbb{R}$ belongs to $C(I;L_2(x_0-1,x_0+1))$.

Now, we have

$$0\equiv\frac{d}{dt}\chi(\overline{X}_1(t;d),t)=\frac{\partial\chi(x,t)}{\partial t}\bigg|_{x=\overline{X}_1(t;d)}+\frac{\partial\chi(x,t)}{\partial x}\bigg|_{x=\overline{X}_1(t;d)}\sqrt{a(\overline{X}_1(t;d),t)}.$$

Take formally one more derivative over t in this relation. We obtain formally:

$$\begin{aligned}0\equiv\frac{d}{dt}[\chi'_t(x,t)+\sqrt{a(x,t)}\chi'_x(x,t)]\bigg|_{x=\overline{X}_1(t;d)}&=\\ =\chi''_{tt}(\overline{X}_1(t;d),t)+a(\overline{X}_1(t,d),t)\chi''_{xx}(\overline{X}_1(t;d),t)&+\\ +2\sqrt{a(\overline{X}_1(t,d),t)}\chi''_{xt}(\overline{X}_1(t;d),t)+\chi'_x(\overline{X}_1(t;d),t)&\times\\ \times\frac{a'_x(\overline{X}_1(t,d),t)\sqrt{a(\overline{X}_1(t;d),t)}+a'_t(\overline{X}_1(t;d),t)}{2\sqrt{a(\overline{X}_1(t;d),t)}}&.\end{aligned}$$

Therefore, since according to the arguments above, all the terms in the right-hand side of this relation, except maybe the second one, are well-defined and are in $L_b(I;L_\infty)$, the second term is still well-defined and $\chi''_{xx}(\cdot,t)\in L_b(I;L_\infty)$. In addition, we derive from the latter relations:

$$\chi'_t(x,t)+\sqrt{a(x,t)}\chi'_x(x,t)=0,\ \eta'_t(x,t)-\sqrt{a(x,t)}\eta'_x(x,t)=0,$$

$$-\chi''_{tt}(x,t)+a(x,t)\chi''_{xx}(x,t)+\frac{1}{2}\chi'_t(x,t)\left\{\frac{a'_t(x,t)}{a(x,t)}-\frac{a'_x(x,t)}{\sqrt{a(x,t)}}\right\}=0$$

and

$$-\eta''_{tt}(x,t)+a(x,t)\eta''_{xx}(x,t)+\frac{1}{2}\eta'_t(x,t)\left\{\frac{a'_t(x,t)}{a(x,t)}+\frac{a'_x(x,t)}{\sqrt{a(x,t)}}\right\}=0$$

for any $t\in I$ fixed for almost all $x\in\mathbb{R}$. The other part of the lemma can be proved by similar arguments.□

Now, make in problem (1)–(2) the change of independent variables by passing from the variables (x,t) to the variables (χ,η). According to the results above, this is a one-to-one correspondence of $I\times\mathbb{R}$ onto its image and its Jacobian $J(x,t)=\dfrac{\partial(\chi,\eta)}{\partial(x,t)}$ is in $L_b(I;\mathbb{R})$ with its inverse one. Then, according to [7], problem (1)–(2) takes the following form (see Subsec. 3.1 in [7]):

$$v''_{\chi\eta}=b_1(\chi,\eta)v+c_1(\chi,\eta)v_\chi+d_1(\chi,\eta)v_\eta+f_1(\chi,\eta,v,v_\chi,v_\eta),\quad v=v(\chi,\eta), \tag{11}$$

$$v\big|_L = u_0,\ \ (v_\chi \chi_t + v_\eta \eta_t)\big|_L = u_1, \tag{12}$$

where $L = \{(\chi,\eta) \in \mathbb{R}^2 :\ \chi = \eta\}$ and, in view of the relations

$$\chi'_t(x,t) + \sqrt{a(x,t)}\chi'_x(x,t) = 0,\ \ \eta'_t(x,t) - \sqrt{a(x,t)}\eta'_x(x,t) = 0,$$

$$-\chi''_{tt}(x,t) + a(x,t)\chi''(x,t) + +\frac{1}{2}\chi'_t(x,t)\left\{\frac{a'_t(x,t)}{a(x,t)} - \frac{a'_x(x,t)}{\sqrt{a(x,t)}}\right\} = 0$$

and

$$-\eta''_{tt} + a(x,t)\eta''_{xx}(x,t) + \frac{1}{2}\eta'_t(x,t)\left\{\frac{a'_t(x,t)}{a(x,t)} + \frac{a'_x(x,t)}{\sqrt{a(x,t)}}\right\} = 0$$

obtained in the proof of lemma 3, we have

$$b_1 = \frac{b}{a_1},\ \ c_1 = \left(\left(c - \frac{1}{2}\left(\frac{a'_t}{a} - \frac{a'_x}{\sqrt{a}}\right)\right)\chi'_t + d\chi'_x\right)/a_1,$$

$$d_1 = \left(\left(c - \frac{1}{2}\left(\frac{a'_t}{a} + \frac{a'_x}{\sqrt{a}}\right)\right)\eta'_t + d\eta'_x\right)/a_1,\ \ f_1 = f/a_1 \text{ and } a_1(\chi,\eta) = 4\chi_t\eta_t. \tag{13}$$

In view of the proof of lemma 3, there exist $0 < c_2 < C_2$ such that

$$c_2 \leqslant -a_1 \leqslant C_2 \tag{14}$$

for an arbitrary $t \in I$ fixed for almost all $x \in \mathbb{R}$.

Lemma 4. *There exist* $0 < c_3 < C_3$ *such that the Jacobian* $J := \det\dfrac{\partial(\chi,\eta)}{\partial(x,t)}$ *satisfies*

$$c_3 \leqslant J \leqslant C_3$$

for any $t \in I$ *fixed for almost all* $x \in \mathbb{R}$. *In addition, for any* $x_0 \in \mathbb{R}$ *one has:* $J(\cdot,t) \in C(I; L_2(x_0 - 1, x_0 + 1))$.

Proof follows from lemma 3.□

Let $A > 0$, $P = \{(t,x) :\ t \in [-A,A],\ x \in \mathbb{R}\}$ and P_1 be its image under the transformation $(x,t) \to (\chi,\eta)$. Then, there exists $A_1 > 0$ such that P_1 contains the domain $G := \{(\chi,\eta) \in \mathbb{R}^2 :\ |\chi - \eta| \leqslant A_1\}$.

According to lemmas 2 and 3, the coefficients b_1, c_1 and d_1, regarded as functions of x and t, belong to $L_b(I; L_\infty)$. We define weak solutions of equation (11) as follows.

