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NONLOCAL QUANTUM ELECTRODYNAMICS

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Нелокальная квантовая электродинамика

Показано, что проблема расходимости квантовой электродинамики первоначально связана с сингулярностью классического электростатического поля. Вид изменения кулоновского потенциала при малых расстояниях приводит к изменению пропагатора фотонного поля, что позволяет нам построить конечную и градиентно-инвариантную квантовую электродинамику. Мы устанавливаем ограничение на величину элементарной длины. Известно, что любая модификация пропагатора спинорного поля приводит к нарушению основных принципов локальной теории. Однако нам удалось построить конечную и градиентно-инвариантную теорию для электромагнитных взаимодействий с лагранжианом квадратного корня от оператора Клейна–Гордона.

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Nonlocal Quantum Electrodynamics

It is shown that an origin of the divergence problem in quantum electrodynamics is associated with a singularity of classical electrostatic field. A modification of its Coulomb potential at small distances leads to the change of the photon propagator which allows us to construct finite and gauge-invariant quantum electrodynamics. We establish restriction on the value of the so-called fundamental length $l \lesssim 10^{-16}$ cm from the experimental data on the measuring anomalous magnetic moment of leptons. It is well known that any modification of the spinor propagator (in particular, electron one) gives rise to many problems connected with verification of basic principles of the theory like gauge invariance, unitarity, causality condition and so on. However, it turns out that square-root modification of the spinor propagator is free from these difficult problems. Here we also construct a finite square-root quantum electrodynamics.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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This paper is dedicated to Professor G. V. Efimov's 80th birthday.

1. INTRODUCTION

A beautiful quantum electrodynamics developed by many physicists of the 20th century (for example, see [1–5]) has played a vital role in the construction of the finite and gauge-invariant so-called standard model [6, 7] of particle physics. What was an initial origin of this theory? It is natural that it was classical electrostatic field theory. Generally speaking, as usual, classical and quantum theories are the models of *point-like particles*. For example, the Newtonian and Coulomb potentials

$$U_N(r) = \frac{G}{4\pi r}, \quad U_C(r) = \frac{e}{4\pi r} \tag{1}$$

are the potentials of the point-like sources of mass and charge, respectively:

$$\rho_N(\mathbf{r}) = m\delta(\mathbf{r}), \quad \rho_C(\mathbf{r}) = e\delta(\mathbf{r}),$$

where $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$ is the Dirac δ function with properties:

$$\int\limits_{-\infty}^{\infty} dx\delta(x) = 1, \quad \int\limits_{-\infty}^{\infty} dx\delta(x)f(x) = f(0),$$

and so on.

It is well known that the inverse Fourier transform of the Coulomb potential for point-like charge is

$$D(\mathbf{p}) = \frac{1}{e}\int d^3r \mathrm{e}^{i\mathbf{pr}} U_C(r) = \frac{1}{\mathbf{p}^2} \tag{2}$$

and its relativistic generalization in four-momentum space

$$D(p) = \frac{1}{-p_0^2 + \mathbf{p}^2 - i\varepsilon} \tag{3}$$

gives the local photon propagator which leads to the divergent theory. Of fundamental importance is the fact that the Coulomb potential (1) satisfies the Laplace equation

$$\Delta U_C(r) = 0, \tag{4}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In principle, any modification of the Coulomb potential at small distances leads to a violation of the Laplace equation (4). Here we find out more simple and natural changing of the Coulomb potential

$$U_C(r) \Rightarrow U_C^l(r) = \frac{e}{4\pi} \frac{1}{\sqrt{r^2 + l^2}}, \tag{5}$$

which does not satisfy the Laplace equation (4) and gives modification of the photon propagator (3):

$$D(p) \Rightarrow D^l(p) = \frac{1}{-p^2 - i\varepsilon} V_l(-p^2 l^2), \tag{6}$$

where

$$V_l(-p^2 l^2) = \frac{1}{2i} \int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{v(\eta)}{\sin \pi\eta} \left[l^2 \left(-p^2 - i\varepsilon \right) \right]^{1+\eta} \quad (1 < \beta < 2), \tag{7}$$

$$v(\eta) = \frac{\pi}{4^{1+\eta}} \frac{1}{\sin \pi\eta \cdot \Gamma(1+\eta)\Gamma(2+\eta)}. \tag{8}$$

Some time ago Markov [8] considered a possibility of changing the metric form

$$S_0 = x^2 + y^2 + z^2 \to x^2 + y^2 + z^2 \pm l^2$$

in his indefinite metric modification of the field theory.

The Poisson equation for the potential (5) takes the form

$$\Delta U = -\frac{3l^2}{\left(r^2 + l^2\right)^{5/2}}.$$

On the other hand, the basic equation for electric stress $\mathbf{E} = -\mathrm{grad}\phi$ with extended charges is

$$\mathrm{div}\mathbf{E} = 4\pi\rho = -\mathrm{div}\ \mathrm{grad}\phi = -\Delta\phi,$$

$$\Delta\phi = -4\pi\rho.$$

It means that in our case electric charge is not located at the single point and is distributed continuously over the whole space with the density

$$\rho = \frac{1}{4\pi} \frac{3l^2}{(r^2 + l^2)^{5/2}}$$

with the normalization

$$\int d^3 r \rho(r) = 1$$

as it should be.

