E5-2015-99

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TRANSFORMATION OF LINEAR SYSTEM OF EVOLUTION EQUATIONS INTO SYSTEM OF GENERALIZED RICCATI EQUATIONS

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Преобразование линейной системы эволюционных уравнений в систему обобщенных уравнений Риккати

Построено отображение между линейной системой эволюционных уравнений, производимых конечномерным оператором, и системой обобщенных уравнений Риккати. Каноническая форма эволюционных уравнений формулируется как система линейных дифференциальных уравнений в частных производных, определяемая сопровождающей матрицей соответствующего конечномерного оператора. Решения этих уравнений даются обобщенными тригонометрическими функциями, которые определяются как коэффициенты разложения экспоненциальной функции в конечный ряд, представленный в виде многочлена с конечным числом корней. Получена нелинейная система дифференциальных уравнений типа Риккати для корней многочлена. Построено отображение от решений обобщенных уравнений Риккати в решения линейных эволюционных уравнений.

Работа выполнена в Лаборатории информационных технологий ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна, 2015

Yamaleev R. M. E5-2015-99

Transformation of Linear System of Evolution Equations into System of Generalized Riccati Equations

A mapping between the linear system of evolution equations, generated by a finite-dimensional operator, and the system of generalized Riccati equations is constructed. The canonical form of evolution equations is extended up to a multivariable system of linear differential equations governed by the companion matrix of the finite-dimensional operator. Solutions of these equations form a set of generalized trigonometric functions which are coefficients of the series of expansion of an exponential function. This series is a polynomial function possessing a definite number of roots. A nonlinear system of differential equations for the roots of the high-order Riccati-type equations is derived. Inverse mapping from the solutions of the obtained system of Riccati equations onto the solutions of the evolution equations is constructed.

The investigation has been performed at the Laboratory of Information Technologies, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna, 2015

1. INTRODUCTION

Let H be a finite-dimensional operator presented by $(n \times n)$ matrix. It is supposed, the operator H is a generator of some evolution process which is described by linear differential equation of the type

$$\frac{d}{dt}\Psi(t) = H\Psi(t), \quad \Psi(0) = \Psi_0, \tag{1.1}$$

whose direct closed-form solution involves computation of the matrix exponential

$$\Psi(t) = \exp{(tH)}\Psi_0.$$

The matrix representation of the finite-dimensional operator H obeys its characteristic polynomial equation

$$f(H) = 0. \tag{1.2}$$

As a matter of convenience, let us suppose that *the characteristic polynomial* coincides with the *minimal polynomial*. We present polynomial $f(X)$ in the form

$$f(X) = X^n + \sum_{k=1}^{n} (-)^k a_k X^{n-k}, \quad a_k \in C. \tag{1.3}$$

Let E be a *companion matrix* of the operator H. The companion matrix satisfies the same characteristic equation (1.2), so that

$$f(E) = 0. \tag{1.4}$$

Besides the evolution equation generated by operator H, one may define an evolution equation governed by the n-order Riccati equation of the form

$$\frac{d}{dt}U = f(U). \tag{1.5}$$

Evidently, the evolution equations (1.1) and (1.5) are closely connected with each other. The main task is to establish an interconnection between solutions of the evolution equations (1.1) and (1.5).

In general, the coefficients of the polynomial $f(U)$ in (1.5) are defined as certain functions of the parameter of evolution. If $f(U)$ is a cubic polynomial, then equation (1.5) is called the Abel (Riccati–Abel) differential equation [1,2]. This kind of equations frequently appears in the modelling of real problems in varied areas. Diverse methods were developed for finding the Abel equations (see, for instance, [3] and references therein). A general exact integration strategy for these equations was first formulated by Liouville [4] and is based on the concept of classes, invariants, and the solution of equivalence problem.

If the coefficients of the polynomial $f(U)$ are given by rational functions, then a classification according to invariant theory of the integrable rational Abel differential equations can be done [5]. Many integrable members of one class can be systematically mapped onto an integrable member of a different class. In [6], it has been found a unified way to find the rational map from the knowledge on the explicitly given first integral.

In [7,8], solutions of the n-order Riccati equation with constant coefficients in a field were expressed in terms of n-order trigonometric functions. The method was based on the theory of generalized trigonometric functions which arise as characteristic functions of multicomplex algebra [9,10]. The fact that the solutions of special kind of the Riccati–Abel equation can be expressed in terms of the third-order trigonometric functions, has been found by P. R. Vein [11]. Recently, in [12, 13] this result has been expanded with several novel findings.

The purpose of the present paper is to establish a mapping between solutions of these two types of evolution equations (1.1) and (1.5). In order to give an idea, we start with the most simple exercise with operator of evolution H defined by (2×2) matrix. In that case, evolution equations given by linear system of differential equations (1.1) are straightforwardly transformed into the Riccati equation (1.5) with quadratic polynomial. Next, we consider the case of evolution generated by (3×3) matrix. By means of this example, we come to the conclusion that the problem of solution of the Riccati–Abel equation is quite distinct from the example with quadratic equation. At the level $n \geqslant 3$, a resolution of the problem requires an extension of the conventional frames of the evolution problem, namely, the evolution equation with single parameter of evolution has to be extended up till the system of $(n-1)$-equation with $(n-1)$ evolution parameter. Then, it is shown, the extended system of linear differential equations is transformed into the system of generalized Riccati equations. Furthermore, under certain conditions, the extended system of evolution equations is reduced to canonical form of n-order Riccati equation.

The paper is set out as follows: In Sec. 2, we recall the principal points of the multicomplex algebra and solutions of evolution equations governed by generator of the algebra. In Sec. 3, the problem of reduction of the linear system of equations to the Riccati–Abel equation is explored. In Sec. 4, the properties of the truncated polynomials are studied. In Sec. 5, a system of evolution equations governed by n-order matrix is transformed into the system of n-order Riccati equations. In Sec. 6, one-to-one mapping between solutions of n-order Riccati equation and the parameter of evolutions is established. In Sec. 7, the developed method is illustrated by analysis of the particular case for $n = 6$.

2. TRIGONOMETRIC FUNCTIONS OF n-ORDER

Let E be $(n \times n)$ companion matrix of the finite-dimensional operator H, and the polynomial $f(X)$ be its characteristic polynomial defined in (1.3). It is supposed that the n-order polynomial $f(X)$ possesses n distinct roots $x_k, k = 1, \ldots, n \in \mathcal{C}$. The *companion matrix* E is explicitly defined as follows:

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -a_n \\ 1 & 0 & 0 & 0 & 0 & a_{n-1} \\ . & . & \ldots & . & . & . \\ 0 & 0 & \ldots & 1 & 0 & -a_2 \\ 0 & 0 & \ldots & 0 & 1 & a_1 \end{pmatrix}. \tag{2.1}$$

The companion matrix E is the representation of equivalence class of all $(n \times n)$ matrices with trace a_1, determinant a_n, and sum of corresponding minors $a_i, i = 2, \ldots, n-1$. Elements of the general complex algebra are defined by the series

$$Z = \sum_{k=0}^{n-1} \mathbf{e}^k q_k, \quad \mathbf{e}^0 = I, \quad Z \in GC_n. \tag{2.2}$$

In matrix representation, the generator $\mathbf{e} \to E$ so, that the element of general complex algebra of n-order is given by $(n \times n)$ matrix of the form

$$Z = \sum_{k=0}^{n-1} E^k q_k, \quad E^0 = I. \tag{2.3}$$

Thus, $Z \in GC_n$ is $(n-1)$-degree polynomial of the form

$$Q(U) = \sum_{k=0}^{n-1} U^k q_k, \; q_{n-1} \neq 0. \tag{2.4}$$

The modulus of $Z \in GC_n$ is conventionally defined by the following determinant [10]:

$$|Z|^n = \mathrm{Det}(\sum_{k=0}^{n-1} E^k q_k). \tag{2.5}$$

The determinant is an n-order multivariable polynomial of n variables: $q_k, k = 0, 1, 2, \ldots, n-1$. However, the algebraic module of the GC_n-number has to be defined via the basic polynomial $f(X)$ of the general complex algebra GC_n. In fact, the following formula for the modulus holds true:

$$|Z|^n = \mathrm{Det}(\sum_{k=0}^{n-1} E^k q_k) = q_{n-1}^n \prod_{k=1}^{n-1} f(u_k), \tag{2.6}$$

where $u_k, k = 1, 2, \ldots, n-1$ are roots of the $(n-1)$-degree polynomial $Q(U)$.