Definition 2. *Let* $G \subset \mathbb{R}^2$ *be an open set. We say that a function* $v = v(\chi,\eta) \in L_{\infty,\mathrm{loc}}(G)$ *is a weak solution of equation (11) in* G *if it has the derivatives* $\dfrac{\partial v}{\partial \chi}, \dfrac{\partial v}{\partial \eta}$ *that belong to* $L_{\infty,\mathrm{loc}}(G)$ *and the derivative* $\dfrac{\partial^2 v}{\partial \chi \partial \eta}$ *that* *belongs to $L_{1,\mathrm{loc}}(G)$ (these derivatives are understood in the sense of distributions so that in particular $\frac{\partial}{\partial \chi}\left(\frac{\partial v}{\partial \eta}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \chi}\right)$) and if $\frac{\partial^2 v}{\partial \chi \partial \eta}$ is equal to the expression in the right-hand side of (11) in the sense of $L_{1,\mathrm{loc}}(G)$.*

Let $v(\chi, \eta)$ be a weak solution of equation (11) and $O \subset G$, where $O = \{(\chi, \eta) : \ \chi_0 - \delta_1 < \chi < \chi_0 + \delta_1, \ \chi_0 - \delta_2 < \eta < \chi_0 + \delta_2\}$ and $\chi_0 \in \mathbb{R}, \ \delta_1 > 0$ and $\delta_2 > 0$ are arbitrary constants. Then, by the Fubini theorem and Sobolev embedding, $v(\chi, \eta)$ is continuous in G and $v'_\chi(\chi, \eta)$ is continuous as a function of $\eta \in (\chi_0 - \delta_2, \chi_0 + \delta_2)$, for almost all fixed $\chi \in (\chi_0 - \delta_1, \chi_0 + \delta_1)$. Therefore, for almost all $(\chi, \chi) \in L \cap O$ (in the sense of the Lebesgue measure on L) there exists a limit $\lim\limits_{\eta \to \chi} v'_\chi(\chi, \eta)$. In addition, the initial conditions (2) determine uniquely the derivative $v'_\chi(\chi, \eta)\big|_{\eta=\chi}$ (in fact, $v'_\chi(\chi, \eta)\big|_{\eta=\chi} = u'_{0x}(x) - \frac{u_1(x)}{\sqrt{a(x,0)}}$ because $\chi \equiv x$ on L). These arguments allow us to accept the following definition.

Definition 3. *We say that a function $v(\chi, \eta)$ is a weak solution of problem (11)–(12) if it is a weak solution of equation (11), it satisfies the condition $v\big|_L = u_0$, the limit of $v'_\chi(\chi, \eta)$ exists as η goes to $\chi \in \mathbb{R}$ and χ is fixed for almost all $\chi \in \mathbb{R}$ and if this limit coincides with $v'_\chi(\chi, \eta)\big|_{\eta=\chi} = u'_{0x}(x) - \frac{u_1(x)}{\sqrt{a(x,0)}}$ for almost all $\chi = x \in \mathbb{R}$ (note that $\chi \equiv x$ on L).*

This definition looks clumsy, but it is sufficient for our goals.

Now, we can rewrite equations (11) and (12) in the following form:

$$v(\chi, \eta) = -\int_\chi^\eta dr \int_r^\eta ds [b_1(r,s)v + c_1(r,s)v'_r + d_1(r,s)v'_s + f_1(r,s,v,v'_r,v'_s)] - \\ - \int_\chi^\eta dr v'_r(r,s)\big|_{s=r} + v(\eta, \eta). \quad (15)$$

By complete analogy, one may interchange χ and η in this representation. In addition, observe that when we pass from the variables (x,t) to (χ, η), the half-plane $\{(x,t) \in \mathbb{R} : \ t > 0\}$ transforms in the half-plane $\{(\chi, \eta) \in \mathbb{R} : \ \chi < \eta\}$.

Make in (15) the inverse change of variables passing from the variables (χ, η) to (x,t). Then, we obtain

$$v(x,t) = -\int_S drdsJ[b_1(\chi,\eta)v(\chi,\eta) + c_1(\chi,\eta)(v_r r_\chi + v_s s_\chi) + \\ + d_1(\chi,\eta)(v_r r_\eta + v_s s_\eta) + f_1(t,x,v,v_r r_\chi + v_s s_\chi, v_r r_\eta + v_s s_\eta)]\Big|_{\substack{\chi=\chi(r,s)\\ \eta=\eta(r,s)}} + \\ + \frac{1}{2}\int_{x_1(x,t)}^{x_2(x,t)} \frac{1}{\sqrt{a(r,0)}} u_1(r)dr + \frac{1}{2}[u_0(x_1(x,t)) + u_0(x_2(x,t))], \quad (16)$$

where S is the curvilinear triangle bounded by the segments of the X axis and of two characteristics $\overline{X}_1$ and $\overline{X}_2$ each of which contains the point (x,t), and $x_1(x,t)$ and $x_2(x,t)$ are the points of intersection of these two characteristics with the X axis, respectively.

Lemma 5. *Let I be a compact interval. Then, there exists $C_4 > 0$ such that*

$$\|b_1(\cdot,t\|_{L_b(I;L_\infty)} + \|c_1(\cdot,t\|_{L_b(I;L_\infty)} + \|d_1(\cdot,t)\|_{L_b(I;L_\infty)} \leqslant C_4$$

and for any $x_0 \in \mathbb{R}$ one has: $b_1(t,\cdot), c_1(t,\cdot), d_1(t,\cdot) \in C(I; L_2(x_0-1, x_0+1))$.

Proof follows from (13), (14) and lemmas 2 and 3.□

Denote

$$(S(t,w))(x,t) = -J(x,t)[b_1(\chi,\eta)w(\chi,\eta) + c_1(\chi,\eta)(w_x x_\chi + w_t t_\chi)+$$
$$+ d_1(\chi,\eta)(w_x x_\eta + w_t t_\eta) + f_1(t,x,w,w_x x_\chi + w_t t_\chi, w_x x_\eta + w_t t_\eta)]\Bigg|_{\substack{\chi=\chi(x,t)\\ \eta=\eta(x,t)}},$$

$$[R(s)g](x,t) = \int\limits_{\overline{X}_1(t,s;x)}^{\overline{X}_2(t,s;x)} g(z)dz$$

and

$$(Pw)(x,t) = \int\limits_0^t \{R(s)[(S(s,w))(\cdot,s)]\}(x,t)ds.$$

Then, equation (16) reads

$$v(x,t) = (Pv)(x,t) + \frac{1}{2}\int\limits_{x_1(x,t)}^{x_2(x,t)} \frac{1}{\sqrt{a(r,0)}} u_1(r)dr + \frac{1}{2}[u_0(x_1(x,t)) + u_0(x_2(x,t))]. \tag{17}$$

Lemma 6. *Let $[-T,T] = I \ni 0$ be a compact interval. Then*

1) For any $t \in I$ S is a continuous operator from $X \cap Y$ in $C(I;L_2) \cap L_b(I;L_\infty)$ and for any ball $B \subset X \cap Y$ there exists $\hat{C} > 0$ such that

$$\|S(\cdot,w_1) - S(\cdot,w_2)\|_{C(I;L_2)\cap L_b(I;L_\infty)} \leqslant \hat{C}\|w_1 - w_2\|_{X\cap Y}$$

for any $w_1, w_2 \in B$;

2) P is a continuous operator from $C(I;L_2) \cap L_b(I;L_\infty)$ in $X \cap Y$ and for any ball $B \subset C(I;L_2) \cap L_b(I;L_\infty)$ there exists a constant $\tilde{C} = \tilde{C}(T) > 0$ such that $\tilde{C}(T) \to +0$ as $T \to +0$ and that

$$\|(Pw_1) - (Pw_2)\|_{X\cap Y} \leqslant \tilde{C}(T)\|w_1 - w_2\|_{C(I;L_2)\cap L_b(I;L_\infty)}$$

for any $w_1, w_2 \in B$.