Therefore, in our scheme, an idealized concept of the point-like charge is absent. Moreover, already in the early developments of quantum mechanics there occur square-root operators. In particular, it was the relativistic relation between energy and momentum in a coordinate space representation that hindered its use [9]. A review of the early and later works is contained in [10]. In bound-state problems of two- and three-quark systems the Salpeter equation is often used [11–13]. Problems associated with binding in very strong fields [14, 15], string theory [16, 17] and astrophysical black holes [18–20] are applicable areas. Green's function for differential equations of infinite order like

$$\sqrt{m^2 - \Box}\Omega(x) = -\delta^{(4)}(x) \tag{9}$$

is treated in [21]. Green's function (9) in momenum p- and x-spaces takes the form

$$\Omega(p) = -\frac{1}{\sqrt{m^2 - p^2 - i\varepsilon}} = \int\limits_{-m}^{m} d\lambda \rho_m(\lambda) \widetilde{S}(\lambda, \widehat{p}) \tag{10}$$

and

$$\Omega(x - y) = \int\limits_{-m}^{m} d\lambda \rho_m(\lambda) S(x - y, \lambda), \tag{11}$$

where the distribution

$$\rho_m(\lambda) = \frac{1}{\pi} \left(m^2 - \lambda^2\right)^{-1/2}$$

has properties like

$$\int\limits_{-m}^{m} d\lambda \rho_m(\lambda) = 1, \quad \int\limits_{-m}^{m} d\lambda \cdot \lambda \cdot \rho_m(\lambda) = 0, \quad \int\limits_{-m}^{m} d\lambda \lambda^2 \rho_m(\lambda) = \frac{1}{2} m^2, \tag{12}$$

and

$$\widetilde{S}(\lambda, \widehat{p}) = \frac{1}{i} \frac{\lambda + \widehat{p}}{\lambda^2 - p^2 - i\varepsilon}, \tag{13}$$

$$S(x-y,\lambda)=\frac{1}{(2\pi)^4}\frac{1}{i}\int d^4p\mathrm{e}^{-ip(x-y)}\frac{\lambda+\widehat{p}}{\lambda^2-p^2-i\varepsilon} \tag{14}$$

are the Dirac spinor propagators in corresponding spaces with random mass λ. Here the relations

$$m^2-p^2=(m-\hat{p})(m+\widehat{p}), \quad \widehat{p}=\gamma^\nu p_\nu$$

and the Feynman parametric formula

$$\frac{1}{a^{n_1}b^{n_2}}=\frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)}\int\limits_0^1 dx x^{n_1-1}(1-x)^{n_2-1}\frac{1}{[ax+b(1-x)]^{n_1+n_2}} \tag{15}$$

are used. In this paper by using formulas (6), (10), (11), (13) and (14) we will construct finite nonlocal and square-root quantum electrodynamics free from ultraviolet divergences.

2. MODIFICATION OF THE COULOMB POTENTIAL AND DERIVATION OF THE NONLOCAL PHOTON PROPAGATOR

We propose the following finite Coulomb potential at small distances:

$$U_C^l(r)=\frac{e}{4\pi}\frac{1}{\sqrt{x^2+y^2+z^2+l^2}}, \tag{16}$$

where l is some parameter dimension of length. Its value may be interpreted as a size of an extended electric charge or as an universal constant like fundamental length in physics. As mentioned above, this modified potential satisfies the Poisson equation. Let us calculate the Fourier transform of the finite potential (16):

$$D_l(\mathbf{p})=\frac{1}{e}\int d^3r\mathrm{e}^{i\mathbf{pr}}\left(\frac{e}{4\pi\sqrt{r^2+l^2}}\right)=\frac{1}{p}\int\limits_0^\infty dr\frac{r}{\sqrt{r^2+l^2}}\sin pr,$$

where $p=|\mathbf{p}|$. By using the Mellin representation this expression takes the form

$$D_l(\mathbf{p})=\frac{l^2}{2\sqrt{\pi}}\cdot\frac{l}{2i}\int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\eta\frac{\left(p^2l^2\right)^\eta}{\sin\pi\eta\Gamma(2+2\eta)}\times\Gamma\left(\frac{3}{2}+\eta\right)\Gamma(-1-\eta), \tag{17}$$

where $1<\beta<2$.

Further, taking into account gamma-function relations:

$$\Gamma(2+2\eta)=\frac{2^{2(1+\eta)-1}}{\sqrt{\pi}}\Gamma(1+\eta)\Gamma\left(\frac{3}{2}+\eta\right),$$

$$\Gamma(\eta)\Gamma(1-\eta) = \frac{\pi}{\sin \pi\eta}$$

and after some elementary calculations, one gets

$$D_l(\mathbf{p}) = \frac{V_l\left(\mathbf{p}^2 l^2\right)}{\mathbf{p}^2}, \tag{18}$$

where

$$V_l(\mathbf{p}^2 l^2) = \frac{1}{2i} \int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{v(\eta)}{\sin \pi\eta} \left[l^2 \mathbf{p}^2\right]^{1+\eta}, \tag{19}$$

$$v(\eta) = \frac{\pi}{4^{1+\eta}} \frac{1}{\sin \pi\eta} \frac{1}{\Gamma(1+\eta)\Gamma(2+\eta)}. \tag{20}$$

From these formulas one can calculate residues at the points $\eta = -1, 0, 1, \ldots$ The result reads

$$D_l(\mathbf{p}) = \frac{1}{\mathbf{p}^2} + \frac{l}{|\mathbf{p}|} K_1(l|p|), \tag{21}$$

where $K_1(x)$ is the modified Bessel function of second kind or the MacDonald function

$$K_1(x) = \frac{\pi}{2} \frac{x}{2} \frac{1}{2i} \int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\zeta \frac{(x/2)^{2\zeta}}{\sin^2 \pi\zeta \Gamma(1+\zeta)\Gamma(2+\zeta)}$$

$(0 < \beta < 1)$, $x = |p|l$. Finally, the modification of the Coulomb potential (16) gives rise to the following nonlocal photon propagator [22]:

$$D^l_{\mu\nu}(x) = \frac{i}{(2\pi)^4} g_{\mu\nu} \int d^4 p \mathrm{e}^{ipx} \frac{V_l\left(-p^2 l^2\right)}{-p^2 - i\varepsilon}, \tag{22}$$

where the form factor $V_l\left(-p^2 l^2\right)$ of the theory is defined by formulas (19) and (20).