Introduce the following n-dimensional vectors:

$$(\mathbf{v}_i^0)_{ij} = \delta_{i,j=i+1}, i, j = 1, \ldots, n; \quad \mathbf{v}_1^a = [(-)^{n-1} a_n, \ldots, -a_2, a_1]^T, \tag{2.7}$$

and form the set of n-component vectors by imposing

$$\mathbf{v}^a{}_{k+1} = E\mathbf{v}^a{}_k, \ k = 1, 2, 3, \ldots, n-2. \tag{2.8}$$

It is convenient to present the *companion matrix* E and its powers $E^p, \ p = 2, \ldots, n-1$ in the basis of vectors $\mathbf{v}_k^a$ as follows:

$$E = [\mathbf{v}_1^0, \mathbf{v}_2^0, \ldots, \mathbf{v}_{n-1}^0, \mathbf{v}_1^a], \tag{2.9}$$

$$E^p = [\mathbf{v}_p^0, \ldots, \mathbf{v}_{n-1}^0, \mathbf{v}_1^a, \ldots, \mathbf{v}_p^a], \ p = 2, 3, \ldots, n-1. \tag{2.10}$$

Denote by $x_i, i = 1, 2, \ldots, n$ the roots of polynomial $f(X)$. Introduce n vectors consisting of degrees of the roots of $f(X)$ by

$$\mathbf{v}_x(i) = [1, x_i, x_i^2, \ldots, x_i^{n-1}]^T. \tag{2.11}$$

Vandermonde's matrix is presented in the basis of these vectors

$$W = [\mathbf{v}_x(1), \mathbf{v}_x(2), \ldots, \mathbf{v}_x(n-1), \mathbf{v}_x(n)]. \tag{2.12}$$

The eigenvalue problem for companion matrix E is formulated as follows:

$$EW = WD(x), \tag{2.13}$$

where $D(x)$ is a diagonal matrix $D_{ij}(x) = x_i \delta_{i,j}$. Correspondingly, the eigenvalue problem for single eigenvalue is written as

$$\bar{\mathbf{v}}_x(i) \ E = x_i \bar{\mathbf{v}}_x(i), \quad \bar{\mathbf{v}}_x(i) = [1, x_i, x_i^2, \ldots, x_i^{n-1}]. \tag{2.14}$$

In the same way as the usual complex algebra is used to describe trigonometry, the *general complex algebra* GC_n induces representations of the set of

n-order trigonometric functions [9]. The Euler formula for the exponential matrix is defined by the series

$$\exp\left(\sum_{k=1}^{n-1} E^k \phi_k\right) = g_0(\phi) + E g_1(\phi) + E^2 g_2(\phi) + \ldots + E^{n-1} g_{n-1}(\phi), \quad (2.15)$$

where ϕ means the set of $(n-1)$ parameters $\phi := (\phi_1, \phi_2, \phi_3, \ldots \phi_{n-1})$.

An evolution generated by matrix E is formulated in a standard way

$$\frac{d}{d\phi}\mathbf{v}^g(\phi) = E\mathbf{v}^g(\phi). \quad (2.16)$$

Solution of this equation is given by the exponential matrix

$$\mathbf{v}^g(\phi) = \exp{(E\phi)}\mathbf{v}^g(\phi = 0), \quad (2.17)$$

where $\mathbf{v}^g(\phi)$ is a vector with components

$$\mathbf{v}^g = [g_0, g_1, g_2, \ldots, g_{n-1}]^T. \quad (2.18)$$

Let $\mathbf{v}^g(0)$ be an initial vector, then solution of Eq. (2.17) is expressed via the exponential matrix as follows:

$$\mathbf{v}^g(\phi + \phi_0) = \exp{(E\phi)}\mathbf{v}^g(\phi_0). \quad (2.19)$$

This formula can be also considered as summation formula for the "g-functions" $g_k(\phi), k = 0, 1, \ldots, n-1$.

The crucial point is the following: besides the evolution governed by equation (2.17), the complete set of differential equations for generalized trigonometric functions (g-functions) consists of the evolution equations generated by degrees of the basic matrix E, they are

$$\frac{\partial}{\partial\phi_k}\mathbf{v}_g(\phi) = E^k\mathbf{v}_g(\phi), \ \phi = (\phi_1, \phi_2, \ldots, \phi_{n-1}), \ k = 1, \ldots, n-1. \quad (2.20)$$

3. EXPRESSION OF SOLUTION OF n-ORDER RICCATI EQUATION IN TERMS OF GENERALIZED TRIGONOMETRY

Let us start with the simple example when $n = 2$. In that case

$$f(X) = X^2 - a_1 X + a_2, \quad (3.1)$$

and

$$E = \begin{pmatrix} 0 & -a_2 \\ 1 & a_1 \end{pmatrix}. \quad (3.2)$$

Algebraic modulus of $Z \in GC_2$ is defined by

$$|Z|^2 = |q_0 + Eq_1|^2 = q_1^2 f(u), \tag{3.3}$$

where u is a solution of linear equation

$$Q(U) = uq_1 + q_0 = 0. \tag{3.4}$$

Exponential of E is defined by the Euler formula

$$\exp(E\phi) = Q(E) = g_1(\phi)E + g_0(\phi)I, \tag{3.5}$$

where functions $g_0(\phi)$ and $g_1(\phi)$ are trigonometric functions obeying the system of differential equations

$$\frac{d}{d\phi}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} 0 & -a_2 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}. \tag{3.6}$$

This system of equations is readily transformed into the Riccati equation of the type

$$f(u) = u^2 - a_1 u - a_2 = \frac{du}{d\phi}, \tag{3.7}$$

with $u(\phi) = -g_0(\phi)/g_1(\phi)$. Obviously,

$$\mathrm{Det}(\exp(E\phi)) = \exp(sp(E)\phi) = \exp(a_1\phi), \tag{3.8}$$

and, according to formula (3.3), we write

$$\exp(a_1\phi) = g_1^2 f(u). \tag{3.9}$$

This formula is necessary in order to construct an inverse mapping from solution of the Riccati equation $u(\phi)$ onto solutions of the matrix equation (3.6). Thus, possessing $u(\phi)$, we define functions g_1 and g_0 as follows:

$$g_1(\phi) = \frac{1}{\sqrt{|f(u)|}}\exp\left(\frac{a_1\phi}{2}\right), \ g_0(\phi) = -u(\phi)g_1(\phi). \tag{3.10}$$

Now, let us use a similar algorithm to construct an interconnection between evolution equations generated by higher-order polynomials and the Riccati equation. In order to give a main idea, firstly, let us consider an evolution equation generated by the third-order companion matrix E obeying the cubic equation

$$E^3 - a_1E^2 + a_2E - a_3I = 0, \quad I \text{ means unit matrix.} \tag{3.11}$$

Expansion of exponential function of E is given by the series

$$\exp(E\phi) = g_0(\phi) + g_1(\phi)E + g_2(\phi)E^2, \tag{3.12}$$

where third-order trigonometric functions are solutions to the system of differential equations

$$\frac{d}{d\phi}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & a_1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}. \tag{3.13}$$

Our aim is to reduce this system of equations to the Riccati–Abel equation of the type

$$\frac{dU}{d\phi} = U^3 - a_1U^2 + a_2U - a_3. \tag{3.14}$$

Here, we have to recall that trigonometric functions of the third order depend on two variables, ϕ_1 and ϕ_2, so that the Euler formula (3.12) has to be written in the form

$$\exp\left(E\phi_1 + E^2\phi_2\right) = g_0(\phi_1,\phi_2) + Eg_1(\phi_1,\phi_2) + E^2g_2(\phi_1,\phi_2). \tag{3.15}$$