Proof is a simple corollary of assumptions (A1)–(A3) and lemmas 2 and 3.□

Lemma 7. *Under the assumptions of theorem 1 for any $R > 0$ there exists $T > 0$ such that for any initial data (u_0, u_1) satisfying $\|u_0\|_{W_2^1 \cap W_\infty^1} + \|u_1\|_{L_2 \cap L_\infty} \leqslant R$ equation (16) has a unique solution $v \in X \cap Y$ in the interval of time $I = [-T, T]$. Denote also by $K_0 \subset \mathbb{R}$ the support $\operatorname{supp}(u_0, u_1)$ of the initial data (u_0, u_1) and by $K(t)$ the set $\bigcup\limits_{x \in K_0} [\overline{X}_2(t; x), \overline{X}_1(t; x)]$. Then, for any $t \in I$, where I is the just taken interval, one has: $\operatorname{supp}(u(\cdot, t)) \subset K(t)$.*

Proof is usual, and it is based on the contraction mapping principle. So, we only sketch it, quite briefly. Observe first that the expression in (16)

$$h(x,t) = \frac{1}{2} \int\limits_{x_1(x,t)}^{x_2(x,t)} \frac{1}{\sqrt{a(r,0)}} u_1(r) dr + \frac{1}{2}[u_0(x_1(x,t)) + u_0(x_2(x,t))]$$

is in $X \cap Y$. Let us set

$$M = \{w \in X \cap Y : \ \|w - h\|_{X \cap Y} \leqslant 1\}.$$

Then M, taken with the distance $\rho(w_1, w_2) = \|w_1 - w_2\|_{X \cap Y}$, is a nonempty complete metric space. In addition, it easily follows from lemma 6 that there exists $T > 0$ depending only on $\|u_0\|_{W_2^1 \cap W_\infty^1}$ and $\|u_1\|_{L_2 \cap L_\infty}$ such that the operator in the right-hand side of (17) maps M into M and is a contraction. Therefore, for this $T > 0$ this map has a unique fixed point in M. Thus, equation (17) has a unique local solution.

To prove the last claim of lemma 7 about the support of a solution $u(x,t)$, it suffices to take in place of the set M above the set

$$M_1 := \{w \in X \cap Y : \ \|w - h\|_{X \cap Y} \leqslant 1 \text{ and } w(x,t) = 0$$

$$\text{for any } t \in I \text{ and } x \in \mathbb{R} \setminus K(t)\}.$$

It is easily seen that the operator in the right-hand side of (17) still maps M_1 into itself and is a contraction for $T > 0$ sufficiently small. This completes our sketch of the proof of lemma 7.□

Lemma 8. *Let $I = (-T_1, T_2)$, where $T_1, T_2 > 0$. Under the assumptions of theorem 1 a function $v(x,t) \in X \cap Y$ is a weak solution of problem (1)–(2) if and only if it is a solution of equation (16).*

Proof. Let a function $v(\cdot, t) \in X \cap Y$ satisfies equation (16). Then, after some calculations, $(v(\cdot, 0), v'_t(\cdot, 0)) = (u_0, u_1)$ in the sense of the space $X \cap Y$. Take an interval $I_1 = (-T_1 + \delta, T_2 - \delta)$, where $\delta \in (0, \min\{T_1; T_2\})$ is sufficiently small, and C^∞-approximations v_ϵ, where $\epsilon > 0$ is sufficiently small, of $v(x,t)$ such that the families $\{v_\epsilon\}$ and $\{v_{\epsilon t}\}$ are bounded, respectively, in $L_b(I_1; W_\infty^1)$ and in $L_b(I_1; L_\infty)$ and that $v_\epsilon \to v$ in $C(I_1; W_2^1) \cap C^1(I_1; L_2)$. Then, by lemma 6 v_ϵ satisfies a slightly perturbed equation (16), with an additional term $f_\epsilon(x,t)$ in the right-hand side such that $\|f_\epsilon(\cdot,t)\|_{L_b(I_1;W^1_\infty)}+\|f'_{\epsilon t}(\cdot,t)\|_{L_b(I_1;L_\infty)} \leqslant$ $\leqslant C$ uniformly in $\epsilon>0$ and that $\|f_\epsilon(\cdot,t)\|_{C(I_1;W^1_2)\cap C^1(I_1;L_2)} \to 0$ as $\epsilon\to+0$. Correspondingly, the function $v^\epsilon(\chi,\eta):=v_\epsilon(x(\chi,\eta),t(\chi,\eta))$ satisfies a slightly perturbed equation (15), with an additional term $f^\epsilon(\chi,\eta)$ in the right-hand side that belongs to $W^1_{2,\mathrm{loc}}((\chi,\eta)(I_1\times\mathbb{R}))$ and goes to 0 in this space as $\epsilon\to+0$. Therefore, this function v^ϵ is a solution of a slightly perturbed equation (11). This perturbed equation (11) contains an additional term $g^\epsilon(\chi,\eta)$ in the right-hand side where, in view of lemmas 2 and 3 and the implicit function theorem, $g^\epsilon(\chi,\eta)=\dfrac{\partial^2 f^\epsilon(\chi,\eta)}{\partial\chi\partial\eta}$ belongs in particular to $L_{1,\mathrm{loc}}((\chi,\eta)(I_1\times\mathbb{R}))$.

Set $g_\epsilon(x,t)=g^\epsilon(\chi(x,t),\eta(x,t))$. Then, in particular $g_\epsilon\in L_{1,\mathrm{loc}}(I_1\times\mathbb{R})$. Take an arbitrary finite $\varphi(\chi,\eta)\in W^1_2((\chi,\eta)(I_1\times\mathbb{R}))$ and observe that

$$\int\limits_{I_1\times\mathbb{R}} J(x,t)\varphi(\chi(x,t),\eta(x,t))g_\epsilon(x,t)dxdt = \int\limits_{(\chi,\eta)(I_1\times\mathbb{R})} \varphi(\chi,\eta)g^\epsilon(\chi,\eta)d\chi d\eta =$$

$$= -\int\limits_{(\chi,\eta)(I_1\times\mathbb{R})} \frac{\partial\varphi(\chi,\eta)}{\partial\eta}\frac{\partial f^\epsilon(\chi,\eta)}{\partial\chi}d\chi d\eta \to 0 \quad \text{as} \quad \epsilon\to+0. \tag{18}$$

In addition, observe that the function $J(x,t)\varphi(\chi(x,t),\eta(x,t))$ runs over the whole space $W^1_{2,\mathrm{loc}}(I_1\times\mathbb{R})$ when $\varphi(\chi,\eta)$ runs over the whole $W^1_{2,\mathrm{loc}}((\chi,\eta)(I_1\times\mathbb{R}))$. To express these two facts, we shall write formally $g_\epsilon\to 0$ in $W^{-1}_{2,\mathrm{loc}}(I_1\times\mathbb{R})$ as $\epsilon\to+0$.

Let us now make in equation (11), written for v^ϵ, the change of variables passing from the variables (χ,η) to (x,t). Then,

$$v_{\epsilon tt} = a(x,t)v_{\epsilon xx} + f_2(t,x,v_\epsilon,v_{\epsilon t},v_{\epsilon x}) + g_\epsilon(x,t), \tag{19}$$

where $f_2(t,x,v,r,s)=b(x,t)v+c(x,t)r+d(x,t)s+f(t,x,v,r,s)$, $g_\epsilon(\cdot,t)\to 0$ in $W^{-1}_{2,\mathrm{loc}}$ and $v_{\epsilon xx}\to v_{xx}$ in $C(I_1;W^{-1}_2)$. Take an arbitrary finite $\varphi\in W^1_2(I_1\times\mathbb{R})$, multiply equation (19) by φ and integrate the result over $I_1\times\mathbb{R}$. Then,

$$\int\limits_{I_1\times\mathbb{R}} dxdt\varphi(x,t)[v_{\epsilon tt}-a(x,t)v_{\epsilon xx}-f_2(t,x,v_\epsilon,v_{\epsilon t},v_{\epsilon x})] + (\varphi,g_\epsilon)_{L_2(I_1\times\mathbb{R})} = 0.$$