Here our theory with the propagator (22) is very similar to the nonlocal theory due to [22] and [23]. Notice that the simple modification of the Coulomb potential (16) leading to the nonlocal photon propagator (22) is cornerstone of the finiteness of classical and quantum electromagnetic fields. For example, now electrostatic self-energy of the extended charge is finite at small distances:

$$W = \frac{e}{2} \int d^3 r \rho(r) U_l(r) = \frac{1}{2} \int d^3 r E^2, \quad E = -\frac{e}{4\pi} \mathrm{grad} \frac{1}{\sqrt{r^2 + l^2}}.$$

Here simple calculation reads

$$W = \frac{e^2}{8\pi}\int\limits_0^\infty dr \frac{r^4}{\left[r^2+l^2\right]^3} = \frac{e^2}{2\Gamma(3)}\frac{1}{l}\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{8\pi} = \frac{3}{32}\frac{\pi}{l}\alpha.$$

Moreover, the nonlocal photon propagator (22) is finite at the origin

$$D^l_{\mu\nu}(0) = g_{\mu\nu}\frac{2\pi^2}{2^4\pi^4}\int\limits_0^\infty dp\, p^3 D_l(p^2) =$$

$$= -g_{\mu\nu}\frac{1}{8\pi}\lim_{\epsilon\to 0}\int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\zeta \frac{\left(l^2/4\right)^{1+\zeta}}{\sin^2\pi\zeta}\frac{\varepsilon^{2\zeta+4}}{\Gamma(1+\zeta)\Gamma(2+\zeta)\Gamma(2\zeta+4)},$$

where $2<\beta<3$.

Calculation of residue at the point $\zeta=-2$ and taking the limit $\varepsilon\to 0$ leads to

$$D^l_{\mu\nu}(0) = \frac{1}{4\pi^2 l^2}g_{\mu\nu} = \text{const}.$$

It turns out that, in principle, due to finiteness of $D^l_{\mu\nu}(0)$ one can calculate vacuum fluctuation diagrams, shown in Fig. 1.

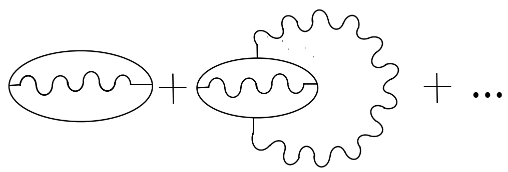

Fig. 1. Primitive Feynman diagrams for vacuum fluctuation

Finally, we indicate one important consequence of the photon propagator (22) with the form factor (19). If we want to calculate high-order divergence integrals over the internal momentum variable p, like

$$\frac{1}{2i}\int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{v(\eta)}{\sin\pi\eta}\int d^4p \frac{\left[p^{2\nu}\right]^\eta}{\left[p^2+A\right]^\lambda}$$

for any order of ν, then we can move integration contour in Fig. 2 to the left through points $\eta=-2,-3,\ldots$, in desired order, since in such type of integrals

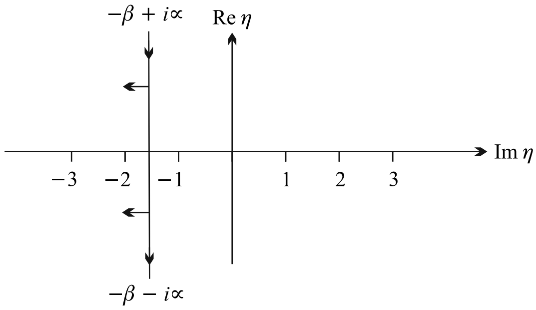

Fig. 2. Integration contour in formula (19)

there are no poles at these points. After integration result we can again move integration contour to the right to calculate residues at the points $\eta = -3, -2, -1, \dots$ and so on. This procedure of analytic continuation over complex variable η plays a vital role in regularization scheme.

3. NONLOCAL QUANTUM ELECTRODYNAMICS

3.1. Introduction. Lagrangian functions of the nonlocal quantum electrodynamics arising from the modification of the Coulomb potential at small distances have structures similar to those in the local theory [24]:

$$\begin{aligned} L(x) = e : \overline{\psi}(x)\widehat{A}(l,x)\psi(x) : + \\ + e(Z_1 - 1) : \overline{\psi}(x)\widehat{A}(l,x)\psi(x) : - \\ - \delta m : \overline{\psi}(x)\psi(x) : + (Z_2 - 1) : \overline{\psi}(x)(i\widehat{\partial} - m)\psi(x) : - \\ - (Z_3 - 1)\frac{1}{4} : F_{\mu\nu}(x)F^{\mu\nu}(x) :, \end{aligned} \quad (23)$$

where

$$\widehat{A}(l,x) = A_\mu(l,x)\gamma^\mu, \quad \widehat{\partial} = \gamma^\mu \frac{\partial}{\partial x_\mu}.$$

Only in our case of the nonlocal theory, renormalization constants $Z_1, Z_2, Z_3, \delta m$ are finite and moreover $Z_1 = Z_2$ due to the Ward–Takahashi identity. Here "chronological" pairing (or *T* product) of the fermionic field operators of electrons has the usual local form:

$$S(x-y) = \langle 0|T[\psi(x)\overline{\psi}(y)]|0\rangle = \frac{1}{(2\pi)^4}\frac{1}{i}\int d^4p \frac{\mathrm{e}^{-ip(x-y)}}{m - \widehat{p} - i\varepsilon}, \quad (24)$$

while "causal" function of the nonlocal electromagnetic field $A_\mu(l,x)$ in (23) takes the form due to formula (22)

$$D^l_{\mu\nu}(x-y) = g_{\mu\nu}D^l(x-y) = -\frac{g_{\mu\nu}}{(2\pi)^4 i}\int d^4p\mathrm{e}^{-ip(x-y)}\frac{V_l(-p^2l^2)}{-p^2-i\varepsilon}, \qquad (25)$$

where $V_l(-p^2l^2)$ is given by formulas (19) and (20).