In [9,10], we have proved that the linear differential equations for trigonometric functions $g_k, k = 0,1,2$ are reduced into canonical form of the Abel equation under the condition $g_2 = 0$. It is worth to emphasize, in that case one has to work with *complete form of the evolution generated by finite dimensional operator*, i.e., one has to take into account all system of differential equations with respect to complete set of parameters $(\phi_1,\ \phi_2)$. So, for evolution generated by the third-order companion matrix E, besides the system of differential equations (3.13) one has to consider equations with respect to the second parameter ϕ_2:

$$\frac{d}{d\phi_2}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 & a_3 & a_3a_1 \\ 0 & -a_2 & a_3 - a_1a_2 \\ 1 & a_1 & a_1^2 - a_2 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}. \tag{3.16}$$

These systems of differential equations are reduced into the Riccati–Abel equation under the condition

$$g_2(\phi_1,\phi_2) = 0. \tag{3.17}$$

This equation implicitly contains functional dependence of the type $\phi_2 = \phi_2(\phi_1)$ and serves as a basic constraint to define solution of the Riccati–Abel equation as follows:

$$U(\phi_2) = -\frac{g_0(\phi_1(\phi_2),\phi_2)}{g_1(\phi_1(\phi_2),\phi_2)}. \tag{3.18}$$

When $n > 2$, this algorithm is generalized straightforwardly [10]. Let $f(X)$ be n-order polynomial with companion matrix E. First of all, it has to be noted that the evolution process generated by $(n \times n)$ matrix E consists of

$(n-1)$ evolution equations generated by matrices $E, E^2, \ldots, E^{n-1}$ with respect to $(n-1)$ parameters of evolution $\phi_1, \phi_2, \ldots, \phi_{n-1}$, correspondingly,

$$\partial_k \Psi(\phi) = E^k \Psi(\phi),\ k = 1, 2, \ldots, n-1; \phi = (\phi_1, \phi_2, \ldots, \phi_{n-1}),\quad \partial_k = \frac{\partial}{\partial \phi_k}. \tag{3.19}$$

The solution of this system is given by the exponential function

$$\exp\left(\sum_{k=1}^{n-1} E^k \phi_k\right) = g_0 + E g_1 + E^2 g_2 + \ldots + E^{n-1} g_{n-1} = Q(E). \tag{3.20}$$

Then in [8] it is proved that by using constraints

$$g_k(\phi) = 0, k = 2, 3, \ldots, n-1, \tag{3.21}$$

the system of linear differential equations (3.19) is reduced into the n-order Riccati equation with respect to parameter of evolution ϕ_{n-1} of the form

$$\frac{d}{d\phi_{n-1}} U = f(U), \tag{3.22}$$

with solution

$$U(\phi_{n-1}) = -\frac{g_0}{g_1}. \tag{3.23}$$

Thus, transformation of the linear system of evolution equations into canonical form of the n-order Riccati equation requires $(n-2)$ constraints. However, we can transform differential equations for g-functions into a system of "Riccati-type" equations. Consider $(n-1)$-degree polynomial in the right-hand side of the Euler formula (3.20)

$$Q(U) = g_0 + U g_1 + U^2 g_2 + \ldots + U^{n-1} g_{n-1}. \tag{3.24}$$

Under constraints $g_k = 0,\ k = 2, 3, \ldots, n-1$ this polynomial has been reduced to the form

$$Q(U) = g_0 + U g_1. \tag{3.25}$$

Then the solution of equation $Q(U) = 0$ turns out to be the solution (3.23) to the n-order Riccati equation (3.22). This observation prompts us an idea that the roots of the polynomial $Q(U) = 0$ free of the constraints (3.21) will obey a system of "Riccati-type" equations.

As an example, consider the case $n = 3$. We have to resolve the following problem: it is necessary to derive differential equations for roots of the polynomial $Q(u)$ making use of differential equations (3.13). Roots of polynomial $Q(u)$ are denoted by U and V. The following **Proposition 3.1** holds true.

Let U, V be solutions of the quadratic equation

$$u^2 + G_{12}(\phi)u + G_{02}(\phi) = 0, \tag{3.26}$$

where $G_{12}(\phi) = g_1/g_2,\ G_{02}(\phi) = g_0/g_2$. Then the functions $U(\phi), V(\phi)$ obey the following system of differential equations:

$$(U - V)\frac{d}{d\phi}U = f(U), \quad (V - U)\frac{d}{d\phi}V = f(V). \tag{3.27}$$

These equations are readily obtained from the following system of linear algebraic equations:

$$\begin{pmatrix} \dfrac{dG_{12}}{d\phi} \\ \dfrac{dG_{02}}{d\phi} \end{pmatrix} = \begin{pmatrix} \dfrac{dG_{12}}{dU} & \dfrac{dG_{12}}{dV} \\ \dfrac{dG_{02}}{dU} & \dfrac{dG_{02}}{dV} \end{pmatrix} \begin{pmatrix} \dfrac{dU}{d\phi} \\ \dfrac{dV}{d\phi} \end{pmatrix}. \tag{3.28}$$

Each of equations of the system (3.27) coincides with the Abel equation of the second kind [2].

Proposition 3.1 is a particular case of the general theorem. In the general case, we have to derive a system of differential equations for roots of the polynomial

$$Q(U) = g_0(\phi) + Ug_1(\phi) + U^2 g_2(\phi) + \ldots + U^{n-1} g_{n-1}(\phi), \tag{3.29}$$

where ϕ means a set of $(n-1)$ parameter: $\phi_1, \phi_2, \ldots, \phi_{n-1}$.

4. TRUNCATED POLYNOMIALS AND THEIR PROPERTIES

The aim of this section is to recall some properties of the truncated polynomials which (in Sec. 5) we shall use in proofs of **Theorem 5.1**.

Introduce n-dimensional vector $\mathbf{v}^u$, by definition

$$\mathbf{v}^u = [1, U, U^2, \ldots, U^{n-1}], \tag{4.1}$$

and form scalar products of this vector with vectors $\mathbf{v}^g$, $\mathbf{v}_k^0$, and $\mathbf{v}_k^a$ which possess the following properties:

Property 4.1

$$Q(U) = \sum_{j=0}^{n-1} U^j g_j = (\mathbf{v}_u \ \cdot \ \mathbf{v}_g). \tag{4.2}$$

Property 4.2

$$(\mathbf{v}^u \cdot \mathbf{v}_k^0) = U^k. \tag{4.3}$$

Lemma 4.3

Let x be one of the roots of polynomial $f(X)$. Then, the following formula holds true:

$$x^{n+p} = (E^p \mathbf{v}_1^a \cdot \mathbf{v}^x) = (\mathbf{v}_{p+1}^a \cdot \mathbf{v}^x). \quad (4.4)$$

Proof.

On making use of formulation of eigenvalue problem for companion matrix E in (2.13) and (2.14), we get

$$x^n = (\mathbf{v}_1^a \cdot \mathbf{v}^x), \quad x^{n+1} = (E\mathbf{v}_2^a \cdot \mathbf{v}^x), \quad x^{n+p} = (E^p \mathbf{v}_1^a \cdot \mathbf{v}^x) = (\mathbf{v}_{p+1}^a \cdot \mathbf{v}^x). \quad (4.5)$$

□

Lemma 4.4

The following formulae hold true:

$$(\mathbf{v}^u \cdot \mathbf{v}_1^a) = U^n - f(U), \quad (4.6)$$

$$(\mathbf{v}^u \cdot \mathbf{v}_p^a) = U^p(U^n - f(U)) + \ f(U) \sum_{k=0}^{p-2} U^k (\mathbf{v}_{n-1}^0 \cdot \mathbf{v}_{p-k-1}^a), \ p > 1. \quad (4.7)$$

Proof.