From this, applying lemma 1 and (18), we obtain that $v_{tt}\in C(I_1;W^{-1}_2)$ and

$$\int\limits_{I_1\times\mathbb{R}} dxdt\varphi(x,t)[v_{tt}-a(x,t)v_{xx}-f_2(t,x,v,v_t,v_x)] = 0.$$

Thus, $v(x,t)$ is a weak solution of problem (1)–(2). Converse is still valid.□

Let us prove the uniqueness of a weak solution of problem (1)–(2). On the contrary, suppose that this problem has two different weak solutions u_1 and u_2. According to lemma 8, u_1 and u_2 are also solutions of equation (16) (or (17)) in an interval of time $I = [-T, T]$, $T > 0$. Without loss of the generality, we may accept that $u_1(\cdot, t) \not\equiv u_2(\cdot, t)$ as elements of $W_2^1 \cap W_\infty^1$ in an arbitrary small right half-neighborhood of the point $t = 0$. But according to lemma 7, an $X \cap Y$-solution of equation (17) is unique in a sufficiently small interval of time $[0, t_0)$. Thus, $u_1(\cdot, t) \equiv u_2(\cdot, t)$ in a right half-neighborhood of the point $t = 0$. This contradiction proves that a weak solution of problem (1)–(2) is unique.

As is well known, a fixed point of a contraction mapping, which is the map in right-hand side of (17) in our case, depends on u_0 and u_1 continuously, in the same sense as the local continuous dependence on (u_0, u_1) in theorem 1. So, locally, in a small neighborhood of the point $t = 0$ we have the continuous dependence of a weak solution of problem (1)–(2) on (u_0, u_1). Now, the continuous dependence for an arbitrary compact interval I of time t on which our weak solution of problem (1)–(2) can be continued the result can be obtained by standard methods by extending it, step by step, for all values of t.

By the standard procedure, our weak solution of problem (1)–(2) can be uniquely continued on a maximal interval $(-T_1, T_2)$ such that either $\limsup\limits_{t \to -T_1+0} [\|u(\cdot, t)\|_{W_2^1 \cap W_\infty^1} + \|u_t(\cdot, t)\|_{L_2 \cap L_\infty}] = +\infty$ or $T_1 = -\infty$ and by analogy for T_2. Our proof of theorem 1 is complete.

3. GLOBAL EXISTENCE. PROOF OF THEOREM 2

Everywhere in this Section, unless otherwise is stated, we accept that the assumptions of theorem 2 are valid. Everywhere in the following: $b \equiv c \equiv d \equiv 0$, $f = f(u)$ in equation (1) (that is, f does not depend on t, x, u_t and u_x) and $f(0) = 0$. We shall prove this result. In view of theorem 1, for this aim, it suffices to show that for any bounded interval $I \ni 0$ on which our weak solution $u(x, t)$ can be continued there exists $C > 0$ such that $\|u(\cdot, t)\|_{X \cup Y} \leqslant C$ for this interval I. First, we shall establish the following three technical results (lemmas 9–11).

Lemma 9. *Let for some coefficient $a = a(x, t)$ and a function $f = f(u)$ that satisfy assumptions (A1)–(A3) problem (1)–(2) have a solution $u(x, t)$ in an interval of time I_1 and this solution belongs to $C(I; W_2^3) \cap C^1(I; W_2^2) \cap C^2(I; W_2^1)$ for any compact interval $I \subset I_1$. Let in addition $uf(u) \leqslant 0$ for any $u \in \mathbb{R}$. Then, for any bounded interval $I \subset I_1$ there exists $C > 0$ such that $\|u\|_{C(I; W_2^1) \cap C^1(I; L_2)} \leqslant C$.*

Proof. Denote $F(u) = \int\limits_0^u f(s)ds$ and $E(u) = \int\limits_{\mathbb{R}} \{\frac{1}{2}[u_t^2(x,t) + a(x,t)u_x^2(x,t)] - F(u(x,t)\}dx$. We have after an integration by parts:

$$\frac{d}{dt}E(u(\cdot,t)) = \int\limits_{\mathbb{R}} [1/2 a_t(x,t)u_x^2(x,t) - a_x(x,t)u_t(x,t)u_x(x,t)]dx \leqslant$$
$$\leqslant (1 + 2a_1^{-1})a_3 E(u(\cdot,t)),$$

therefore

$$E(u(\cdot,t)) \leqslant E(u(\cdot,0))e^{(1+2a_1^{-1})a_3 t},$$

where $a_1 = a_1(I) > 0$ is the constant in (3) and $a_3 = a_3(I) > 0$ is a constant such that $\|\nabla a(\cdot,t)\|_{L_b(I;L_\infty)} \leqslant a_3$. Now

$$\|u(\cdot,t\|_{L_2}^2 = \int\limits_{\mathbb{R}} \left\{ \int\limits_0^t u'_r(x,r)dr + u_0(x) \right\}^2 dx \leqslant$$
$$\leqslant 2\left\{ \|u_0\|_{L_2}^2 + t\int\limits_0^t \int\limits_{\mathbb{R}} {u'}_r^2(x,r)dx\ dr \right\}. \square$$

Lemma 10. *Let $s \geqslant 3$ be integer, $f(\cdot) \in C_{\mathrm{loc}}^{s-1}(\mathbb{R})$, $f(0) = 0$ and let a coefficient $a(x,t)$ that satisfies assumption (A1) obey in addition the following two hypotheses.*

1) For any $x_0 \in \mathbb{R}$ and a bounded interval I $a(\cdot,t) \in C(I; W_2^s(x_0 - 1, x_0 + 1)) \cap C^1(I; W_2^{s-1}(x_0 - 1, x_0 + 1))$.

2) Let for any bounded interval I one has: $a_t^{(k)}(\cdot,t) \in L_b(I; W_\infty^{s-k})$ for $k = 0, 1$.

Then, for any $(u_0, u_1) \in W_2^s \times W_2^{s-1}$ there exists $T > 0$ that depends only on $\|u_0\|_{W_2^s} + \|u_1\|_{W_2^{s-1}}$ and a unique solution of problem (1)–(2) of the class $C(I; W_2^s) \cap C^1(I; W_2^{s-1})$, where $I = [-T, T]$.

Proof in fact repeats our proof of theorem 1. So, we establish only the main idea of this proof. Since it is clear that a W_2^s-solution in this lemma is also a weak solution of this problem, in view of lemma 8, we have to prove only that equation (16) has a unique local W_2^s-solution the life time of which is bounded from below by a positive constant that depends only on $\|u_0\|_{W_2^s} + \|u_1\|_{W_2^{s-1}}$.

For a bounded interval $I \ni 0$ denote $X^s = C(I; W_2^s) \cap C^1(I; W_2^{s-1})$. Observe that, as in lemmas 2 and 3, there exist $0 < c < C$ such that

$$c \leqslant \overline{X}'_{it}(t,s;x), \overline{X}'_{ix}(t,s;x) \leqslant C, \quad i = 1, 2,$$

for any $t \in I, s \in [0,t]$ and $x \in \mathbb{R}$ and that for any such t, s and x there exist partial derivatives in x of the functions $\overline{X}_i(t,s;x)$, $\overline{X}'_{it}(t,s;x)$ and $\overline{X}'_{ix}(t,s;x)$, where $i = 1,2$, of the orders $0,1,...,s-1$ and that each of these derivatives belongs to $L_b(I;L_\infty)$ and, for any $x_0 \in \mathbb{R}$, to $C(I;L_2(x_0-1,x_0+1))$. By analogy, there exist partial derivatives in x of the orders $0,1,...,s-1$ of the coefficients b_1, c_1 and d_1 and of the Jacobian J and each of these derivatives belongs to the same spaces.