3.2. The Electron Self-Energy in NQED. The complete electron propagator in NQED is given by the sum

$$\left[-i(2\pi)^{-4}S_l^{'}(p)\right] = \left[-i(2\pi)^{-4}S(p)\right] + \left[i(2\pi)^{-4}S(p)\right]\left[i(2\pi)^4\Sigma_l(p)\right]\times$$
$$\times\left[-i(2\pi)^{-4}S(p)\right] + \ldots,$$

where

$$S(p) = \frac{m+\widehat{p}}{m^2-p^2-i\varepsilon}.$$

The sum is trivial and gives

$$S_l^{'}(p) = \left[m-\widehat{p}-\Sigma_l-i\varepsilon\right]^{-1}.$$

In lowest order there is a one-loop contribution to Σ_l, given in Fig. 3:

$$-i:\overline{\psi}(x)\Sigma_l(x-y)\psi(y):,$$

where

$$\Sigma_l(x-y) = -ie^2\gamma_\mu S(x-y)\gamma_\mu D^l(x-y). \qquad (26)$$

Passing to the momentum representation and going to the Euclidean metric by using $k_0 \to \exp(i\pi/2)k_4$, one gets

$$\widetilde{\Sigma}_l(p) = \frac{e^2}{(2\pi)^4}\int d^4k_E\frac{V_l(k_E^2l^2)}{k_E^2}\gamma_\mu^{(E)}\frac{m-\widehat{p}_E+\widehat{k}_E}{m^2+(p_E-k_E)^2}\gamma\mu^{(E)}.$$

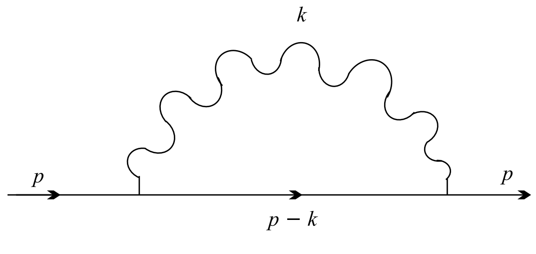

Fig. 3. Diagram of self-energy of an electron in NQED

Here $p_E = (-ip_0, \mathbf{p})$, $\gamma^{(E)} = (-i\gamma_0, \boldsymbol{\gamma})$ and $k_E = (k_4, \mathbf{k})$. Taking into account the Mellin representation (19) for the form factor $V_l(k_E^2 l^2)$ and after some calculations, we have

$$\widetilde{\Sigma}_l(p) = -\frac{e^2}{8\pi}\frac{1}{2i}\int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{1}{\sin^2 \pi\eta}\frac{v(\eta)(m^2e^2)^{1+\eta}}{\Gamma(2+\eta)} F(\eta, p) \tag{27}$$

where

$$F(\eta, p) = \frac{1}{\Gamma(-\eta)}\int\limits_0^1 du \left(\frac{1-u}{u}\right)^{1+\eta}\left(1-\frac{p^2}{m^2}u\right)^{1+\eta}(2m - \widehat{p}u) \tag{28}$$

is a regular function in the half-plane $\mathrm{Re}\,\eta > -2$.

Assuming the value m^2l^2 to be small, one can obtain (after calculation of residues at the points $\eta = -1, 0$)

$$\begin{aligned}\widetilde{\Sigma}_l(p) = {}& \frac{e^2}{8\pi^2}\int\limits_0^1 du(2m - \widehat{p}u)\ln\left(1-\frac{p^2}{m^2}u\right) + \\ &+ \frac{e^2}{8\pi^2}\left\{\ln\left(\frac{m^2l^2}{4}\right)\left(2m - \frac{1}{2}\widehat{p}\right) + \widehat{p}\left(\frac{1}{2}+\psi(1)\right) - 4m\psi(1)\right\} + \\ &+ \frac{me^2}{32\pi^2}(m^2l^2)\left[\ln^2\left(\frac{m^2l^2}{4}\right) - \ln\left(\frac{m^2l^2}{4}\right)(3+4\psi(1)) + \right. \\ &\left. + 4\psi(1)(1+\psi(1)) + 2 - \frac{1}{3}\pi^2\right], \end{aligned} \tag{29}$$

where $\psi(1) = -C,\ C = 0.57721566490\ldots$ is the Euler number.

3.3. Vertex Function and Anomalous Magnetic Moment of Leptons in NQED. Let us consider Feynman diagram shown in Fig. 4. The following matrix element corresponds to this diagram:

$$\mathrm{ie} : \overline{\psi}(x)\Gamma_\mu^l(x, z/y)\psi(z)A_\mu(y) : . \tag{30}$$