Since $\mathbf{v}_u$ is not an eigenvector of E, then instead of Eq. (4.5) we have

$$E^+ \mathbf{v}^u = U\mathbf{v}^u + f(U)\mathbf{v}_{n-1}^0, \quad (4.8)$$

where $\mathbf{v}_{n-1}^0 = [0, 0, 0 \ldots, 1]^T$. The set of Eqs. (4.5) are extended as follows:

$$\begin{aligned}
(\mathbf{v}_1^a \cdot \mathbf{v}^u) &= U^n - f(U), \\
(\mathbf{v}_2^a \cdot \mathbf{v}^u) &= U(U^n - f(U)) + f(U)(\mathbf{v}_1^a \cdot \mathbf{v}_{n-1}^0), \\
(\mathbf{v}_3^a \cdot \mathbf{v}^u) &= (\mathbf{v}_2^a E^+ \cdot \mathbf{v}^u) = (\mathbf{v}_2^a \cdot E^+ \mathbf{v}^u) = \\
&= (\mathbf{v}_2^a \cdot (U\mathbf{v}^u + \mathbf{v}_{n-1}^0))f(U) = \\
&= U(\mathbf{v}_2^a \cdot \mathbf{v}^u) + (\mathbf{v}_2^a \cdot \mathbf{v}_{n-1}^0)f(U) = \\
&= U \ (\ U(U^n - f(U)) + f(U)((\mathbf{v}_1^a \cdot \mathbf{v}_{n-1}^0) + (\mathbf{v}_2^a \cdot \mathbf{v}_{n-1}^0)) = \\
&= U^2(U^n - f(U)) + U(f(U)(\mathbf{v}_1^a \cdot \mathbf{v}_{n-1}^0) \ ) + f(U)(\mathbf{v}_2^a \cdot \mathbf{v}_{n-1}^0).
\end{aligned} \quad (4.9)$$

At the final step of this algorithm, we come to the general formula of the form

$$(\mathbf{v}^u \cdot \mathbf{v}_p^a) = U^p(U^n - f(U)) + \ f(U) \sum_{k=0}^{p-2} U^k (\mathbf{v}_{n-1}^0 \cdot \mathbf{v}_{p-k-1}^a)). \quad (4.10)$$

□

Lemma 4.5

Let U be one of the roots of polynomial $Q(U)$, then the following formula holds true:

$$(\mathbf{v}_u \cdot E^p \mathbf{v}_g) = -f(U)(U^{n-2} + G_{n-2}U^{n-3} + \ldots + G_{n-k}U^{n-k-1} + \ldots + G_1), \quad p = 1, \ldots, n-1. \tag{4.11}$$

Proof.

Let us begin with the case $p = 1$. In that case, formula (4.11) is reduced to

$$(\mathbf{v}^u \cdot E\mathbf{v}^g) = -U^n + (\mathbf{v}^u \cdot \mathbf{v}_1^a) = -f(U). \tag{4.12}$$

From definition of the companion matrix E in the basis of vectors $\mathbf{v}_k^0$ (see, (2.11), (2.12)), it follows that the vector $E\mathbf{v}^g$ can be presented as a sum of the following vectors:

$$E\mathbf{v}^g = [\mathbf{v}_1^0 G_0 + \mathbf{v}_2^0 G_1 + \ldots + \mathbf{v}_{n-1}^0 G_{n-2}] + [\mathbf{v}_1^a]. \tag{4.13}$$

Form a scalar product of this vector with vector $\mathbf{v}^u$, then

$$(\mathbf{v}^u \cdot (\mathbf{v}_1^0 G_0 + \mathbf{v}_2^0 G_1 + \ldots + \mathbf{v}_{n-1}^0 G_{n-2})) + (\mathbf{v}^u \cdot \mathbf{v}_1^a). \tag{4.14}$$

By taking into account (4.3), this series is transformed as follows:

$$(\mathbf{v}^u \cdot E\mathbf{v}^g) = (\ G_0 U + G_1 U^2 + \ldots + G_{n-2}U^{n-1}\ ) + (\mathbf{v}^u \cdot \mathbf{v}_1^a). \tag{4.15}$$

Since U is one of the roots of the polynomial $Q(U)$, equation $Q(U) = 0$ can be written in the form

$$G_0 U + G_1 U^2 + \ldots + G_{n-2}U^{n-1} = \\ = U(G_0 U + G_1 U^2 + \ldots + G_{n-2}U^{n-2} = -U^{n-1}) = -U^n. \tag{4.16}$$

By using this formula in (4.15), we obtain

$$(\mathbf{v}^u \cdot E\mathbf{v}^g) = -U^n + (\mathbf{v}^u \cdot \mathbf{v}_1^a). \tag{4.17}$$

Now take into account that

$$(\mathbf{v}^u \cdot \mathbf{v}_1^a) = a_1 U^{n-1} - a_2 U^{n-2} + \ldots + a_n(-1)^n = -f(U) + U^n. \tag{4.18}$$

Hence, in the final formula, the term U^n is removed, and the right-hand side of Eq. (4.17) takes the form

$$(\mathbf{v}^u \cdot E\mathbf{v}^g) = -U^n + (\mathbf{v}^u \cdot \mathbf{v}_1^a) = -f(U). \tag{4.19}$$

Now, consider the general case when $p > 1$. Write the following series:

$$E^p \mathbf{v}^g = [\mathbf{v}_p^0 g_0 + \mathbf{v}_{p+1}^0 g_1 + \ldots + \mathbf{v}_{n-1}^0 g_{n-p-1} + \mathbf{v}_1^a g_{n-p} + \ldots + \mathbf{v}_p^a g_{n-1}]. \quad (4.20)$$

Divide both sides of this equation by $g_{n-1} \neq 0$ and form a scalar product of this vector with $\mathbf{v}_u$ which we present as a sum of two brackets:

$$(\mathbf{v}^u \cdot (\mathbf{v}_p^0 G_0 + \mathbf{v}_{p+1}^0 G_1 + \ldots + \mathbf{v}_{n-1}^0 G_{n-p-1})) + ((\mathbf{v}^u \cdot \mathbf{v}_1^a) G_{n-p} + \ldots). \quad (4.21)$$

On making use of formula

$$(\mathbf{v}^u \cdot \mathbf{v}_p^0) = U^p,$$

the series inside the first brackets in (4.21) is presented by the following polynomial:

$$U^p G_0 + G_1 U^{p+1} + \ldots + G_{n-p-1} U^{n-1} = \\ = U^p (G_0 + G_1 U + \ldots + U^{n-p-1} G_{n-p-1}). \quad (4.22)$$

Since U satisfies equation $Q(U) = 0$, the following identity is true:

$$G_0 + G_1 U + \ldots + G_{n-p-1} U^{n-p-1} = \\ = -(G_{n-p} U^{n-p} + G_{n-p+1} U^{n-p+1} + \ldots + U^{n-1}). \quad (4.23)$$

By using this identity, we get

$$U^p (G_0 + G_1 U + \ldots + U^{n-p-1} G_{n-p-1}) = \\ = -U^p \; (G_{n-p} U^{n-p} + G_{n-p+1} U^{n-p+1} + \ldots + U^{n-1}). \quad (4.24)$$

Now evaluate the series inside the second brackets in (4.21)

$$((\mathbf{v}^u \cdot \mathbf{v}_1^a) G_{n-p} + \ldots) = (\mathbf{v}^u \cdot \mathbf{v}_1^a) G_{n-p} + (\mathbf{v}^u \cdot \mathbf{v}_2^a) G_{n-p+1} + \ldots + (\mathbf{v}^u \cdot \mathbf{v}_p^a). \quad (4.25)$$

By using formulae (4.6), (4.7) and by collecting together the resulting expressions of both brackets, we come to the following expression:

$$(\mathbf{v}^u \cdot E^p \mathbf{v}_g) = -f(U)(U^{n-2} + G_{n-2} U^{n-3} + \ldots + G_{n-k} U^{n-k-1} + \ldots + G_1). \quad (4.26)$$

□

To proceed, it is necessary to recall some features of the following m-degree polynomial

$$B(Y) := Y^m + \sum_{k=0}^{m-1} Y^k b_k. \quad (4.27)$$

Lemma 4.6

Let $y_k, k = 1, .., m$ be a set of roots of $B(Y)$. Define the following truncated polynomials:

$$B_p(y) = y_i^{m-p} + \sum_{k=0}^{m-p-1} y_i^k b_{k+p},\ \ i = 1, \dots, m-1;\ p = 1, 2, \dots, m-1. \quad (4.28)$$

The truncated polynomial $B_p(y)$ implicitly is independent of y_i and equals to

$$B_p(y) = b_p|_{y_i=0}. \quad (4.29)$$

Proof.