By this observation, S is a continuous operator from X^s in X^{s-1} and for any ball $B \subset X^s$ there exists $C = C(B) > 0$ such that

$$\|S(w_1) - S(w_2)\|_{X^{s-1}} \leqslant C\|w_1 - w_2\|_{X^s}$$

for any $w_1, w_2 \in B$. By this, P is a continuous operator from X^{s-1} in X^s and for any ball $B \subset X^{s-1}$ there exists $C_1 = C_1(B,T) > 0$ with $C_1(B,T) \to +0$ as $T \to +0$ such that

$$\|P(w_1) - P(w_2)\|_{X^s} \leqslant C_1\|w_1 - w_2\|_{X^{s-1}}$$

for any $w_1, w_2 \in B$.

Now, one can prove our lemma completely as the first part of lemma 7, the existence and uniqueness of a local solution of equation (16).□

Lemma 11. *Let the assumptions of lemma 10 be valid with $s = 3$, $(u_0, u_1) \in W_2^3 \times W_2^2$ and $(-T_3', T_3'')$ be the maximal interval of time t on which the corresponding W_2^3-solution can be continued (here $T_3', T_3'' > 0$). Clearly, for our initial data we have in addition the existence and uniqueness of a weak solution $u(x,t)$ of problem (1)–(2) in an interval of time $I_1 \ni 0$ and clearly, $(-T_3', T_3'') \subset I_1$ and this weak solution $u(x,t)$ coincides with the W_2^3-solution in the whole interval $(-T_3', T_3'')$. Then, we have the following.*

1) In fact, $(-T_3', T_3'') = I_1$.

2) Let we have initial data $(u_0^h, u_1^h) \in W_2^3 \times W_2^2$, a coefficient a^h and a function $f^h = f^h(u)$ in (1) that depend on a parameter $h \in (0,1]$. Suppose that for any $h \in (0,1]$ this coefficient a^h satisfies assumption (A1) and assumptions 1) and 2) in lemma 10 with $s = 3$. Let for any bounded interval I the constants a_1, a_2 in (3) do not depend on $h \in (0,1]$ and the norms of $a^h(x,t)$ in the spaces of functions indicated in (A1) are bounded uniformly in h. Let in addition for any bounded open interval $\hat{I} \subset \mathbb{R}$ the norm of f^h in $C^1(\hat{I})$ is bounded uniformly with respect to $h \in (0,1]$ and $f^h(0) = 0$ for any $h \in (0,1]$. Now, suppose that $(u_0^h, u_1^h) \to (u_0, u_1)$ in $W_2^1 \times L_2$ as $h \to +0$, that the quantity $\|u_0^h\|_{W_\infty^1} + \|u_1^h\|_{L_\infty}$ is bounded uniformly with respect to $h \in (0,1]$ and that for any bounded interval I and $x_0 \in \mathbb{R}$ the coefficients a^h and their gradients $\nabla a^h(x,t)$ converge, respectively, to a and $\nabla a(x,t)$ in $C(I; L_2(x_0-1, x_0+1))$ as $h \to +0$ and for any bounded interval $J \subset \mathbb{R}$ the functions $f^h(\cdot)$ converge to $f(\cdot)$ in $C^1(J)$. Denote by $u^h(x,t)$ the corresponding W_2^3-solution of problem (1)–(2)

taken with $a = a^h$, $f = f^h$ and with the initial data $(u_0, u_1) = (u_0^h, u_1^h)$ and by $u(x,t)$ the weak solution of this problem taken with the limit coefficient a, with the limit function f and with the initial data (u_0, u_1). Then, for any compact interval I on which our weak solution $u(x,t)$ can be continued for any $h > 0$ sufficiently small, the W_2^3-solution $u^h(x,t)$ can be continued on this interval I and one has that

$$\|u^h(\cdot,t) - u(\cdot,t)\|_{C(I;W_2^1)\cap C^1(I;L_2)} \to 0$$

as $h \to +0$ and that there exists $\overline{C} > 0$ such that

$$\|u^h(\cdot,t)\|_{L_b(I;W_\infty^1)} + \|u_t^h(\cdot,t)\|_{L_b(I;L_\infty)} \leqslant \overline{C}$$

for any $h > 0$ sufficiently small.

Proof. As for claim 1), we shall prove only that $[0, T_3'') = I_1 \cap [0,+\infty)$ because the relation $(-T_3', 0] = I_1 \cap (-\infty, 0]$ can be proved by complete analogy. Let $I_2 \ni 0$ be an arbitrary bounded interval on which our weak solution can be continued and $I = I_2 \cap [0,+\infty)$. We need to prove only that there exists a constant $C = C(I) > 0$ such that $\|u(\cdot,t)\|_{W_2^3} + \|u_t(\cdot,t)\|_{W_2^2} \leqslant C(I)$ for any $t \in I$ for which our W_2^3-solution $u(x,t)$ is determined. From (16), we obtain

$$\begin{aligned}
&\|u(\cdot,t)\|_{W_2^3} + \|u_t(\cdot,t)\|_{W_2^2} \leqslant \\
&\leqslant C \int_0^t ds \Bigg\| \int_{\overline{X}_1(t,s;\cdot)}^{\overline{X}_2(t,s;\cdot)} dy J(y,s)[b_1(\chi,\eta)u(\chi,\eta) + c_1(\chi,\eta)(u_y y_\chi + u_s s_\chi) + \\
&\qquad + d_1(\chi,\eta)(u_y y_\eta + u_s s_\eta) + f_1(u(\chi,\eta))]\Big|_{\substack{\chi=\chi(y,s)\\ \eta=\eta(y,s)}} \Bigg\|_{W_2^3} + \\
&+ \sum_{i=1}^{2} \int_0^t ds \Big\{ \Big\| \overline{X}'_{it}(t,s;\cdot) J(\overline{X}_i(t,s;\cdot),s)[b_1(\chi,\eta)u(\chi,\eta) + c_1(\chi,\eta)(u_y y_\chi + u_s s_\chi) + \\
&\qquad + d_1(\chi,\eta)(u_y y_\eta + u_s s_\eta) + f_1(u(\chi,\eta))]\Big|_{\substack{\chi=\chi(\overline{X}_i(t,s;\cdot),s)\\ \eta=\eta(\overline{X}_i(t,s,\cdot),s)}} \Big\|_{W_2^2},
\end{aligned} \tag{20}$$

where the constant $C > 0$ does not depend on $t \in I$, $s \in [0,t]$ and x. Observe that, as in lemmas 2 and 3 and in the proof of the previous lemma 10, there exist $0 < c < C$ such that

$$c \leqslant \overline{X}'_{it}(t,s;x), \overline{X}'_{ix}(t,s;x) \leqslant C, \quad i = 1,2,$$

for any $t \in I, s \in [0,t]$ and $x \in \mathbb{R}$ and that for any such t, s and x there exist partial derivatives in x of the functions $\overline{X}_i(t,s;x)$, $\overline{X}'_{it}(t,s;x)$ and $\overline{X}'_{ix}(t,s;x)$, where $i = 1, 2$, of orders 1 and 2 and that each of these derivatives belongs to $L_b(I; L_\infty)$ and, for any $x_0 \in \mathbb{R}$, to $C(I; L_2(x_0 - 1, x_0 + 1))$. By analogy, there exist partial derivatives in x of orders $0, 1$ and 2 of the coefficients b_1, c_1 and d_1 and of the Jacobian J and each of these derivatives belongs to the same spaces.