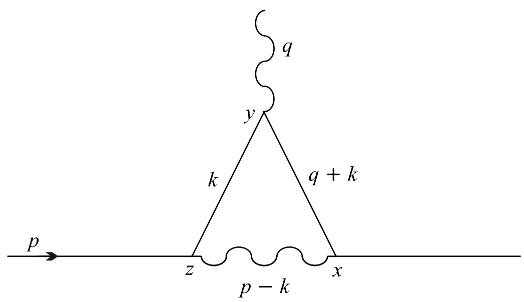

Fig. 4. Vertex function in NQED

Analogously, in the momentum space and in the Euclidean metric, the vertex function takes the form

$$\widetilde{\Gamma}^{l}_{\mu}(p_1,p) = -\frac{e^2}{(2\pi)^4}\int d^4k_E \frac{V_l\left((p_E-k_E)^2l^2\right)}{(p_E-k_E)^2}\gamma_\nu \times \\ \times \frac{m-\widehat{k}_E-\widehat{q}_E}{m^2+(k_E+p_E)^2}\gamma_\mu \frac{m-\widehat{k}_E}{m^2+k_E^2}\gamma_\nu. \quad (31)$$

Again passing to the Minkowski metric and using the generalized Feynman parameterization formula (15), one gets

$$\widetilde{\Gamma}^{l}_{\mu}(p_1;p) = \frac{e^2}{8\pi}\frac{1}{2i}\int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{v(n)}{\sin^2\pi\eta}\frac{(m^2l^2)^{1+\eta}}{\Gamma(2+\eta)}F\mu(\eta;p_1,p), \quad (32)$$

where

$$F_\mu(\eta;p_1,p) = \gamma_\mu F_1(\eta;p_1,p) + F_2(\eta;p_1,p).$$

Here

$$F_1(\eta;p_1,p) = \frac{1}{\Gamma(-\eta)}\int\limits_0^1\int\limits_0^1\int\limits_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma)\alpha^{-1-\eta}Q^{1+\eta},$$

$$F_2(\eta;p_1,p) = \frac{1}{\Gamma(-1-\eta)}\int\limits_0^1\int\limits_0^1\int\limits_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma)\alpha^{-1-\eta}Q^{\eta}\times \\ \times \frac{1}{m^2}\left[m^2\gamma_\mu - 2mq_\mu + 4m(\beta q_\mu - \alpha p_\mu) + (\alpha\widehat{p}-\beta\widehat{q})\gamma_\mu\widehat{q} + \right. \\ \left. + (\alpha\widehat{p}-\beta\widehat{q})\gamma\mu(\alpha\widehat{p}-\beta\widehat{q})\right], \quad (33)$$

$$Q = \beta+\gamma-\alpha\gamma\frac{p^2}{m^2} - \beta\gamma\frac{q^2}{m^2} - \alpha\beta\frac{(p+q)^2}{m^2}. \quad (34)$$

Let us calculate the vertex function (32) for two cases: first, when $q=0$ and p has an arbitrary value; second, when q is an arbitrary quantity and p, p_1 are situated on the m-mass shell. In the first case, assuming $q=0$ in formula (33) and after some standard calculations, one gets

$$F_\mu(\eta;p_1,p) = \frac{1}{\Gamma(-\eta)}\int\limits_0^1 du\left(\frac{1-u}{u}\right)^{1+\eta}\left(1-u\frac{p^2}{m^2}\right)^{1+\eta}\times \\ \times \left[u\gamma_\mu + \frac{2(1+\eta)up_\mu(2m-u\widehat{p})}{m^2-up^2}\right]. \quad (35)$$

Comparing this formula with the expression (28) for the self-energy of the electron, it is easily seen that

$$F_\mu(\eta; p_1, p) = -\frac{\partial}{\partial p_\mu} F(\eta; p). \tag{36}$$

From this identity, we can obtain a very important conclusion. In nonlocal QED constructed using the modification of the Coulomb potential, the Ward–Takahashi identity is valid:

$$\widetilde{\Gamma}^l_\mu(p, p) = -\frac{\partial}{\partial p_\mu} \widetilde{\Sigma}_l(p). \tag{37}$$

In the second case, one can put

$$\overline{u}(\mathbf{p}_1)\widetilde{\Gamma}^l_\mu(p_1, p)u(\mathbf{p}) = \overline{u}(\mathbf{p}_1)\Lambda_\mu(q)u(\mathbf{p}), \tag{38}$$

where $u(\mathbf{p}_1)$ and $u(\mathbf{p})$ are solutions of the Dirac equation

$$(\widehat{p} - m)u(\mathbf{p}) = 0, \quad \overline{u}(\mathbf{p}_1)(\widehat{p}_1 + m) = 0.$$

Substituting the vertex function (32) into (38) and after some transformations, we have

$$\overline{u}(\mathbf{p}_1)F_\mu(\eta; p_1, p)u(\mathbf{p}) = u(\mathbf{p}_1)\Lambda_\mu(\eta; q)u(\mathbf{p}). \tag{39}$$

Here

$$\Lambda_\mu(\eta; q) = \gamma_\mu f_1(\eta; q^2) + \frac{i}{2m}\sigma_{\mu\nu}q_\nu f_2(\eta; q^2),$$

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu),$$

$$f_j(\eta; q^2) = \frac{1}{\Gamma(-\eta)} \int\limits_0^1 \int\limits_0^1 \int\limits_0^1 d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma)\alpha^{-1-\eta}L^\eta \times$$
$$\times\, g_j(\alpha, \beta, \gamma, q^2), \quad j = 1{,}2,$$

$$L = \varepsilon\alpha + (1 - \alpha)^2 - \beta\gamma\frac{q^2}{m^2}, \tag{40}$$

$$g_1(\alpha, \beta, \gamma, q^2) = (1 - \alpha)^2(-\eta) + 2\alpha(1 + \eta) -$$
$$- \frac{q^2}{m^2}\left[\beta\gamma + (1 + \eta)(\alpha + \beta)(\alpha + \gamma)\right] \times$$
$$\times\, g_2(\alpha, \beta, \gamma, q^2) = 2\alpha(1 - \alpha)(1 + \eta).$$