Since y_i satisfies the equation $B(Y) = 0$, then

$$u_i(u_i^{m-1} + \sum_{k=2}^{m-2} u_i^k b_{k+1} + b_1) = -b_0. \quad (4.30)$$

According to Vieta's formula, coefficient b_0 is given by the product of the roots

$$b_0 = (-1)^m\ y_1 y_2 \dots y_i \dots y_m. \quad (4.31)$$

Use this expression instead of b_0 which admits to remove the factor y_i from both the sides of Eq. (4.29). In this way, we come to the equation which does not contain y_i,

$$\left(y_i^{m-1} + \sum_{k=1}^{m-2} y_i^k b_{k+1} + b_1 \right) = y_1 \dots y_{i-1} y_{i+1} \dots y_m (-1)^m. \quad (4.32)$$

In the right-hand side, we have the coefficient b_1 with $y_i = 0$, hence,

$$\left(y_i^{m-1} + \sum_{k=1}^{m-2} y_i^k b_{k+1} + b_1 \right) = b_1|_{y_i=0}. \quad (4.33)$$

Next, re-write this equation as follows:

$$y_i \left(y^{m-2} + \sum_{k=2}^{m-3} y^k b_{k+2} + b_2 \right) = b_1|_{y_i=0} - b_1. \quad (4.34)$$

Notice that

$$b_1|_{y_i=0} - b_1 = y_i b_2|_{y_i=0}, \quad (4.35)$$

and remove the factor y_i from the identity. We get,

$$\left(y^{m-2} + \sum_{k=2}^{m-3} y^k b_{k+2} + b_2 \right) = b_2|_{y_i=0}. \quad (4.36)$$

By continuing this process at the pth step, we arrive to the following equation:

$$\left(y^{m-p} + \sum_{k=p}^{m-p-1} y^k b_{k+p} + b_p\right) = b_p|_{y_i=0}. \tag{4.37}$$

□

Lemma 4.7
Derivative of polynomial $B(Y)$ at $Y = y_i$ is equal to polynomial of the form

$$\frac{dB(Y)}{dY}\Big|_{Y=y_i} = y_i^{m-1} + \sum_{k=1}^{m-1} y_i^{k-1}(b_k|_{y_i=0}). \tag{4.38}$$

Proof.
Differentiate $B(Y)$ with respect to Y,

$$\frac{dB(Y)}{dY}\Big|_{Y=y_i} = my_i^{m-1} + \sum_{k=1}^{m-1}(k-1)y_i^{k-1}b_k. \tag{4.39}$$

This polynomial can be presented as a sum of polynomials as follows:

$$\frac{dB(Y)}{dY}\Big|_{Y=y_i} = \sum_{j=0}^{m-1}\left(y_i^{m-1} + \sum_{k=1+j}^{m-1} y_i^{k-1}b_k\right). \tag{4.40}$$

By applying Lemma 1 for all polynomials inside brackets, we obtain

$$\frac{dB(Y)}{dY}\Big|_{Y=y_i} = \sum_{j=0}^{m-1} y_i^j \left(y_i^{m-2} + \sum_{k=1+j}^{m-2} y_i^{k-1}b_k\right) = \\ = y_i^{m-1} + \sum_{k=1}^{m-1} y_i^{k-1}(b_k|_{y_i=0}). \tag{4.41}$$

□

Lemma 4.8
Consider triangle matrix M_{ij} of the form

$$M_{ij} = \delta_{ij} + \sum_{k=1}^{n/2-1}(\mathbf{v}_{n-1}^0 \cdot \mathbf{v}_k^a)\delta_{i+k,j}, \\ M_{ij} = 0,\ i < j. \tag{4.42}$$

The matrix inverse to M_{ij} has the form:

$$M^{-1}_{ij} = \delta_{ij} + \sum_{k=1} \delta_{i,j+k}(-)^i a_k, \; i \geqslant j, \\ M^{-1}_{ij} = 0, \; i < j. \tag{4.43}$$

5. SYSTEM OF n-ORDER RICCATI DIFFERENTIAL EQUATIONS FOR ROOTS OF THE POLYNOMIAL

The aim of this section is to derive differential equations for roots of the polynomial $Q(U)$ through the agency of system of differential equations for its coefficients.

Theorem 5.1

Let the set of functions $u_k(\phi), \; k = 1, 2, 3, \ldots, n-1; \; \phi = (\phi_1, \phi_2, \ldots, \phi_{n-1})$ be a set of the roots of the polynomial

$$Q(U) = \sum_{j=0}^{n-1} U^j g_j(\phi), \tag{5.1}$$

where coefficients $g_j(\phi), j = 0, 1, 2, \ldots, n-1$ are solutions of evolution equations:

$$\partial_i \; g_j = \sum_{m=1}^{n} (E^i)^m_j g_{m-1}, \; i = 1, \ldots, n-1. \tag{5.2}$$

Then, the functions $u_k(\phi), k = 1, \ldots, n-1$ obey the following system of nonlinear equations:

$$F(u_m) \sum_{k=1}^{n-p} a_{n-k-p} \partial_k \; u_m = A_p \; f(u_m), \quad m = 1, \ldots, n-1, \tag{5.3}$$

where $F(u_m)$ is $(n-2)$-degree truncated polynomial of the form

$$F(u_m) = \frac{dQ(U)}{dU}\Big|_{U=u_m} = u_m^{n-2} + \sum_{k=0}^{n-3} u_m^k A_k(m) = \prod_{k=1, k\neq m}^{n-1} (u_m - u_k), \tag{5.4}$$

and $A_p(m)$ is p-th coefficient of the polynomial $F(u_m)$.

Proof.

The calculations are essentially simplified if the generating polynomial $f(X)$ is taken in a reduced form, i.e.,

$$f(X) = X^n + (-1)^{n-1}a_{n-1}X + (-1)^n a_n, \quad a_k = 0, k = 1, 2, \ldots, n-2. \tag{5.5}$$

Firstly, let us proceed the proofs with the reduced polynomial.

Step 1.

Differentiate equation $Q(U) = 0$ with respect to parameters ϕ_k, $k = 1, \ldots, n-1$:

$$\partial_k Q(U) = \partial_k \left(g_{n-1}U^{n-1} + \sum_{j=1}^{n-1} U^{j-1}g_{j-1} \right) = \partial_k(\mathbf{v}_u \cdot \mathbf{v}_g) = 0.$$

Here U is one of the roots of polynomial $Q(U)$. Equation is presented as a sum of two parts,

$$\partial_k Q(U) = ((\mathbf{v}_g \cdot \partial_k \mathbf{v}_u)) + \left(\mathbf{v}_u \cdot \frac{\partial}{\partial \psi_k} \mathbf{v}_g \right) = 0. \tag{5.6}$$

Divide this equation by $g_{n-1} \neq 0$ and denote fractions by

$$G_j = \frac{g_j}{g_{n-1}}, \; j = 0, 1, 2, \ldots, n-2.$$

For pth root $U = u_l$ the first part is written as follows (see, Lemma 4.7):

$$(\;(n-1)U^{n-2} + \sum_{j=2}^{n-1} (j-1)U^{j-2}G_{j-1}\;)\partial_k U = F(u_l)\partial_k U, \tag{5.7}$$

where

$$F(u_l) = \prod_{m=1, m\neq l}^{n-2} (u_l - u_m). \tag{5.8}$$

The second part contains derivations of the g-functions

$$\frac{1}{g_{n-1}}(\mathbf{v}_u \cdot \partial_k \mathbf{v}_g) = \frac{1}{g_{n-1}} \left((\partial_k g_{n-1})U^{n-1} + \sum_{j=1}^{n-1} U^{j-1}(\partial_k g_{j-1}) \right). \tag{5.9}$$

Step 2.