The expression in the right-hand side of (20) contains the norms of the kind

$$\Bigg\| \int\limits_{\overline{X}_1(t,s;\cdot)}^{\overline{X}_2(t,s;\cdot)} dy J(y,s)[b_1(\chi,\eta)u(\chi,\eta) + c_1(\chi,\eta)(u_y y_\chi + u_s s_\chi) + \\ + d_1(\chi,\eta)(u_y y_\eta + u_s s_\eta) + f_1(u(\chi,\eta))]\Big|_{\substack{\chi=\chi(y,s)\\ \eta=\eta(y,s)}} \Bigg\|_{W_2^3}$$

and

$$\Bigg\| \overline{X}'_{it}(t,s;\cdot) J(\overline{X}_i(t,s;\cdot),s)[b_1(\chi,\eta)u(\chi,\eta) + c_1(\chi,\eta)(u_y y_\chi + u_s s_\chi) + \\ + d_1(\chi,\eta)(u_y y_\eta + u_s s_\eta) + f_1(u(\chi,\eta))]\Big|_{\substack{\chi=\chi(\overline{X}_i(t,s;\cdot),s)\\ \eta=\eta(\overline{X}_i(t,s;\cdot),s)}} \Bigg\|_{W_2^2}, \quad i = 1, 2.$$

All these expressions can be estimated in the same way, therefore we shall do this below only for the terms of the second kind. The observation is that each of these latter terms can be estimated from above by a sum of L_2-norms of

$$\begin{aligned} & u(y,s)\big|_{y=\overline{X}_i(t,s;x)},\ u_y(y,s)\big|_{y=\overline{X}_i(t,s;x)},\ u_s(y,s)\big|_{y=\overline{X}_i(t,s;x)}, \\ & u_{sy}(y,s)\big|_{y=\overline{X}_i(t,s;x)},\ u_{yy}(y,s)\big|_{y=\overline{X}_i(t,s;x)}, \\ & u_{syy}(y,s)\big|_{y=\overline{X}_i(t,s;x)},\ \text{and } u_{yyy}(y,s)\big|_{y=\overline{X}_i(t,s;x)},\ i = 1, 2, \end{aligned} \tag{21}$$

multiplied by some coefficients, denoted by $k(t, s, x)$. These coefficients k, being regarded as functions of the arguments s and x with a fixed t, are bounded in $L_b([0, t]; L_\infty)$ uniformly with respect to $t \in I$. Therefore, we need to obtain only upper bounds for the L_2-norms of the expressions in (21). We have

$$\int\limits_{\mathbb{R}} [u_{yyy}(y,s)\big|_{y=\overline{X}_1(t,s;x)}]^2 dx = \\ = \int\limits_{\mathbb{R}} dy [\overline{X}'_{1x}(t,s;x(y))]^{-1} u^2_{yyy}(y,s) \leqslant C_2 \|u_{yyy}(\cdot,s)\|^2_{L_2},$$

where the constant $C_2 > 0$ does not depend on u, t, s and x above. The L_2-norms of all other expressions in (21) can be estimated by complete analogy.

Summarizing the arguments above, we obtain finally from (20) for any $t \in I$:

$$\|u(\cdot,t)\|_{W_2^3} + \|u_t(\cdot,t)\|_{W_2^2} \leqslant C_2 \int_0^t ds[\|u(\cdot,s)\|_{W_2^3} + \|u_s(\cdot,s)\|_{W_2^2}] + C_3, \quad (22)$$

where positive constants C_2 and C_3 do not depend on $t \in I$. Now, claim 1) of our lemma for $t > 0$ follows by applying the Gronwell lemma. The case $t < 0$ can be treated by complete analogy. So, claim 1) is proved.

Let us prove claim 2). Let $I \ni 0$ be a compact interval on which our weak solution $u(x,t)$ can be continued and let $I_+ = I \cap [0,+\infty)$ and $I_- = I \cap (-\infty, 0]$. Observe that, due to the results above, the coefficients b_1^h, c_1^h and d_1^h, that correspond to $a = a^h(x,t)$, are bounded in $L_b(I_+; L_\infty)$ uniformly with respect to $h \in (0,1]$ and that these coefficients converge, respectively, to b_1, c_1 and d_1 at least in the sense that, for any $x_0 \in \mathbb{R}$,

$$\sup_{t\in I_+}[\|b_1^h - b_1\|_{L_2(x_0-1,x_0+1)} + \|c_1^h - c_1\|_{L_2(x_0-1,x_0+1)} + \\ + \|d_1^h - d_1\|_{L_2(x_0-1,x_0+1)}] \to 0$$

as $h \to +0$.

Denote

$$g(x,t) = [b_1(\chi,\eta)u(\chi,\eta) + c_1(\chi,\eta)(u_x x_\chi + u_t t_\chi) + d_1(\chi,\eta)(u_x x_\eta + u_t t_\eta)]\Big|_{\substack{\chi=\chi(x,t)\\ \eta=\eta(x,t)}}$$

and

$$g^h(x,t) = [b_1^h(\chi,\eta)u^h(\chi,\eta) + c_1^h(\chi,\eta)(u_x^h x_\chi + u_t^h t_\chi) + \\ + d_1^h(\chi,\eta)(u_x^h x_\eta + u_t^h t_\eta)]\Big|_{\substack{\chi=\chi(x,t)\\ \eta=\eta(x,t)}}.$$

Let us consider the difference between two samples of equation (16) written, respectively, for $u^h(x,t)$ and for our weak solution $u(x,t)$. Then, in view of lemmas 2 and 3, it follows as when deriving (20) and (22) that

$$\|u^h(\cdot,t) - u(\cdot,t)\|_{W_2^1} + \|u_t^h(\cdot,t) - u_t(\cdot,t)\|_{L_2} \leqslant$$

$$\leqslant \gamma(h) + C_1 \int_0^t ds \Bigg\{ \sum_{i=1}^{2} [\sup_{t,s,x} |\overline{X}_{it}^{h\,\prime}(t,s;x)| + \sup_{t,s,x} |\overline{X}_{ix}^{h\,\prime}(t,s;x)|] \times$$

$$\times\|g^h(\overline{X}_i^h(t,s;\cdot),s)-g(\overline{X}_i(t,s;\cdot),s)\|_{L_2}+\left\|\int\limits_{\overline{X}_1^h(t,s;\cdot)}^{\overline{X}_2^h(t,s;\cdot)} dy(g^h(y,s)-g(y,s))\right\|_{L_2}+$$

$$+\sum_{i=1}^{2}[\sup_{t,s,x}|\overline{X}_{it}^{h\,\prime}(t,s;x)-\overline{X}_{it}{}'(t,s;x)|+$$

$$\left.+\sup_{t,s,x}|\overline{X}_{ix}^{h\,\prime}(t,s;x)-\overline{X}_{ix}{}'(t,s;x)|]\|g(\cdot,s)\|_{L_2}\right\}\leqslant$$

$$\leqslant\gamma_1(h)+C_2\int\limits_0^t ds[\|u^h(\cdot,s)-u(\cdot,s)\|_{W_2^1}+\|u_s^h(\cdot,s)-u_s(\cdot,s)\|_{L_2}],$$

where positive constants C_1 and C_2 do not depend on $t\in I_+$ and h, $\gamma(h),\gamma_1(h)\to+0$ as $h\to+0$, all the supremums are taken over $t\in I_+, s\in[0,t]$ and $x\in\mathbb{R}$ and where we applied the mean continuity of the Lebesgue integral.