To avoid infrared divergences in the vertex function, we have introduced here the parameter $\varepsilon = m_{\rm ph}^2/m^2$, taking into account the "mass" of the photon. Finally, one gets

$$\Lambda_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu} q_\nu F_2(q^2), \tag{41}$$

where

$$F_j(q^2) = \frac{e^2}{8\pi}\frac{1}{2i}\int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{v(\eta)}{\sin^2\pi\eta}\frac{(m^2l^2)^{1+\eta}}{\Gamma(2+\eta)} f_j(\eta; q^2). \tag{42}$$

It is easy to verify that the vertex function $\Lambda_\mu(q)$ satisfies the gauge-invariant condition:

$$q_\mu \overline{u}(\mathbf{p}_1)\Lambda_\mu(q)u(\mathbf{p}) = 0. \tag{43}$$

Let us write the first terms of the decomposition for the functions $F_1(q^2)$ and $F_2(q^2)$ over small parameters m^2l^2 and q^2/m^2:

$$F_2(q^2) = -\frac{\alpha}{2\pi}\left[1 + \frac{m^2l^2}{6}\left(\ln\frac{m^2l^2}{4} - 2\psi(1) + \frac{1}{6}\right)\right] + O\left(\frac{q^2}{m^2}\right), \tag{44}$$

$$\begin{aligned} F_1(q^2) = -\frac{\alpha}{4\pi}\Bigg\{3\left[\ln\frac{m^2l^2}{4} - 2\psi(1) - \frac{3}{2}\right] + \\ + m^2l^2\left[\ln\frac{m^2l^2}{4} - 2\psi(1) - \frac{1}{3}\right]\Bigg\} + O\left(\frac{q^2}{m^2}\right). \end{aligned} \tag{45}$$

From this first formula we can see that corrections to the anomalous magnetic moment (AMM) for leptons are given by

$$\Delta\mu = \frac{\alpha}{2\pi}\left[1 + \frac{m^2l^2}{6}\left(\ln\frac{m^2l^2}{4} + \frac{1}{6} - 2\psi(1)\right)\right]. \tag{46}$$

We see that the first term in (46) is exactly famous Schwinger correction obtained in local QED. From the experimental values of the AMM of the electron and muon ([25–27] and [28])

$$\Delta\mu^{(e)}_{\rm exp} = \frac{\mu_e}{\mu_B} - 1 = \frac{1}{2}(q-2) = (1159652180.73(0.28))\cdot 10^{-12} \tag{47}$$

and

$$\Delta\mu^{(e)}_{\rm exp} = \frac{\mu_\mu}{(e\hbar/2m_\mu)} - 1 = \frac{1}{2}(g_\mu - 2) = (116592089(63))\cdot 10^{-11}, \tag{48}$$

one gets the following restriction on the value of the universal parameter (or the fundamental length) l:

$$l \lesssim 7.0 \cdot 10^{-17} \text{ cm for } \Delta\mu_{\rm exp}^{(e)}, \tag{49}$$

$$l \lesssim 2.6 \cdot 10^{-17} \text{ cm for } \Delta\mu_{\rm exp}^{(\mu)}. \tag{50}$$

Recent theoretical calculations of the AMM of the electron and muon have been carried by [29].

3.4. Vacuum Polarization. Since in our scheme the propagator $S(x-y)$ of the charged lepton spinor is not changed, the diagrams of the vacuum polarization, i. e., closed spinor propagators (see Fig. 5) of the leptons in our nonlocal QED are studied in the same way as in the local theory. For completeness we calculate it in e^2-order by using d-dimensional regularization procedure [30]. The result reads in the momentum space:

$$\widetilde{\prod}^{\rho\sigma}(q) = (q^2 g^{\rho\sigma} - q^\rho q^\sigma)\widetilde{\prod}(q^2), \tag{51}$$

where

$$\widetilde{\prod}(q^2) = \frac{e^2}{2\pi^2}\int\limits_0^1 dx(1-x)\ln\left(1+\frac{q^2x(1-x)}{m^2}\right). \tag{52}$$

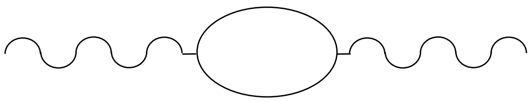

Fig. 5. The vacuum polarization in NQED

The physical importance of the vacuum polarization in NQED can be explored by considering its effects on the scattering of two charged particles of spin $1/2$.

4. THE SQUARE-ROOT NONLOCAL QUANTUM ELECTRODYNAMICS

The purpose of this section is to study nonlocal interactions of the charged square-root spinors with nonlocal photons within our scheme. Thus, the Lagrangian corresponding to the equation

$$\sqrt{m^2 - \Box}\phi(x) = 0 \tag{53}$$

is given by

$$L_\phi^0 = \phi^*(x)\sqrt{m^2 - \Box}\phi(x). \tag{54}$$

Instead of (54) we consider the Lagrangian density

$$L_\psi^0 = -N\left\{\overline{\psi}(x,\lambda_1)(-\widehat{\partial})\psi(x,\lambda_2) + L_{1\psi}^0\right\} \tag{55}$$

for the $\psi(x,\lambda)$ field. Here the following notation is used:

$$\begin{gathered} L_{1\psi}^0 = \Psi(x,\lambda_1)U(\lambda_1,\lambda_2)\Psi(x,\lambda_2), \\ N = \int\limits_{-m}^{m}\int\limits_{-m}^{m} d\lambda_1 d\lambda_2 \rho(\lambda_1)\rho(\lambda_2), \quad \widehat{\partial} = i\gamma^\mu \frac{\partial}{\partial x_\mu}, \\ \overline{\Psi}_(x,\lambda_1) = (0, \overline{\psi}(x,\lambda_1)), \\ \Psi(x,\lambda_2) = \begin{pmatrix} \psi(x,\lambda_2) \\ 0 \end{pmatrix}, \\ U(\lambda_1,\lambda_2) = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix}. \end{gathered} \tag{56}$$