Let us start with differential equation with respect to variable ϕ_1. The derivatives of g-functions with respect to ϕ_1 are given by formula

$$(\mathbf{v}^u \;\cdot\; \partial_1 \mathbf{v}^g) = (\mathbf{v}^u \;\cdot\; E\mathbf{v}^g). \tag{5.10}$$

According to formula (2.11), we write

$$\partial_1 \mathbf{v}^g = E\mathbf{v}^g = [\mathbf{v}_1^0 G_0 + \mathbf{v}_2^0 G_1 + \ldots + \mathbf{v}_{n-1}^0 G_{n-2}] + [\mathbf{v}_1^a].$$

Form a scalar product with vector $\mathbf{v}_u$ as

$$(\mathbf{v}^u \cdot (\mathbf{v}_1^0 G_0 + \mathbf{v}_2^0 G_1 + \ldots + \mathbf{v}_{n-1}^0 G_{n-2})) + (\mathbf{v}^u \cdot \mathbf{v}_1^a), \tag{5.11}$$

and take into account formula (4.3) (see, Property 4.2). In this way we obtain

$$\begin{aligned}(\mathbf{v}^u \cdot E\mathbf{v}^g) = (\ G_0 U + G_1 U^2 + \ldots + G_{n-2} U^{n-1}\ ) + (\mathbf{v}^u \cdot \mathbf{v}_1^a) = \\ = -U^n + (\mathbf{v}^u \cdot \mathbf{v}_1^a). \end{aligned}\tag{5.12}$$

Now recall formula (4.6) (see, Lemma 4.4)

$$(\mathbf{v}_u \cdot \mathbf{v}_1^a) = a_1 U^{n-1} - a_2 U^{n-2} + \ldots + a_n (-1)^n = -f(U) + U^n.$$

By taking into account this formula, we come to conclusion that

$$(\mathbf{v}^u \cdot \partial_1 \mathbf{v}^g) = (\mathbf{v}^u \cdot E\mathbf{v}^g) = -U^n + (\mathbf{v}^u \cdot \mathbf{v}_1^a) = -f(U). \tag{5.13}$$

Finally, we come to the following equation:

$$F(u_l)\partial_k u_l = f(u_l), \tag{5.14}$$

with $F(u_l)$ defined in (5.8).

Step 3.

Next, consider the case $p > 1$. On making use of formula

$$E^p = [\mathbf{v}_p^0, \mathbf{v}_{p+1}^0, \ldots, \mathbf{v}_{n-1}^0, \mathbf{v}_1^a, \mathbf{v}_2^a, \ldots, \mathbf{v}_p^a],$$

the derivatives of g-functions with respect to ϕ_p can be represented as follows:

$$\partial_p \mathbf{v}_g = E^p \mathbf{v}_g = [\mathbf{v}_p^0 g_0 + \mathbf{v}_{p+1}^0 g_1 + \ldots + \mathbf{v}_{n-1}^0 g_{n-p-1} + \mathbf{v}_1^a g_{n-p} + \ldots + \mathbf{v}_p^a g_{n-1}]. \tag{5.15}$$

Divide this equation by $g_{n-1} \neq 0$ and calculate the scalar product

$$(\mathbf{v}^u \cdot (\partial_p \mathbf{v}^g)) = (\mathbf{v}^u \cdot E^p \mathbf{v}^g).$$

The result includes two parts, P_I and P_{II}:

$$\begin{aligned} P_I + P_{II}, \quad P_I = (\mathbf{v}_u \cdot (\mathbf{v}_p^0 G_0 + \mathbf{v}_{p+1}^0 G_1 + \ldots + \mathbf{v}_{n-1}^0 G_{n-p-1})), \\ P_{II} = (\mathbf{v}_u \cdot \mathbf{v}_1^a G_{n-p} + \ldots). \end{aligned}\tag{5.16}$$

The first part is calculated by taking into account formula (4.6). We get

$$P_I = U^p G_0 + G_1 U^{p+1} + \ldots + G_{n-p-1} U^{n-1} =$$
$$= U^p (G_0 + G_1 U + \ldots + U^{n-p-1} G_{n-p-1}). \tag{5.17}$$

Since U is one of the roots of equation $Q(U) = 0$, the following equation holds true:

$$G_0 + G_1 U + \ldots + G_{n-p-1} U^{n-p-1} =$$
$$= -(G_{n-p} U^{n-p} + G_{n-p+1} U^{n-p+1} + \ldots + U^{n-1}). \tag{5.18}$$

Hence, the expression for P_I takes the form

$$P_I = -(G_{n-p} U^n + G_{n-p+1} U^{n+1} + \ldots + G_{n-1} U^{n+p-1}). \tag{5.19}$$

Now calculate the second part of the sum, P_{II}, which has the form

$$P_{II} = (\mathbf{v}_u \cdot \mathbf{v}_1^a) G_{n-p} + (\mathbf{v}_u \cdot \mathbf{v}_2^a) G_{n-p+1} + \ldots + (\mathbf{v}_u \cdot \mathbf{v}_p^a). \tag{5.20}$$

Firstly, let us consider the most simple case when polynomial $f(X)$ has the form defined in (5.1) with $a_k = 0, k = 1, 2, 3, \ldots, n-2$. In that case, from formulae (4.6), (4.7) of Lemma 4.4 it follows:

$$\begin{gathered} (\mathbf{v}_u \cdot \mathbf{v}_1^a) = U^n - f(U), \\ (\mathbf{v}_u \cdot \mathbf{v}_2^a) = U^{n+1} - U f(U), \ldots, (\mathbf{v}_u \cdot \mathbf{v}_p^a) = U^{n+p-1} - f(U). \end{gathered} \tag{5.21}$$

By replacing scalar products $(\mathbf{v}_u \cdot \mathbf{v}_p^a)$ in (5.20) according to these equations, we get

$$P_{II} = (U^n - f(U)) G_{n-p} + (U^{n+1} - U f(U)) G_{n-p+1} + \ldots + (U^{n+p-1} - f(U)). \tag{5.22}$$

Now join the results of two calculations:

$$P_I + P_{II} = -(G_{n-p} U^n + G_{n-p+1} U^{n+1} + \ldots + U^{n+p-1}) +$$
$$+ (U^n - f(U)) G_{n-p} + (U^{n+1} - U f(U)) G_{n-p+1} + \ldots (U^{n+p-1} - f(U)) =$$
$$= (-f(U)) G_{n-p} + (-U f(U)) G_{n-p+1} + \ldots + (-f(U)) = -f(U) A_p, \tag{5.23}$$

where by A_p we denoted the polynomial (see, Lemma 4.6)

$$A_p(U) = U^{p-1} + U^{p-2} G_{n-2} + \ldots + U G_{n-p+1} + G_{n-p}. \tag{5.24}$$

Notice, for $p = 1$, $A_1(U) = 1$. Joining obtained equations into unique system, we arrive to the following system of equations for function $U(\phi)$:

$$\prod_k (U - u_k) \frac{\partial}{\partial \psi_p} U = f(U)(U^{p-1} + U^{p-2} G_{n-2} + \ldots + U G_{n-p+1} + G_{n-p}). \tag{5.25}$$

In notations introduced in (5.8) and (5.24), this system of equations is written as

$$F(u_l) \frac{\partial}{\partial \psi_p} u_l = f(U)\ A_p(u_l), \quad l, p = 1, 2, 3, \ldots, n-1. \tag{5.26}$$

From formula (5.8) for function $F(u_l)$ it follows that

$$\prod_k (U - u_k) = \sum_p U^{n-p-1}(U^{p-1} + U^{p-2} G_{n-2} + \ldots + U G_{n-p+1} + G_{n-p}) =$$
$$= \sum_p U^{n-p-1} A_p(U). \tag{5.27}$$

In this way we come to the following equation:

$$\sum_{p=1}^{n-1} U^{n-p-1} \partial_p U = f(U). \tag{5.28}$$

Step 4.