Now, the first relation in claim 2) of our lemma for $t>0$ follows by applying the Gronwell lemma. The case $t<0$ can be treated by complete analogy. The second estimate in claim 2) of our lemma can now be obtained from an estimate derived from (16) for $\|u^h(\cdot,t)\|_{W_\infty^1}+\|u_t(\cdot,t)\|_{L_\infty}$ by analogy with (22). Our proof of lemma 11 is complete.□

Lemma 12. *Let $a(x,t)$ and $f=f(u)$ be as in lemma 10. Then, for a W_2^3-solution $u(x,t)$ of problem (1)–(2) and integer $n\geqslant 2$ one has*

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}}\Big\{(|u_t(x,t)|+a^{1/2}(x,t)|u_x(x,t)|)^{2n}+$$

$$\left.+(|u_t(x,t)|-a^{1/2}(x,t)|u_x(x,t)|)^{2n}\right\}=$$

$$=\sum_{k=0}^{n-1}2n\binom{2n-1}{2k}\int\limits_{\mathbb{R}}dxa^k(x,t)f(u(x,t))u_t^{2n-2k-1}(x,t)u_x^{2k}(x,t)-$$

$$-\sum_{k=1}^{n}k\binom{2n}{2k-1}\int\limits_{\mathbb{R}}dxa^{k-1}(x,t)a_x(x,t)u_t^{2n-(2k-1)}(x,t)u_x^{2k-1}(x,t)+$$

$$+\sum_{k=1}^{n}k\binom{2n}{2k}\int\limits_{\mathbb{R}}dxa^{k-1}(x,t)a_t(x,t)u_t^{2n-2k}(x,t)u_x^{2k}(x,t),\qquad(23)$$

where $\binom{n}{m}=\frac{n!}{m!(n-m)!}$ $(n\geqslant m)$.

Proof. Consider the expression $\frac{d}{dt}\int\limits_{\mathbb{R}} dx u_t^{2n}(x,t)$. We have

$$\frac{d}{dt}\int\limits_{\mathbb{R}} dx u_t^{2n}(x,t) = 2n\int\limits_{\mathbb{R}} u_t^{2n-1}(x,t)[a(x,t)u_{xx}(x,t) + f(u(x,t))] =$$

$$= 2n\int\limits_{\mathbb{R}} dx\{u_t^{2n-1}(x,t)f(u(x,t)) - u_t^{2n-1}(x,t)u_x(x,t)a_x(x,t)\} -$$

$$-2n(2n-1)\int\limits_{\mathbb{R}} dx a(x,t)u_t^{2n-2}(x,t)u_x(x,t)u_{tx}(x,t) =$$

$$= 2n\int\limits_{\mathbb{R}} dx\{u_t^{2n-1}(x,t)f(u(x,t)) - a_x(x,t)u_t^{2n-1}(x,t)u_x(x,t)\} -$$

$$-\binom{2n}{2}\frac{d}{dt}\int\limits_{\mathbb{R}} dx a(x,t)u_t^{2n-2}(x,t)u_x^2(x,t) +$$

$$+\binom{2n}{2}\int\limits_{\mathbb{R}} dx\frac{\partial}{\partial t}[a(x,t)u_t^{2n-2}(x,t)]u_x^2(x,t) =$$

$$= 2n\int\limits_{\mathbb{R}} dx\{u_t^{2n-1}(x,t)f(u(x,t)) - a_x(x,t)u_t^{2n-1}(x,t)u_x(x,t)\} -$$

$$-\binom{2n}{2}\frac{d}{dt}\int\limits_{\mathbb{R}} dx a(x,t)u_t^{2n-2}(x,t)u_x^2(x,t) +$$

$$+\binom{2n}{2}\int\limits_{\mathbb{R}} dx a_t(x,t)u_t^{2n-2}(x,t)u_x^2(x,t) +$$

$$+(2n-2)\binom{2n}{2}\int\limits_{\mathbb{R}} dx a(x,t)u_t^{2n-3}(x,t)u_x^2(x,t)[a(x,t)u_{xx}(x,t) + f(u(x,t))] =$$

$$= 2n\int\limits_{\mathbb{R}} dx\{u_t^{2n-1}(x,t)f(u(x,t)) - a_x(x,t)u_t^{2n-1}(x,t)u_x(x,t)\} -$$

$$-\binom{2n}{2}\frac{d}{dt}\int\limits_{\mathbb{R}} dx a(x,t)u_t^{2n-2}(x,t)u_x^2(x,t) +$$

$$+\binom{2n}{2}\int\limits_{\mathbb{R}} dx a_t(x,t)u_t^{2n-2}(x,t)u_x^2(x,t) +$$

$$+(2n-2)\binom{2n}{2}\int\limits_{\mathbb{R}} dxa(x,t)u_t^{2n-3}(x,t)u_x^2(x,t)f(u(x,t))-$$

$$-(2n-3)\binom{2n}{3}\int\limits_{\mathbb{R}} dxa^2(x,t)u_t^{2n-4}(x,t)u_x^3(x,t)u_{tx}(x,t)-$$

$$-2\binom{2n}{3}\int\limits_{\mathbb{R}} dxa(x,t)a_x(x,t)u_t^{2n-3}(x,t)u_x^3(x,t) =$$

$$= 2n\int\limits_{\mathbb{R}} dx\{u_t^{2n-1}(x,t)f(u(x,t)) - a_x(x,t)u_t^{2n-1}(x,t)u_x(x,t)\}-$$

$$-\binom{2n}{2}\frac{d}{dt}\int\limits_{\mathbb{R}} dxa(x,t)u_t^{2n-2}(x,t)u_x^2(x,t)+$$

$$+\binom{2n}{2}\int\limits_{\mathbb{R}} dxa_t(x,t)u_t^{2n-2}(x,t)u_x^2(x,t)+$$

$$+(2n-2)\binom{2n}{2}\int\limits_{\mathbb{R}} dxa(x,t)u_t^{2n-3}(x,t)u_x^2(x,t)f(u(x,t))-$$

$$-2\binom{2n}{3}\int\limits_{\mathbb{R}} dxa(x,t)a_x(x,t)u_t^{2n-3}(x,t)u_x^3(x,t)-$$

$$-\binom{2n}{4}\frac{d}{dt}\int\limits_{\mathbb{R}} dxa^2(x,t)u_t^{2n-4}(x,t)u_x^4(x,t)+$$

$$\binom{2n}{4}\int\limits_{\mathbb{R}} dxu_x^4(x,t)\frac{\partial}{\partial t}[a^2(x,t)u_t^{2n-4}(x,t)].$$

Continue this process. Then, finally we obtain relation (23).□

Lemma 13. *Let assumption (A1) be valid,* $b \equiv c \equiv d \equiv 0$, $f = f(u) \in C^1_{\mathrm{loc}}(\mathbb{R})$, $f(0) = 0$, $uf(u) \leqslant 0$ *for any* $u \in \mathbb{R}$ *and let* $u(x,t)$ *be a weak solution of equations (1) and (2) that can be continued on an interval* $I \ni 0$. *Then, for any bounded interval* $I_1 = (-T_1, T_2) \subset I$ $(T_1, T_2 > 0)$ *there exists* $C > 0$ *such that* $\|u(\cdot,t)\|_{W^1_\infty} + \|u_t(\cdot,t)\|_{L_\infty} \leqslant C$ *for any* $t \in I_1$.