The equations of motion

$$\begin{gathered} \int\limits_{-m}^{m} d\lambda\rho(\lambda)(\widehat{\partial} - \lambda)\psi(x,\lambda) = 0. \\ \int\limits_{-m}^{m} d\lambda\rho(\lambda)\left(i\frac{\partial\overline{\psi}(x,\lambda)}{\partial x_\mu}\gamma^\mu + \lambda\overline{\psi}(x,\lambda)\right) = 0 \end{gathered} \tag{57}$$

for $\psi(x,\lambda)$ fields can be obtained from the action

$$A = \int d^4x L_\psi^0(x)$$

by using independent variations over the fields $\psi(y,\lambda)$ and $\overline{\psi}(y,\lambda)$ and by taking the differentiation $\delta L_{1\psi}^0/\delta\overline{\psi}(y,\lambda)$ and $\delta(L_{1\psi}^0)^T/\delta\psi(y,\lambda)$. Here we have used the following obvious relations:

$$\frac{\delta\overline{\psi}(x,\lambda_i)}{\delta\overline{\psi}(y,\lambda)} = \frac{\delta\psi(x,\lambda_i)}{\delta\psi(y,\lambda)} = \delta^{(4)}(x-y)\delta(\lambda_i - \lambda)$$

and definition

$$(L_{1\psi}^0)^T = \overline{\Psi}(x,\lambda_1)U^T(\lambda_1,\lambda_2)\Psi(x,\lambda_2).$$

It is easily seen that the propagator of the field $\phi(x)$ in (53) is given by Eq. (9) or

$$\Omega(x) = -\frac{1}{\sqrt{m^2 - \Box}}\delta^{(4)}(x) =$$
$$\text{v}\frac{1}{i}\int\limits_{-m}^{m} d\lambda\rho(\lambda)\frac{1}{\lambda + \widehat{\partial}}\delta^{(4)}(x) = \int\limits_{-m}^{m} d\lambda\rho(\lambda)S(x,\lambda). \quad (58)$$

In the momentum representation, expression (58) takes the form

$$\widetilde{\Omega}(p) = \int\limits_{-m}^{m} d\lambda\rho(\lambda)\widetilde{S}(\lambda,\widehat{p}), \quad (59)$$

where

$$\widetilde{S}(\lambda,\widehat{p}) = \frac{1}{i}\frac{\lambda + \widehat{p}}{\lambda^2 - p^2 - i\varepsilon} \quad (60)$$

is the spinor propagator with random "mass" λ in momentum space.

Our next goal is to study Feynman diagrams in nonlocal square-root quantum electrodynamics with Green's functions (22), (58)–(60).

In the "square-root" NQED the S-matrix can be constructed by the usual rule:

$$S = \text{Expec}\; T\exp\left[\int d^4x L_{in}(x)\right], \quad (61)$$

where

$$\begin{aligned} L_{in}(x) =& eN\left\{\overline{\psi}(x,\lambda_1)\widehat{A}_l(x)\psi(x,\lambda_2)\right\}, \\ \widehat{A}_l =& \gamma^\mu A^l_\mu(x) \end{aligned} \quad (62)$$

and N is given by (56). The symbol T is defined by

$$\langle 0|T\left[\psi(x,\lambda_1)\overline{\psi}(y,\lambda_2)\right]|0\rangle = \delta(\lambda_1 - \lambda_2)S(x-y,\lambda_1)/\rho(\lambda_1) \quad (63)$$

for the spinor fields. For example, at least for connected diagrams in the momentum space one assumes

$$\text{Expec}\left\{\widetilde{\Omega}(p)\right\} = \int\limits_{-m}^{m} d\lambda\rho(\lambda)\widetilde{S}(\widehat{p},\lambda),$$

$$\text{Expec}\left\{\gamma^{\nu_1}\widetilde{\Omega}(p_1)\gamma^{\nu_2}\widetilde{\Omega}(p_2)\gamma^{\nu_3}\right\} = \int\limits_{-m}^{m} d\lambda\rho(\lambda)\left\{\gamma^{\nu_1}\widetilde{S}(\widehat{p}_1,\lambda)\gamma^{\nu_2}\widetilde{S}(\widehat{p}_2,\lambda)\gamma^{\nu_3}\right\} \quad (64)$$

and so on.

The gauge invariance of the "square-root" NQED means that matrix elements of the S-matrix (61) defining the concrete electromagnetic processes have a definite structure, and algebraical relations exist between them. In particular, in the momentum representation, the so-called vacuum polarization diagram like (in Fig. 4) in the second order of the perturbation theory has the form

$$\widetilde{\prod}_{\mu\nu}^{l,s}(k) = (k_\mu k_\nu - g_{\mu\nu}k^2)\widetilde{\prod}^{l,s}(k^2) \tag{65}$$

and the relation

$$\frac{\partial\widetilde{\sum}_l^s(p)}{\partial p_\mu} = -\widetilde{\Gamma}_\mu^{l,s}(p,q)|_{q=0} \tag{66}$$

is valid between the vertex function $\widetilde{\Gamma}_\mu^{l,s}(p,q)$ and the self-energy of the "square-root" electron $\widetilde{\Sigma}_l^s(p)$. The relation (66) generalizes the Ward–Takahashi identity in QED. Here, in accordance with (64), we have

$$\widetilde{\Sigma}_l^s(p) = \frac{-ie^2}{(2\pi)^4}\int\limits_{-m}^{m} d\lambda\rho(\lambda)\int d^4kD_l(k^2)\gamma^\mu S(\widehat{p}-\widehat{k},\lambda)\gamma^\mu \tag{67}$$