In the general case when $a_k \neq 0, k = 1, 2, \ldots, n$, we have to use general formulae for the scalar product $\mathbf{v}_u$ with $\mathbf{v}_k^a$. In the general case instead of formulae

$$(\mathbf{v}_u \cdot \mathbf{v}_p^a) = U^{n+p-1} - f(U),\ a_k = 0$$

for $k = 1, 2, \ldots, a_{n-2}$, and $a_{n-1} \neq 0, a_n \neq 0$, we have to use the formula

$$((\mathbf{v}_u \cdot \mathbf{v}_p^a) = -U^{p-1}(U^n - f(U)) + f(U) U^{p-2}(v_{n-1}^0 \cdot v_1^a) + \ldots +$$
$$+ U^{p-k-1}(v_{n-1}^0 \cdot v_k^a) + \ldots + (v_{n-1}^0 \cdot v_3^a)). \tag{5.29}$$

By using these formulae in

$$(\mathbf{v}_u \cdot \mathbf{v}_1^a) G_{n-p} + (\mathbf{v}_u \cdot \mathbf{v}_2^a) G_{n-p+1} + \ldots + (\mathbf{v}_u \cdot \mathbf{v}_p^a), \tag{5.30}$$

we come to the following system of equations for function $U(\phi)$:

$$F(U) \partial_k U = f(U) \sum_{l=1}^{n-1} M_{kl} A_l(U), \tag{5.31}$$

where matrix M_{kl} is defined in (4.42). The matrix inverse to M_{ij} according to Lemma 4.8 has the form

$$M^{-1}_{ij} = \delta_{ij} + \sum_{k=1} \delta_{i,j+k} (-)^i a_i, \ i \geqslant j,$$

$$M^{-1}_{ij} = 0, \ i < j. \tag{5.32}$$

By applying the inverse matrix to the system of equations (5.24), we come to the following set of equations:

$$F(U) \sum_{k=1}^{n-p} a_{n-k-p} \partial_k \ U = A_p \ f(U), \quad p = 1, 2, 3, \ldots, n-1. \tag{5.33}$$

To proceed, multiply pth equation by function U^p and summarize into one equation

$$F(U) \ \sum_k U^k \sum_{k=1}^{n-p} a_{n-k-p} \partial_k \ U = \sum_p A_p \ f(U). \tag{5.34}$$

By using the identity

$$\sum_p A_p U^p = F(U),$$

we arrive to the following equation:

$$\sum_{k=0}^{n-2} U^{n-2-k} (\sum_{i=0}^{i=k} \ (-1)^i \ a_i \partial_{k+1-i}) \ U = \ f(U), \tag{5.35}$$

which also can be written in the form

$$\sum_{k=1}^{n-1} \ (\sum_{i=0}^{n-k-1} \ (-1)^{i+1} a_{n-k-i-1} U^i) \partial_k \ U = \ f(U). \tag{5.36}$$

□

6. THE SYSTEM OF EQUATIONS INVERSE TO THE SYSTEM OF n-ORDER RICCATI EQUATIONS

The n-order Riccati equation

$$\frac{dU}{d\phi} = f(U), \tag{6.1}$$

with constant coefficient, is directly integrated with respect to inverse function $\phi = \phi(U)$ by

$$d\phi = \frac{dU}{f(U)}. \tag{6.2}$$

Thus, in order to integrate the Riccati equation, one has to construct an inverse equation [14]. The system of n-order Riccati equations also admits an inverse system of equations, where the set of variables $\psi_k, k = 1, 2, 3, \ldots, n-1$ are functions of the roots $u_k, k = 1, 2, 3, \ldots, n-1$. Define a mapping $u_k \to \psi_k$ given by the following Jacobian matrix:

$$J(\frac{Du}{D\psi}) = \begin{pmatrix} \frac{\partial u_1}{\partial \psi_1} & \ldots & \frac{\partial u_1}{\partial \psi_{n-1}} \\ \ldots & \ldots & \ldots \\ \frac{\partial u_{n-1}}{\partial \psi_1} & \ldots & \frac{\partial u_{n-1}}{\partial \psi_{n-1}} \end{pmatrix}. \tag{6.3}$$

The differential of u_i is given by

$$du_i = \sum_{k=1}^{n-1} \frac{\partial u_i}{\partial \psi_k}\ d\psi_k,\ i = 1, 2, \ldots, n-1. \tag{6.4}$$

Firstly, let us consider the most simple case of the reduced polynomial $f(X)$ defined in (5.5). In that case, we have to use the reduced evolution equations (5.26),

$$\frac{\partial u_i}{\partial \psi_p} = f(u_i)\ A_p[\prod_{k \neq i}(u_i - u_k)]^{-1},\ i, p = 1, 2, 3, \ldots, n-1. \tag{6.5}$$

The right-hand sides of these equations are denoted by

$$A_{k,p} = A_p[\prod_{k \neq i}(u_i - u_k)]^{-1}. \tag{6.6}$$

In the Jacobian matrix (6.3), replace derivatives according to equations (6.5),

$$J(\frac{Du}{D\psi}) = \begin{pmatrix} f(u_1)A_{1,1} & \ldots & f(u_1)A_{1,n-1} \\ \ldots & \ldots & \ldots \\ f(u_{n-1})A_{n-1,1} & \ldots & f(u_{n-1})A_{n-1,n-1} \end{pmatrix}. \tag{6.7}$$

It is seen, if in this matrix we put $f(u_i) = 1$, then the inverse matrix is nothing else than the Vandrmonde matrix

$$V = \begin{pmatrix} u_1^{n-2} & \ldots & u_{n-1}^{n-2} \\ \ldots & \ldots & \ldots \\ u_1 & \ldots & u_{n-1} \\ 1 & \ldots & 1 \end{pmatrix}. \tag{6.8}$$

Consequently, the inverse matrix has the following form:

$$J^{-1}(\frac{Du}{D\psi}) = J(\frac{D\psi}{Du}) == \begin{pmatrix} \frac{u_1^{n-2}}{f(u_1)} & \dots & \frac{u_{n-1}^{n-2}}{f(u_{n-1})} \\ \dots & \dots & \dots \\ \frac{u_1}{f(u_1)} & \dots & \frac{u_{n-1}}{f(u_{n-1})} \\ \frac{1}{f(u_1)} & \dots & \frac{1}{f(u_{n-1})} \end{pmatrix}. \tag{6.9}$$

In this way, we come to the following system of equations for function $\phi_k(u_i)$:

$$d\psi_k = \sum_{i=1}^{n-1} \frac{u_i^{n-k-1}}{f(u_i)}\, du_i,\ \ k = 1, 2, \dots, n-1. \tag{6.10}$$

In the general case, elements of Jacobian matrix in (6.3) are defined by equations (5.31). We have

$$J(\frac{Du}{D\psi}) =$$
$$= \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,n-1} \\ A_{2,1} & A_{2,2} & \dots & A_{2,n-1} \\ \dots & \dots & \dots & \dots \\ A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,3} \end{pmatrix} \begin{pmatrix} 1 & a_1 & \dots & M_{1,n-1} \\ 0 & 1 & a_1 \dots & M_{2,n-1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_1 \\ 0 & 0 & \dots & 1 \end{pmatrix}. \tag{6.11}$$

Jacobian of inverse transformation is defined by matrix

$$J^{-1}(\frac{Du}{D\psi}) = J(\frac{D\psi}{Du}) = \begin{pmatrix} \partial_{u_1}\phi_1 & \partial_{u_2}\phi_1 & \dots & \partial_{u_{n-1}}\phi_1 \\ \dots & \dots & \dots & \dots \\ \partial_{u_1}\phi_{n-2} & \partial_{u_2}\phi_{n-2} & \dots & \partial_{u_{n-1}}\phi_{n-2} \\ \partial_{u_1}\phi_{n-1} & \partial_{u_2}\phi_{n-1} & \dots & \partial_{u_{n-1}}\phi_{n-1} \end{pmatrix}.$$

On making use of Lemma 4.8 and formulae (6.7) and (6.8), we get

$$J(\frac{D\psi}{Du}) =$$
$$= \begin{pmatrix} 1 & -a_1 & \dots & a_{n-2}(-1)^n \\ 0 & 1 & \dots & a_{n-3}(-1)^{n-1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -a_1 \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} u_1^{n-2} & u_2^{n-2} & \dots & u_{n-1}^{n-2} \\ u_1^{n-1} & u_2^{n-1} & \dots & u_{n-1}^{n-1} \\ \dots & \dots & \dots & \dots \\ u_1 & u_2 & \dots & u_{n-1} \\ 1 & 1 & \dots & 1 \end{pmatrix}. \tag{6.12}$$