Proof. We shall establish our proof only for $t \in [0, T_2)$ because for $t \in (-T_1, 0]$ it can be made by complete analogy. First, let $f = f(u)$, the coefficient

$a(x,t)$ and the initial data (u_0, u_1) be smooth as in lemma 10 and let $u(x,t)$ be the corresponding W_2^3-solution of problem (1)–(2). Then, we have from (23):

$$\frac{1}{2}\frac{d}{dt}\left\{\int\limits_{\mathbb{R}} dx[(|u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n} + \right.$$

$$\left. +(|u_t(x,t)| - a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n}]^{\frac{1}{2n}}\right\} \leqslant$$

$$\leqslant \frac{1}{2}\left(\||u_t(x,t)| + a^{\frac{1}{2}}|u_x(x,t)|\|_{L_{2n}}^{2n} + \||u_t(x,t)| - a^{\frac{1}{2}}|u_x(x,t)|\|_{L_{2n}}^{2n}\right)^{\frac{1}{2n}-1} \times$$

$$\times\left\{\sum_{k=0}^{n-1}\int\limits_{\mathbb{R}} dx|f(u(x,t))|\binom{2n-1}{2k}|u_t(x,t)|^{2n-2k-1}a^k(x,t)|u_x(x,t)|^{2k} + \right.$$

$$+\sum_{k=1}^{n}\int\limits_{\mathbb{R}} dx\binom{2n}{2k-1}\frac{|a_x(x,t)|}{a^{\frac{1}{2}}(x,t)}|u_t(x,t)|^{2n-(2k-1)}a^{k-\frac{1}{2}}(x,t)|u_x(x,t)|^{2k-1} +$$

$$\left. +\sum_{k=1}^{n}\int\limits_{\mathbb{R}} dx\binom{2n}{2k}\frac{|a_t(x,t)|}{a(x,t)}|u_t(x,t)|^{2n-2k}a^k(x,t)|u_x(x,t)|^{2k}\right\} =$$

$$= \frac{1}{2}\left(\||u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|\|_{L_{2n}}^{2n} + \right.$$

$$\left.\||u_t(x,t)| - a^{\frac{1}{2}}(x,t)|u_x(x,t)|\|_{L_{2n}}^{2n}\right)^{\frac{1}{2n}-1} \times (I + II + III) \qquad (24)$$

(note that the divisor in the right-hand side of (24) does not vanish by the proved uniqueness of a solution). In (24), we shall estimate the terms I, II and III separately. We have by the Sobolev embedding and lemma 9:

$$I(t) \leqslant \frac{1}{2}C_1(T_2)\int\limits_{\mathbb{R}} dx[(|u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n-1} +$$

$$+ (|u_t(x,t)| - a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n-1}],$$

where the constant $C_1 > 0$ does not depend on $n > 0$ integer sufficiently large. Applying the Hölder inequality for sums, we obtain from this estimate:

$$I(t) \leqslant C_1 2^{\frac{1}{2n}}\left\{\int\limits_{\mathbb{R}} dx[(|u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n} + \right.$$

$$\left. +(|u_t(x,t)| - a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n}]\right\}^{1-\frac{1}{2n}}. \qquad (25)$$

For $II(t)$, we have by analogy:

$$II(t) \leqslant \frac{a_3}{2\sqrt{a_1}} \int\limits_{\mathbb{R}} dx[(|u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n} -$$

$$- (|u_t(x,t)| - a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n}]. \quad (26)$$

Finally, for $III(t)$, by analogy:

$$III(t) \leqslant \frac{a_3}{2a_1} \int\limits_{\mathbb{R}} dx[(|u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n} +$$

$$+ (|u_t(x,t)| - a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n}]. \quad (27)$$

Thus, from (24)–(27), for $t \in [0, T_2]$,

$$\left\{ \int\limits_{\mathbb{R}} dx[(|u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n} + \right.$$

$$\left. +(|u_t(x,t)| - a^{\frac{1}{2}}(x,t)|u_x(x,t)|)^{2n}] \right\}^{\frac{1}{2n}} \leqslant$$

$$\leqslant \left\{ \int\limits_{\mathbb{R}} dx[(|u_t(x,0)| + a^{\frac{1}{2}}(x,0)|u_x(x,0)|)^{2n} + \right.$$

$$\left. +(|u_t(x,0)| - a^{\frac{1}{2}}(x,0)|u_x(x,0)|)^{2n}] \right\}^{\frac{1}{2n}} +$$

$$+ 2C_1T_2 + C_3 \int\limits_0^t ds \left\{ \int\limits_{\mathbb{R}} dx[(|u_s(x,s)| + a^{\frac{1}{2}}(x,s)|u_x(x,s)|)^{2n} + \right.$$

$$\left. +(|u_s(x,s)| - a^{\frac{1}{2}}(x,s)|u_x(x,s)|)^{2n}] \right\}^{\frac{1}{2n}}, \quad (28)$$

where the constant $C_3 > 0$ does not depend on $t \in [0, T_2)$ and on $n > 0$ integer sufficiently large. Now, under the assumptions of our lemma, in view of lemma 11, for a weak solution estimate (28) can be obtained by taking the limit over sequences of smooth (u_0, u_1), coefficients $a(x,t)$ and functions $f(u)$

and the corresponding W_2^3-solutions of problem (1)–(2) converging, respectively, to nonsmooth (u_0, u_1), $a(x,t)$, $f(u)$ and a weak solution $u(x,t)$ in the sense indicated in lemma 11.

From (28), we have for a weak solution $u(x,t)$ of problem (1)–(2) by applying the Gronwell lemma:

$$\||u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|\|_{L_{2n}} \leqslant C_4, \tag{29}$$

where the constant $C_4 > 0$ does not depend on $t \in [0, T_2)$ and on $n > 0$ integer sufficiently large. For $t \in (-T_1, 0]$, estimate (29) still holds by analogous arguments. From (29), by taking the limit $n \to \infty$,

$$\||u_t(x,t)| + a^{\frac{1}{2}}(x,t)|u_x(x,t)|\|_{L_b((-T_1,T_2);L_\infty)} \leqslant C_4,$$

and lemma 13 is proved.□

Now, the result in theorem 2 follows from theorem 1 and lemmas 9 and 13.□

REFERENCES

1. *Courant R., Lax P. D.* On Nonlinear Partial Differential Equations with Two Independent Variables // Comm. Pure Appl. Math. 1949. V. 2. P. 255–274.
2. *Hughes Ht. J. R., Kato T., Marsden J. E.* Well-Posed Quasi-Linear Second-Order Hyperbolic Systems with Applications to Nonlinear Elastodynamics and General Relativity // Arch. Ration. Mech. Anal. 1976–1977. V. 63–64. P. 273–293.
3. *John F.* Nonlinear Wave Equations. Formation of Singularities. AMS, RI, Providence, 1990.
4. *Kato T.* Quasi-Linear Equations of Evolution, with Applications to Partial Differential Equations // Proc. Symp. of Dandee, 1974; Lecture Notes in Math. V. 448. Springer-Verlag, 1975. P. 25–70.
5. *Lax P.D.* Hyperbolic Partial Differential Equations. AMS, RI, Providence, 2006.
6. *Shatah J., Struwe M.* Geometric Wave Equations. Courant Lecture Notes. V. 2. AMS, RI, Providence, 2000.
7. *Tricomi F. G.* Lezione Sulle Equazioni a Derivate Parziale / Editrice Gheroni Torino, Via Cario Alberto. 1954. V. 13.; Russian translation: *Trikomi F.* Lectures on Partial Differential Equations. M.; Inostr. Literatura, 1957.

Received on April 10, 2012.

Корректор *Т. Е. Попеко*

Подписано в печать 17.05.2012.
Формат 60 × 90/16. Бумага офсетная. Печать офсетная.
Усл. печ. л. 1,68. Уч.-изд. л. 2,26. Тираж 275 экз. Заказ № 57652.

Издательский отдел Объединенного института ядерных исследований
141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@jinr.ru
www.jinr.ru/publish/