and

$$\widetilde{\Gamma}_\mu^{l,s}(p,q) = \frac{ie^2}{(2\pi)^4}\int d^4kD_l((p-k)^2l^2)\times \text{Expec}\left\{\gamma^\nu\widetilde{\Omega}(q+k)\gamma^\mu\widetilde{\Omega}(k)\gamma^\nu\right\} =$$

$$= \frac{ie^2}{(2\pi)^4}\int\limits_{-m}^{m} d\lambda\rho(\lambda)\int d^4kD_l((p-k)^2l^2)\gamma^\nu S(\widehat{q}+\widehat{k},\lambda)\gamma^\mu\times\widetilde{S}(\widehat{k},\lambda)\gamma^\nu, \tag{68}$$

where

$$\widetilde{S}(\widehat{p},\lambda) = \frac{1}{(\lambda-\widehat{p})}$$

and

$$D_l(k^2,l^2) = \frac{V_l(k^2l^2)}{-k^2-i\varepsilon}.$$

Fort the proof of the relation (66), consider the identity

$$\frac{\partial\widetilde{S}(\widehat{p},\lambda)}{\partial p_\mu} = \widetilde{S}(\widehat{p},\lambda)\gamma^\mu\widetilde{S}(\widehat{p},\lambda). \tag{69}$$

Further, it is easy to verify the identity (66) by differentiating (67) over p_μ and making use of the equality (69) as well as by choosing other momentum variables in (68) and assuming $q=0,\ p^{'}=p+q=p$. The relations of the type

$$q_\mu\widetilde{\Gamma}_\mu^{l,s}(p,q)|_{p'^2=p^2=\lambda^2} = 0$$

follow from the definition

$$q_\mu \text{Expec}\left\{\widetilde{\Omega}(p_1)\gamma^\mu\widetilde{\Omega}(p_2)\right\} = q_\mu \int\limits_{-m}^{m}\int\limits_{-m}^{m} d\lambda_1 d\lambda_2 \rho(\lambda_1)\rho(\lambda_2)\times$$

$$\times \widetilde{S}(\widehat{p}_1,\lambda_1)\gamma^\mu \widetilde{S}(\widehat{p}_2,\lambda_2)\frac{\delta(\lambda_1-\lambda_2)}{\rho(\lambda_1)} = \widetilde{\Omega}(p_1)-\widetilde{\Omega}(p_2) =$$

$$= \int\limits_{-m}^{m} d\lambda\rho(\lambda)\left[\widetilde{S}(\widehat{p}_1,\lambda)-\widetilde{S}(\widehat{p}_2,\lambda)\right] \quad (70)$$

if $q = p_1 - p_2$.

Now let us demonstrate that the gauge invariance of the vacuum polarization diagram in the “square-root” NQED and its matrix element is given by

$$\widetilde{\prod}^{s}_{\mu\nu}(k) = e^2 \text{Expec}\left\{\int d^d p\,\text{Tr}\left\{\gamma^\mu\widetilde{\Omega}(p+k)\gamma^\nu\widetilde{\Omega}(p)\right\}\right. =$$

$$= e^2\int\limits_{-m}^{m} d\lambda\rho(\lambda)\int d^d p\,\text{Tr}\left\{\gamma^\mu\widetilde{S}(\widehat{p}+\widehat{k},\lambda)\gamma^\nu\widetilde{S}(\widehat{p},\lambda)\right\}. \quad (71)$$

Here we have used the d-dimensional gauge-invariant regularization procedure due to [31] and the definition (64). After some calculations we obtain the same structure as in (65):

$$\prod^{s}_{\mu\nu}(k) = \frac{8i\pi^{d/2}}{\Gamma(2)}\Gamma\left(2-\frac{1}{2}d\right)(k_\mu k_\nu - k^2 g_{\mu\nu})\times$$

$$\times\int\limits_{-m}^{m} d\lambda\rho(\lambda)\int\limits_{0}^{1} dx\, x(1-x)\left[\lambda^2 - k^2 x(1-x)\right]^{d/2-2}, \quad (72)$$

which is manifestly gauge-invariant. Calculation of the matrix elements for $\widetilde{\Sigma}^s_l(p)$ and $\widetilde{\Gamma}^{l,s}_\mu(p,q)$ can be carried out by the same method as in (27) and (32), where we have to change $m \to \lambda$.

In conclusion, we notice that similar modification of the Newtonian potential (1) and (5) gives rise to a finite quantum gravitational theory with the causal Green’s function for the graviton:

$$G^c_{\mu\nu,\rho\sigma}(x) = \frac{-1}{(2\pi)^4 i}\int d^4p\,\mathrm{e}^{-ipx}\widetilde{\prod}_{\mu\nu,\rho\sigma}(p)\frac{V_l(-p^2l^2)}{-p^2-i\varepsilon},$$

where the projecting tensor $\widetilde{\prod}_{\mu\nu,\rho\sigma}(p)$ is given by the expression

$$\widetilde{\prod}_{\mu\nu,\rho\sigma}(p) = d_{\mu\rho}(p)d_{\nu\sigma}(p) + d_{\mu\sigma}(p)d_{\nu\rho}(p) - \frac{2}{3}d_{\mu\nu}(p)d_{\rho\sigma}(p),$$

$$d_{\mu\nu}(p) = g_{\mu\nu} - p_\mu p_\nu / p^2,$$

and $V_l(-p^2 l^2)$ is defined by the same formula, (7). Here l should be changed by the Planck length:

$$l \to l_{\mathrm{Pl}} = \sqrt{\frac{\hbar G_N}{c^3}} = 1.62 \cdot 10^{-33} \text{ cm},$$

where G_N is the Newtonian constant.

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