7. EXAMPLE

It is useful to illustrate the method in the case of evolution equation generated by polynomial given in an explicit form. Let us consider an evolution generated by the polynomial of sixth order. The generating polynomial of $n = 6$ order is written in the form

$$f(X) = X^6 - a_1X^5 + a_2X^4 - a_3X^3 + a_4X^2 - a_5X + a_6. \tag{7.1}$$

Let U be one of the roots of the polynomial $Q(U)$. Then the function U obeys the following system of equations:

$$F(U)\partial_1 U = A_1\ f(U), \tag{7.2}$$

$$F(U)\partial_2 U = (A_2 + a_1A_1)f(U),$$

$$F(U)\partial_3 U = (A_3 + A_2a_1 + (a_1^2 - a_2)A_1)f(U),$$

$$F(U)\partial_4 U = (A_4 + a_1A_3 + (a_1^2 - a_2)A_2 + (a_3 - 2a_1a_2 + a_1^3)A_1)f(U),$$

$$F(U)\partial_5 U = (A_5 + A_4a_1 + (a_1^2 - a_2)A_3 + (a_3 - 2a_1a_2 + a_1^3)A_2 + \\ + (-a_4 + 2a_1a_3 - 3a_1^2a_2 + a_2^2 + a_1^4)A_1)f(U),$$

where

$$A_1 = 1,\ -A_2 = V + W + Y + Z,\ A_3 = VW + VY + VZ + WY + WZ + YZ,$$

$$-A_4 = WVY + WVZ + WYZ + VYZ,\ A_5 = WVYZ, \tag{7.3}$$

and polynomial $F(U)$ is defined by

$$F(U) = (U-V)(U-W)(U-Y)(U-Z) = A_1U^4 + A_2U^3 + A_3U^2 + A_4U + A_5. \tag{7.4}$$

The system (7.2) is written in the matrix form as follows:

$$F(U)\ \partial_k\ U = \sum_{j=1}^{5} M_{kj}\ A_j,\ k = 1, 2, 3, 4, 5, \tag{7.5}$$

with matrix

$$M_{kj} = \\ = \begin{pmatrix} 1 & a_1 & -a_2 + a_1^2 & a_3 - 2a_1a_2 + a_1^3 & -a_4 + 2a_1a_3 - 3a_1^2a_2 + a_2^2 + a_1^4 \\ 0 & 1 & a_1 & -a_2 + a_1^2 & a_3 - 2a_1a_2 + a_1^3 \\ 0 & 0 & 1 & a_1 & -a_2 + a_1^2 \\ 0 & 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.6}$$

Notice that the inverse matrix has a more simple form:

$$M_{ij}^{-1} = \begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & a_4 \\ 0 & 1 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (7.7)$$

By using the inverse matrix, the system of Eqs. (7.5) is transformed to

$$\begin{gathered} F(U)\ (\partial_5 - a_1\partial_4 + a_2\partial_3 - a_3\partial_2 + a_4\partial_1)U = A_5\ f(U), \\ F(U)\ (\partial_4 - a_1\partial_3 + a_2\partial_2 - a_3\partial_1)U = A_4\ f(U), \\ F(U)\ (\partial_3 - a_1\partial_2 + a_2\partial_1)U = A_3\ f(U), \\ F(U)(\partial_2 - a_1\partial_1)U = A_2\ f(U), \\ F(U)\partial_1 U = A_1\ f(U). \end{gathered} \qquad (7.8)$$

Now, we collect these equations into one equation by taking into account the identity (7.4). In this way we come to the equation containing unique unknown U:

$$\begin{pmatrix} 1 & U & U^2 & U^3 & U^4 \end{pmatrix} \begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & a_4 \\ 0 & 1 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_5 \\ \partial_4 \\ \partial_3 \\ \partial_2 \\ \partial_1 \end{pmatrix} U = f(U). \qquad (7.9)$$

This matrix equation can be written in the form either

$$\begin{gathered} (U^4\partial_1 + U^3(\partial_2 - a_1\partial_1) + U^2(\partial_3 - a_1\partial_2 + a_2\partial_1) + \\ + U(\partial_4 - a_1\partial_3 + a_2\partial_2 - a_3\partial_1) + \\ + (\partial_5 - a_1\partial_4 + a_2\partial_3 - a_3\partial_2 + a_4\partial_1))\ U = f(U), \end{gathered} \qquad (7.10)$$

or

$$\begin{gathered} (\ (U^4 - a_1U^3 + a_2U^2 - a_3U + a_4)\partial_1 + \\ + (U^3 - a_1U^2 + a_2U - a_3)\partial_2 + \\ + (U^2 - a_1U + a_2)\partial_3 + \\ + (U - a_1)\partial_4 + \partial_5)\ U = f(U). \end{gathered} \qquad (7.11)$$

The Jacobian of inverse mapping is defined by the matrix

$$J^{-1} = \begin{pmatrix} \partial_u\phi_1 & \partial_v\phi_1 & \partial_w\phi_1 & \partial_y\phi_1 & \partial_z\phi_1 \\ \partial_u\phi_2 & \partial_v\phi_2 & \partial_w\phi_2 & \partial_y\phi_2 & \partial_z\phi_2 \\ \partial_u\phi_3 & \partial_v\phi_3 & \partial_w\phi_3 & \partial_y\phi_3 & \partial_z\phi_3 \\ \partial_u\phi_4 & \partial_v\phi_4 & \partial_w\phi_4 & \partial_y\phi_4 & \partial_z\phi_4 \\ \partial_u\phi_5 & \partial_v\phi_5 & \partial_w\phi_5 & \partial_y\phi_5 & \partial_z\phi_5 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & a_4 \\ 0 & 1 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} U^4/f(u) & V^4/f(v) & W^4/f(w) & Y^4/f(y) & Z^4/f(z) \\ U^3/f(u) & V^3/f(v) & W^3/f(w) & Y^3/f(y) & Z^3/f(z) \\ U^2/f(u) & V^2/f(v) & W^2/f(w) & Y^2/f(y) & Z^2/f(z) \\ U/f(u) & V/f(v) & W/f(w) & Y/f(y) & Z/f(z) \\ 1/f(u) & 1/f(v) & 1/f(w) & 1/f(y) & 1/f(z) \end{pmatrix}. \qquad (7.12)$$

By using the notations $u_1 = u,\ u_2 = v,\ u_3 = w,\ u_4 = y,\ u_5 = z$, the matrix equation is reduced to

$$\frac{\partial\phi_1}{\partial u_i} = \frac{1}{f(u_i)}(u_i^4 - a_1u_i^3 + a_2u_i^2 - a_3u_i + a_4),$$

$$\frac{\partial\phi_2}{\partial u_i} = \frac{1}{f(u_i)}(u_i^3 - a_1u_i^2 + a_2u_i + a_3),$$

$$\frac{\partial\phi_3}{\partial u_i} = \frac{1}{f(u_i)}(u_i^2 - a_1u_i + a_2),$$

$$\frac{\partial\phi_4}{\partial u_i} = \frac{1}{f(u_i)}(u_i - a_1),$$

$$\frac{\partial\phi_5}{\partial u_i} = \frac{1}{f(u_i)}. \qquad (7.13)$$

Matrix form of this system of equations is

$$\frac{\partial}{\partial u_i}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \frac{1}{f(u_i)}\begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & a_4 \\ 0 & 1 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 1 & -a_1 & a_2 \\ 0 & 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u_i^4 \\ u_i^3 \\ u_i^2 \\ u_i \\ 1 \end{pmatrix}. \qquad (7.14)$$

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Received on November 20, 2015.

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Подписано в печать 25.12.2015.
Формат 60 × 90/16. Бумага офсетная. Печать офсетная.
Усл. печ. л. 1,81. Уч.-изд. л. 2,49. Тираж 235 экз. Заказ № 58718.

Издательский отдел Объединенного института ядерных исследований
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